Summary and more

Summary and less!

Summary and more or less?

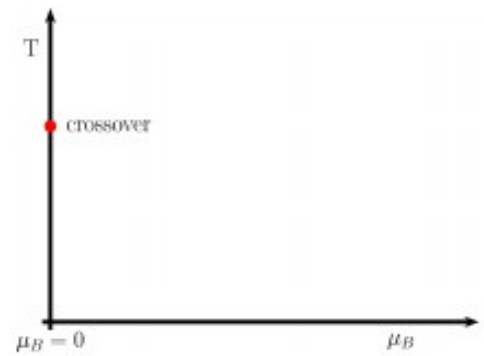
More or less a Summary !

How to study the QCD phase diagram...

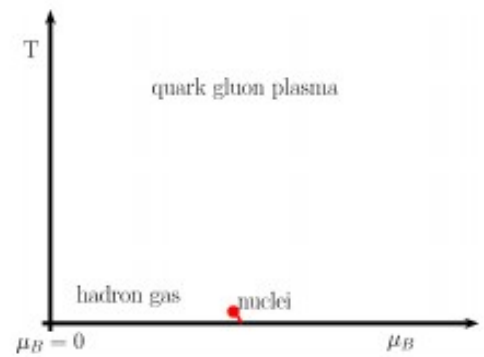
... be brave and solve

$$Z(T,\mu_B) = \int \mathcal{D}(A,q,q^\dagger)\mathrm{e}^{-S^E_{\mathrm{QCD}}}$$

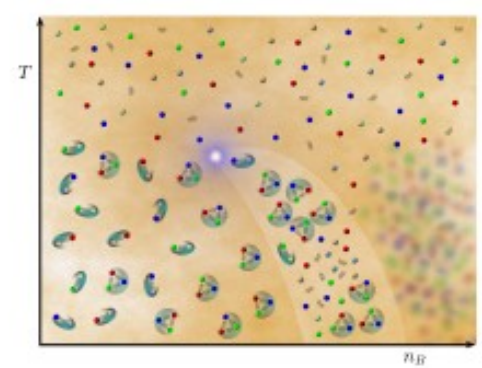
ab initio and nonperturbatively,



... be strong and collide heavy ions at ultrarelativistic energies,



... be creative and study effective models of QCD.



State-of-the-art:

the thermal conditions at freeze-out are determined by comparing experimental results for ratios of particle yields with the Hadron Resonance Gas (HRG) model

the goal:

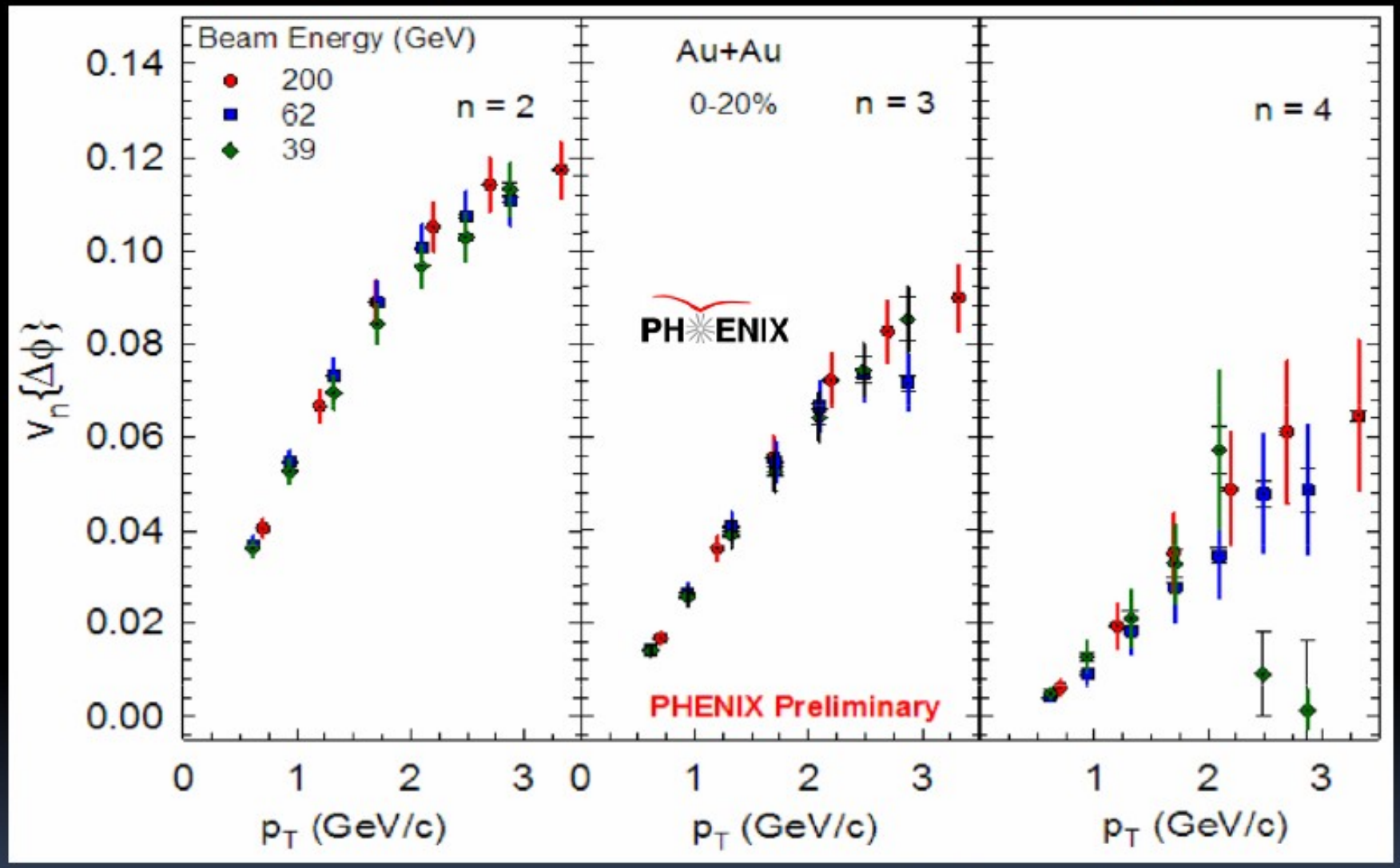
eventually we should get rid of model calculations and should be able to determine the freeze-out parameters directly by comparing experimental results with (lattice) QCD calculations

Squeezing out the middle man?

Boxing in the phase structure of QCD

- The baseline: Is it more hadronic or more partonic
- The subtleties: Critical point, cross over, etc:

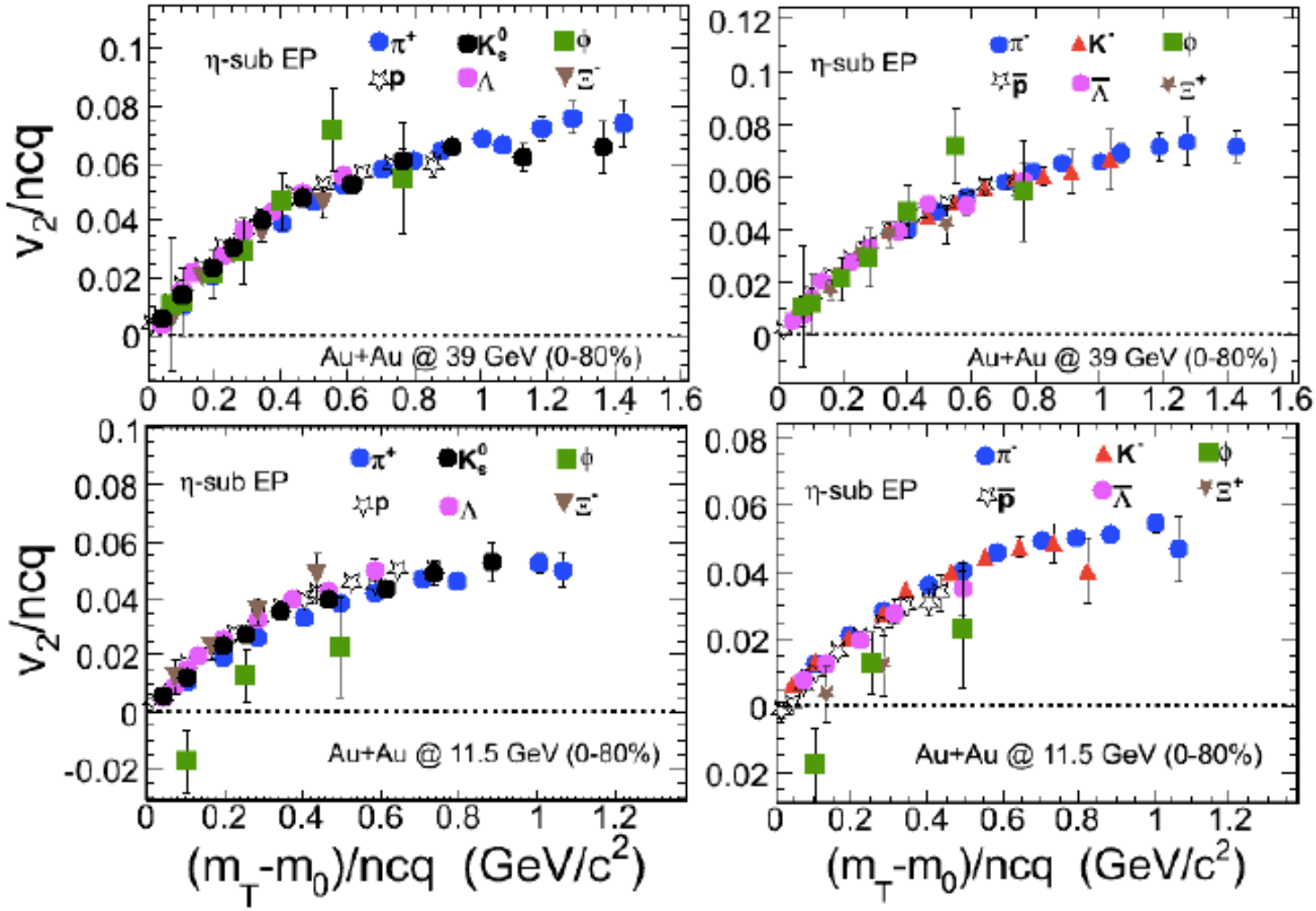
v_2, v_3, v_4 as a function of $\sqrt{s_{NN}}$



v_2, v_3, v_4 are independent of $\sqrt{s_{NN}}$ for 39, 62.4, 200 GeV

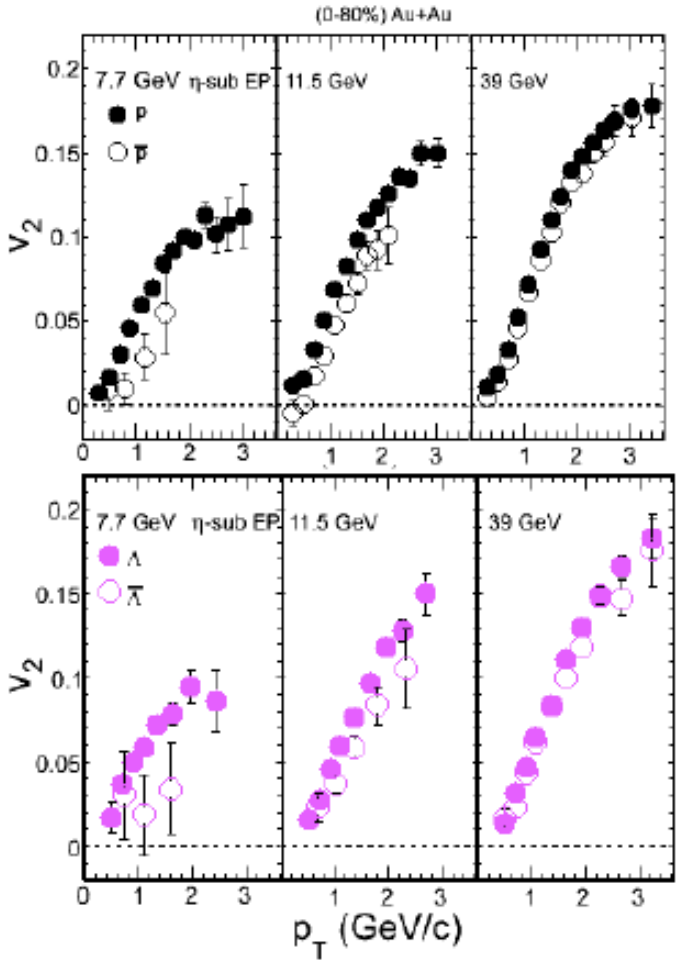
Mitchell

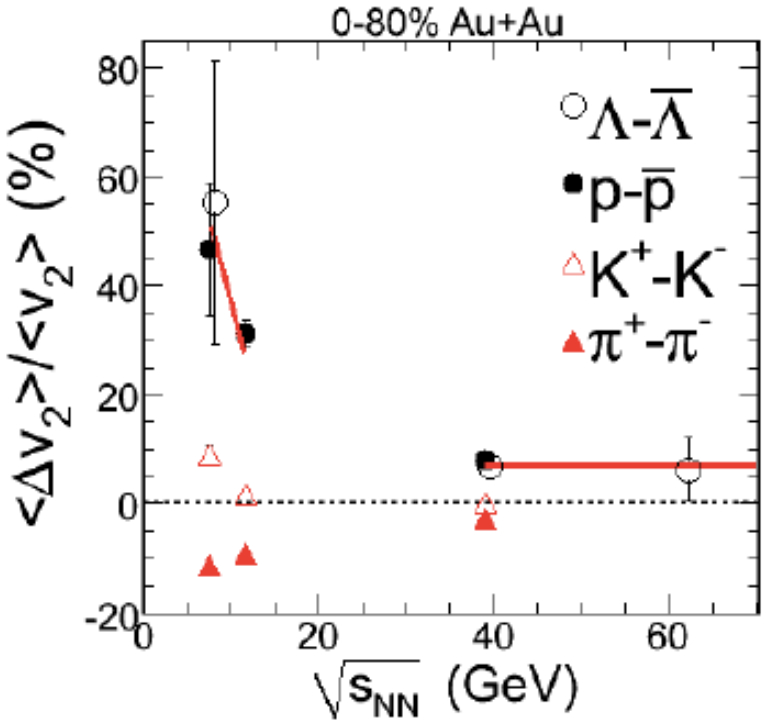
φ-meson v_2 vs. $\sqrt{s_{NN}}$



- The φ-meson v_2 falls off trend from other hadrons at 11.5 GeV
- An effect of 2.6σ

(anti-)Particle v_2 vs. $\sqrt{s_{NN}}$





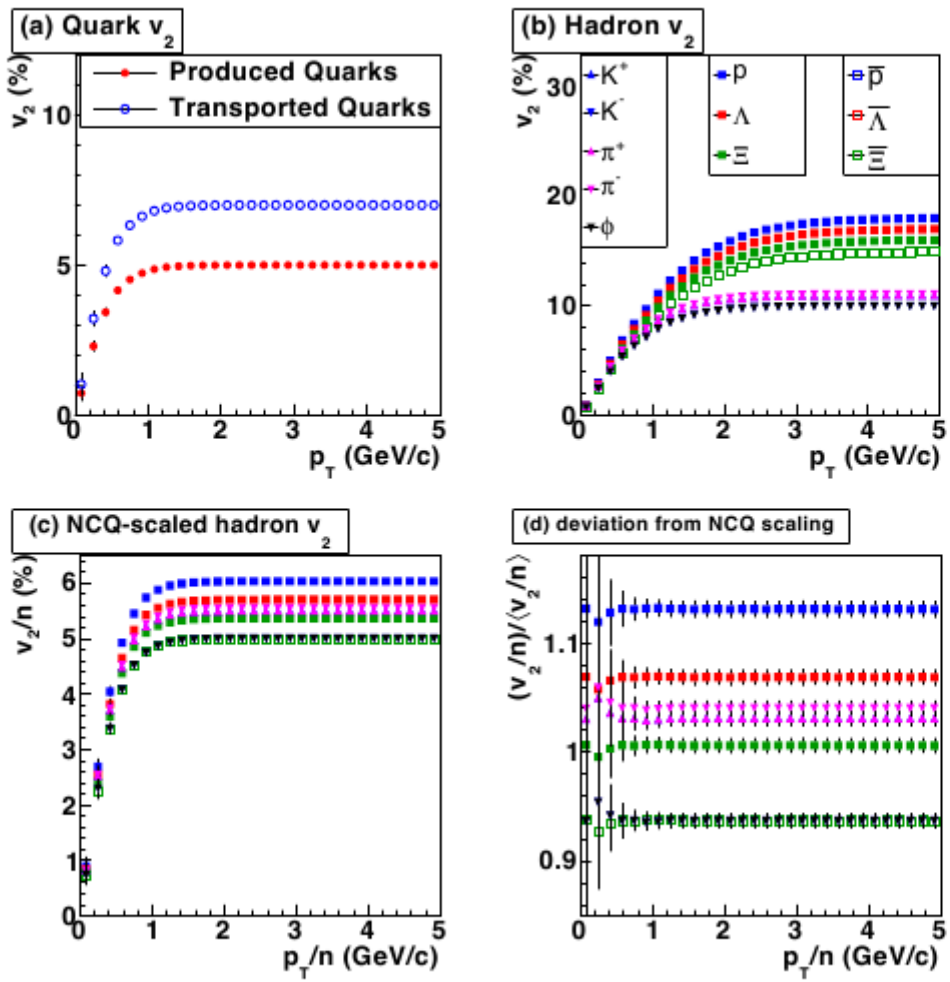
At $\sqrt{s_{NN}} \leq 11.5$ GeV:

- v_2(baryon) > v_2(anti-baryon)
- $v_2(\pi^+) < v_2(\pi^-)$
- $v_2(K^-) < v_2(K^+)$

STAR: Quark Matter 2011

Hadronic interactions appear dominant

Constituent quark coalescence, accounting for stopping...



NCQ scaling break-down

$v_2[\pi^+] < v_2[\pi^-]$ ✓

$v_2[K^+] > v_2[K^-]$ ✓

$v_2[p] > v_2[\bar{p}]$ ✓

$v_2[\Lambda] > v_2[\bar{\Lambda}]$ ✓

$v_2[\bar{\Xi}^+] < v_2[\Xi^-]$ ✓

J.C. Dunlop, MAL, P. Sorensen arXiv:1107.3078

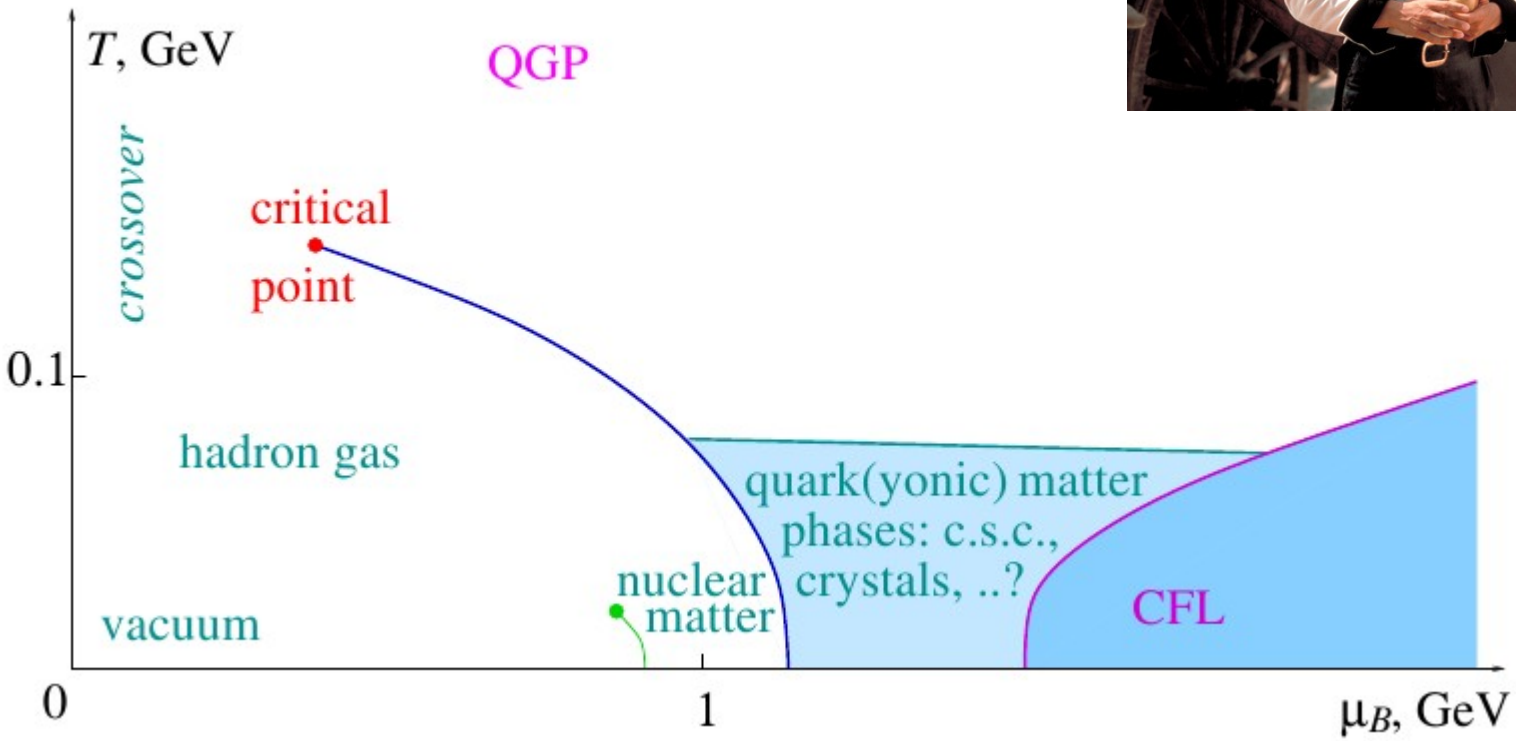
Testable idea!!!

Simple absorption may also work...

Is the matter hadronic or partonic
at
lower energies?

YES!!!

The villain(s)



Higher moments (cumulants) and ξ

- Consider probability distribution for the order-parameter field:

$$P[\sigma] \sim \exp\{-\Omega[\sigma]/T\}\,,$$

$$\Omega = \int d^3x \left[\frac{1}{2}(\boldsymbol{\nabla}\sigma)^2 + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \ldots\right]. \qquad \Rightarrow \quad \xi = m_\sigma^{-1}$$

- Moments (connected) of $\boldsymbol{q} = 0$ mode $\sigma_V \equiv \int d^3x\,\sigma(x)$:

$$\kappa_2 = \langle\sigma_V^2\rangle = VT\,\xi^2\,; \qquad \kappa_3 = \langle\sigma_V^3\rangle = 2VT^2\,\lambda_3\,\xi^6\,;$$

$$\kappa_4 = \langle\sigma_V^4\rangle_c \equiv \langle\sigma_V^4\rangle - 3\langle\sigma_V^2\rangle^2 = 6VT^3\,[\,2(\lambda_3\xi)^2 - \lambda_4\,]\,\xi^8\,.$$

- Tree graphs. Each propagator gives ξ^2.

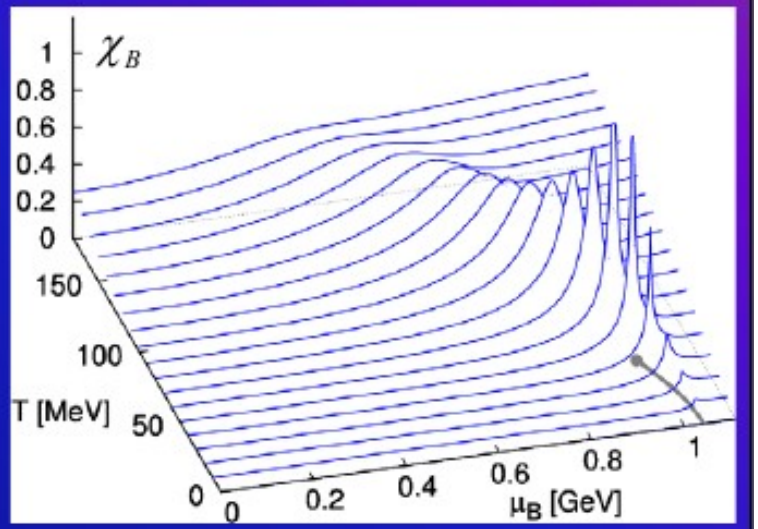
- Scaling requires "running": $\lambda_3 = \tilde{\lambda}_3 T(T\xi)^{-3/2}$ and $\lambda_4 = \tilde{\lambda}_4(T\xi)^{-1}$, i.e.,

$$\kappa_3 = \langle\sigma_V^3\rangle = 2VT^{3/2}\,\tilde{\lambda}_3\,\xi^{4.5}\,; \quad \kappa_4 = 6VT^2\,[\,2(\tilde{\lambda}_3)^2 - \tilde{\lambda}_4\,]\,\xi^7\,.$$

Physical Meaning of 3rd Fluc. Moment

χ_B : Baryon number susceptibility

in general, has a peak along phase transition



$\Rightarrow \dfrac{\partial \chi_B}{\partial \mu_B}$ changes the sign at QCD phase boundary !

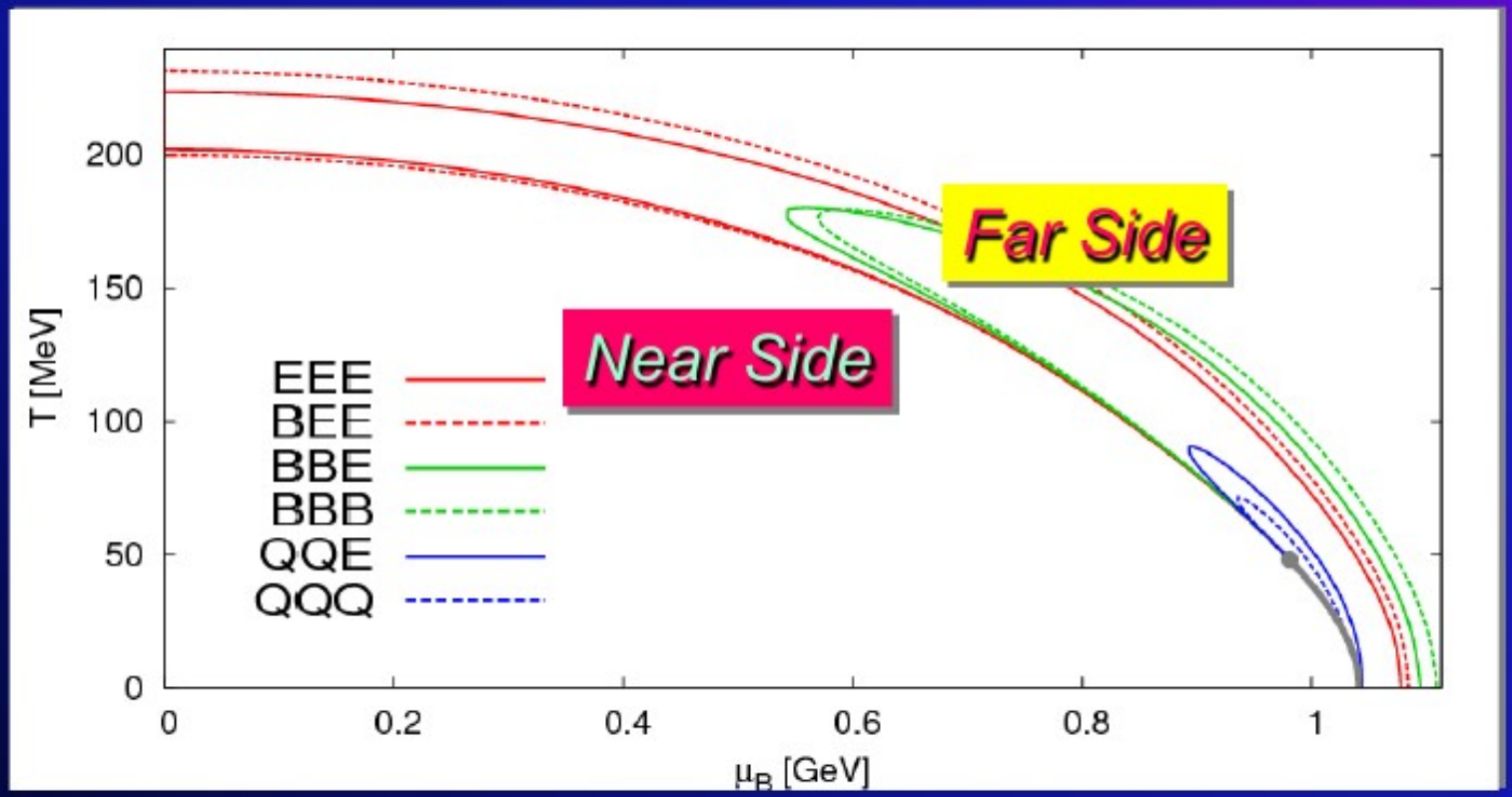
- In the Language of fluctuation moments:

$$\chi_B = -\frac{1}{V}\frac{\partial^2 \Omega}{\partial \mu_B^2} = \frac{\left\langle (\delta N_B)^2 \right\rangle}{VT}$$

$$\frac{\partial \chi_B}{\partial \mu_B} = -\frac{1}{V}\frac{\partial^3 \Omega}{\partial \mu_B^3} = \frac{\left\langle (\delta N_B)^3 \right\rangle}{VT^2} \equiv m_3(\mathrm{BBB})$$

more information than usual fluctuation

Comparison of Various Moments



2-flavor NJL with standard parameters

G=5.5GeV^{-2}
m_q=5.5MeV
Λ=631MeV

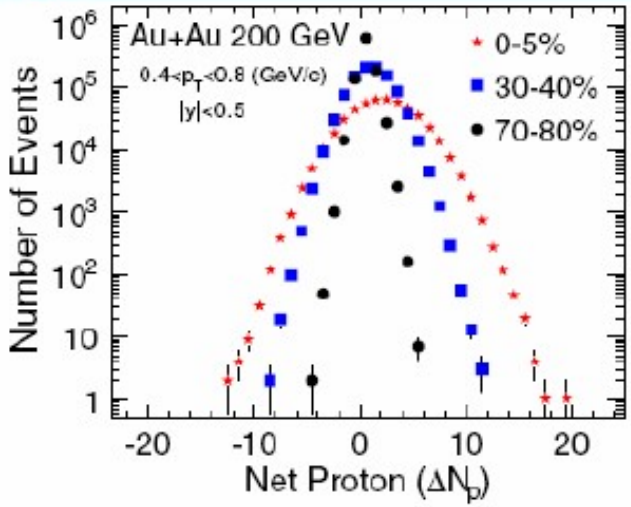
- Different moments have different regions with negative moments

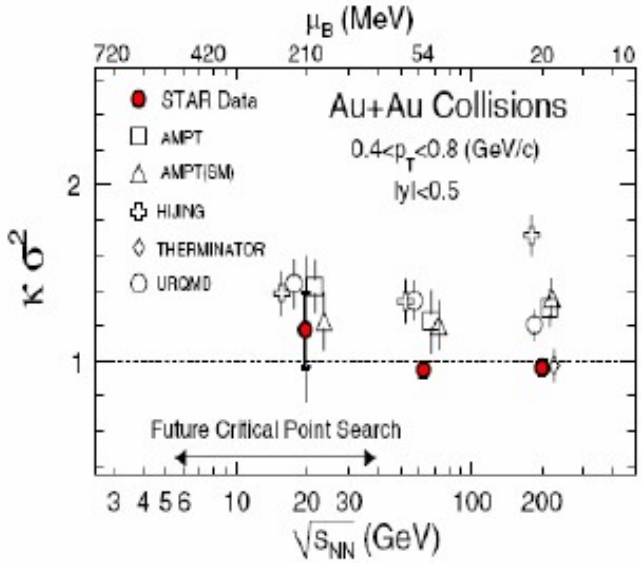
By comparing the signs of various moments, possible to pin down the origin of moments

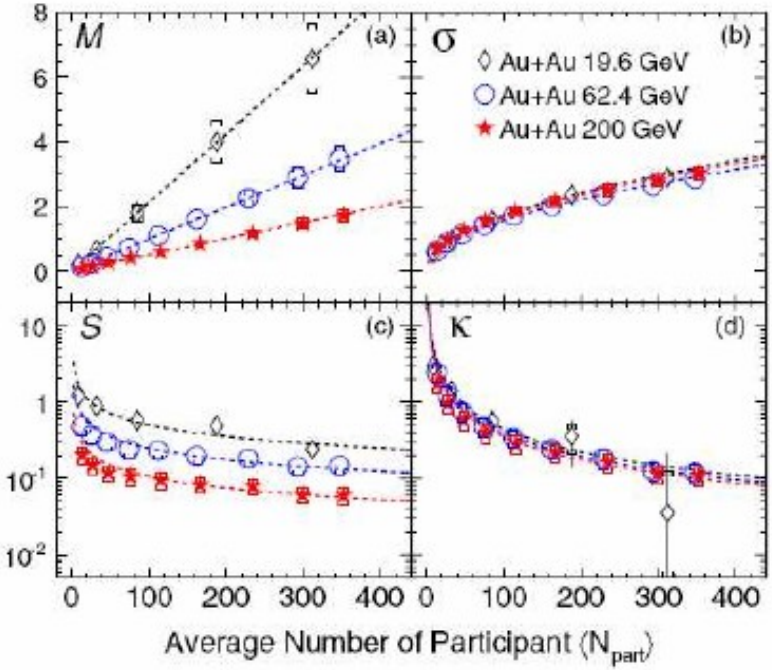
- Negative m_3(EEE) region extends to T-axis (in this particular model)
- Sign of m_3(EEE) may be used to estimate heat conductivity

M. Asakawa (Osaka University)

First result on higher moments of net-proton



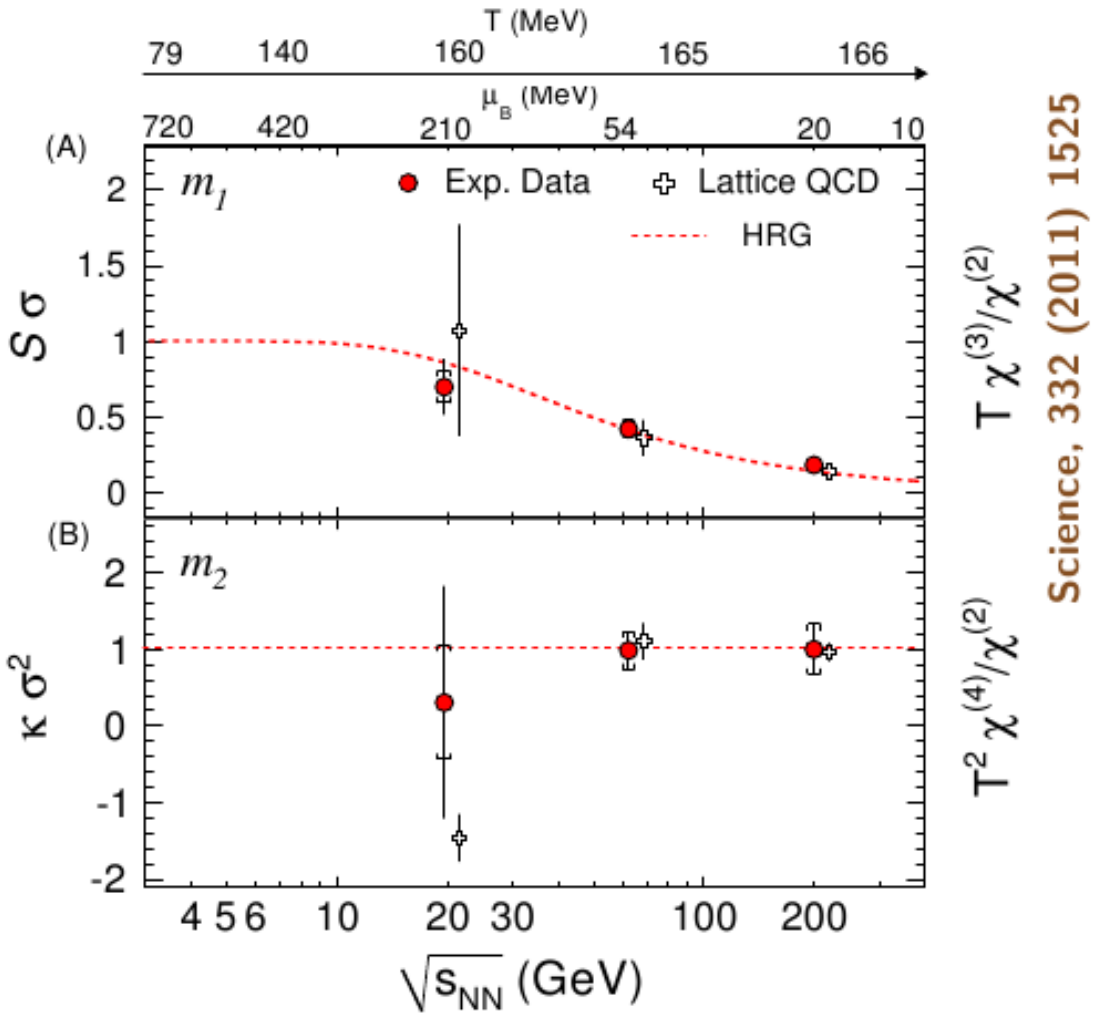




STAR Collaboration, PRL 105 (2010) 022302.

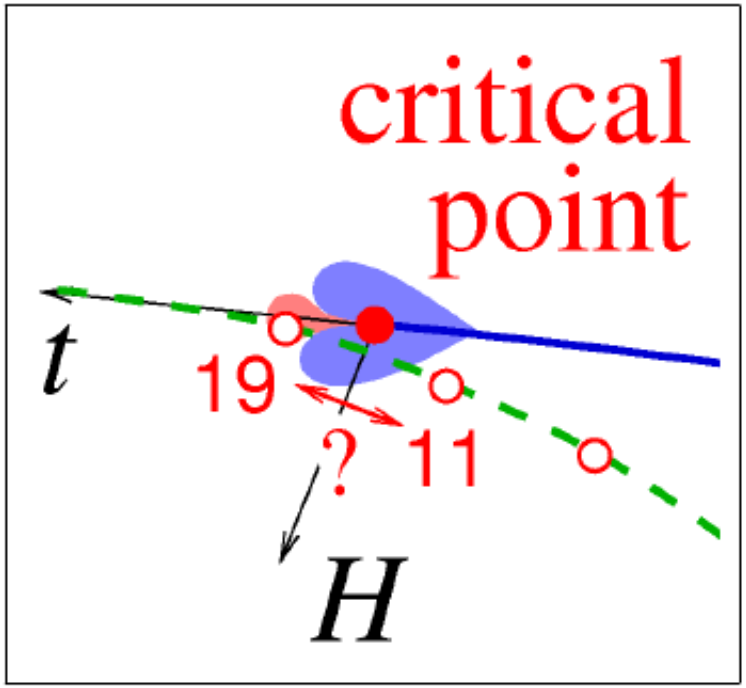
- STAR first results on higher moments analysis are up to fourth order.
- Using ratios used to establish base line measurements for the QCD critical point search.
- **This talk: C_6 / C_2 and C_4 / C_2.**

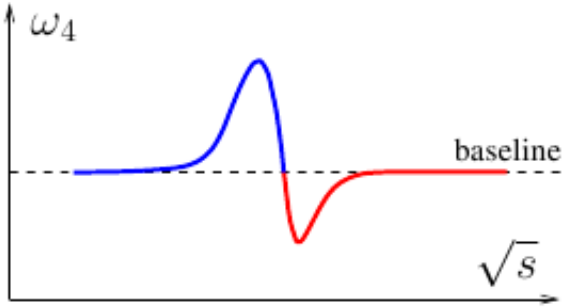
Checking the match



Science, 332 (2011) 1525

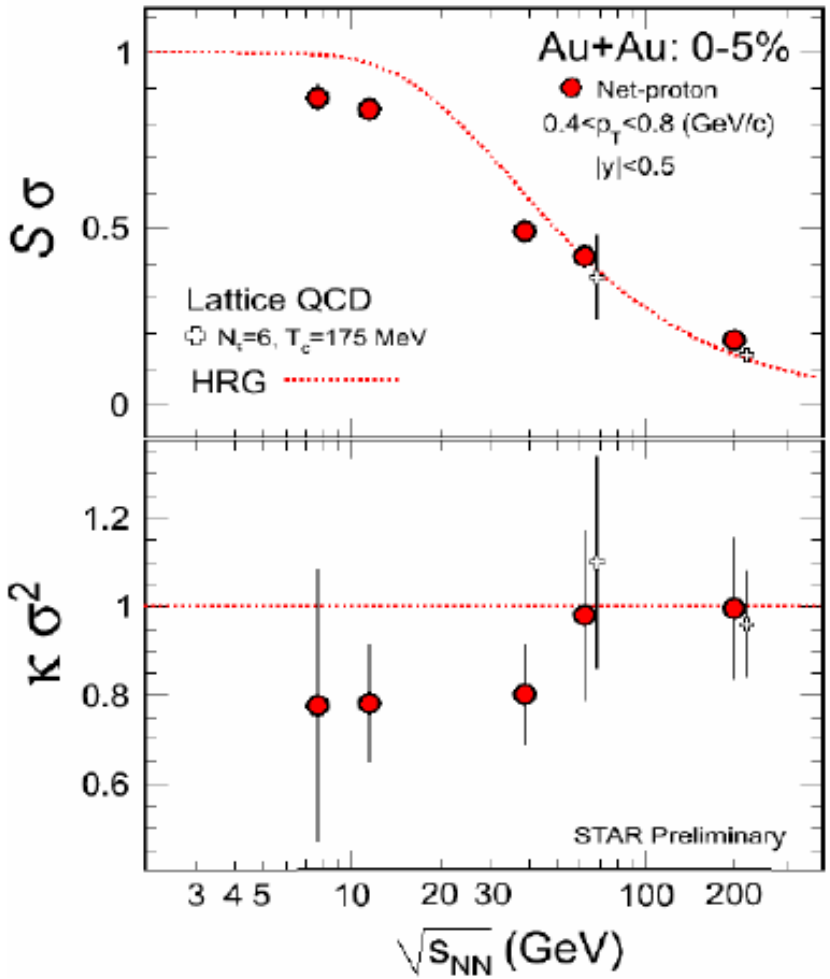
A scenario





- If the kurtosis stays significantly below Poisson value in 19 GeV data, the logical place to take a closer look is between 19 and 11 GeV.

Results (I): Energy dependence of product moments



X. Luo, SQM 2011

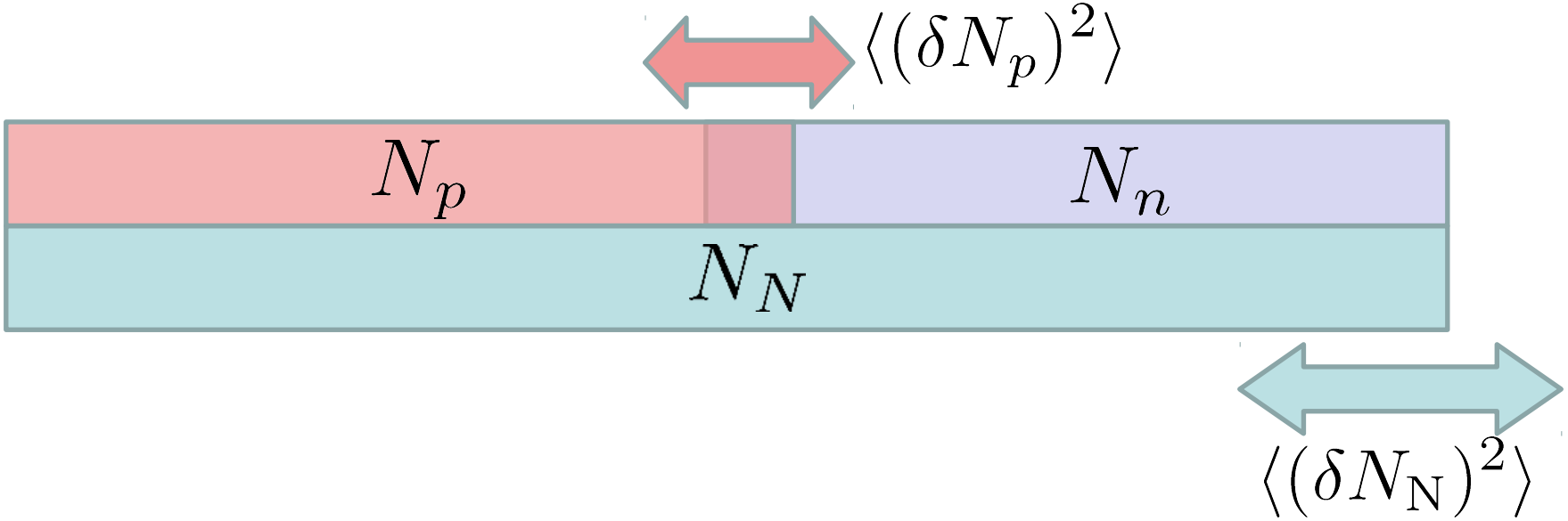
- Data are compared with HRG and Lattice QCD.

The return of the middle man

Protons vs Baryons

All protons are baryons but not all baryons are protons

Baryon & Proton Number Fluctuations



- In, general, fluctuations of *N*N and *N*p are different.
- Due to the isospin fluctuations, *N*p fluctuations tend to be close to equilibrium ones than *N*N fluctuations.

Kitazawa

Simple model

Consider only neutron, protons, and charged pions

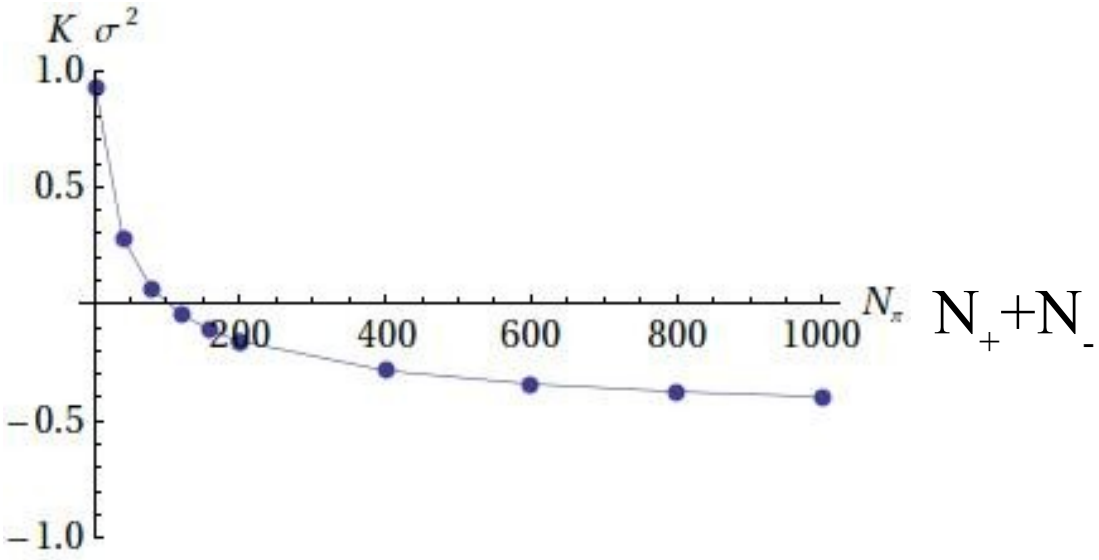
Conserved quantities: electr. Charge Q and baryon number B

Still fluctuations: Protons ↔ neutrons and $\pi^+ \leftrightarrow \pi^-$

$$Z = \sum_{n_p, n_n} \frac{P^{n_p} N^{n_n}}{n_p! n_n!} \delta_{B, n_p + n_n} \sum_{k_+, k_-} \frac{\pi_+^{k_+} \pi_-^{k_-}}{k_+! k_-!} \delta_{Q, n_p + k_+ - k_-}$$

Would be binomial if we ignore charge conservation

Simple Model



$N_+ + N_-$

Effect of Isospin Distribution

(1) $N_B^{(\mathrm{net})} = N_B - N_{\bar{B}}$ deviates from the equilibrium value.

(2) Boltzmann (Poisson) distribution for $N_B, N_{\bar{B}}$.

$$
\begin{cases}
2\langle(\delta N_p^{(\mathrm{net})})^2\rangle = \frac{1}{2}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^2\rangle + \frac{1}{2}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^2\rangle_{\mathrm{free}} \\
2\langle(\delta N_p^{(\mathrm{net})})^3\rangle = \frac{1}{4}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^3\rangle + \frac{3}{4}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^3\rangle_{\mathrm{free}} \\
2\langle(\delta N_p^{(\mathrm{net})})^4\rangle_c = \frac{1}{8}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^4\rangle_c + \frac{7}{8}\langle(\delta N_{\mathrm{B}}^{(\mathrm{net})})^4\rangle_{c,\mathrm{free}}
\end{cases}
$$

genuine info. noise

For free gas

$$2\langle(\delta N_p^{(\mathrm{net})})^n\rangle_c = \langle(\delta N_{\mathrm{N}}^{(\mathrm{net})})^n\rangle_c$$

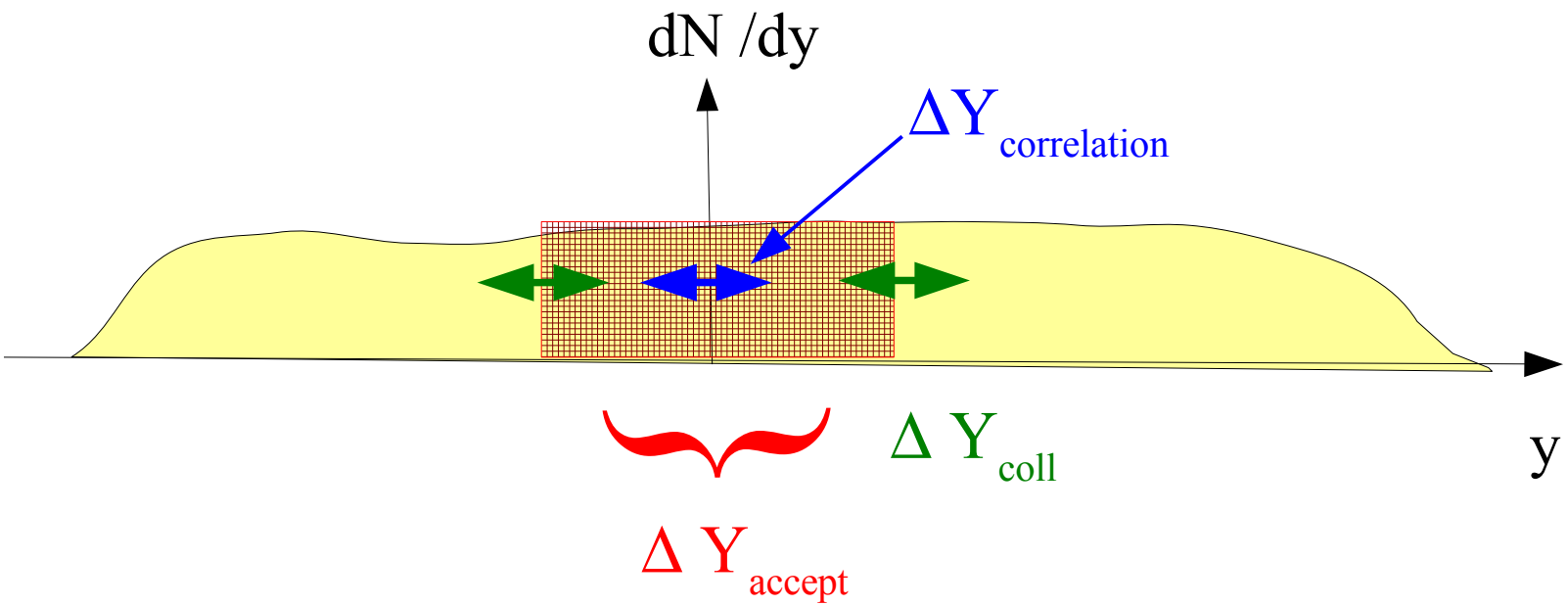
Kitizawa

Baryon number conservation

- Affects all susceptibilities: Variance, Kurtosis....
- Proton Fluctuations are also affected
  - Distinguish from Isospin fluctuations

- Still LARGE baryon DENSITY fluctuations

“Charge” fluctuations

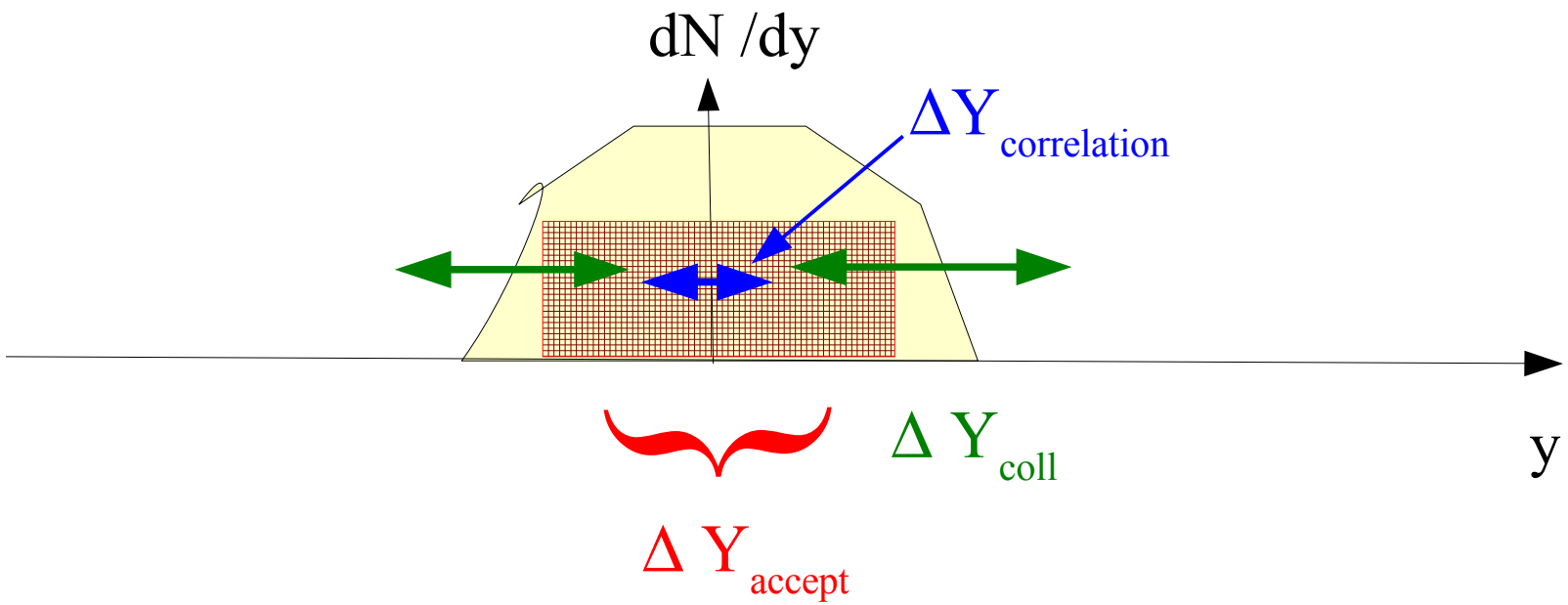


Condition for “charge” fluctuations:

0)$\Delta Y_{corrrelation} << \Delta Y_{accept}$ **(catch the physics)**

2)$\Delta Y_{total} >> \Delta Y_{accept} >> \Delta Y_{coll}$ **(keep the physics)**

“Charge” fluctuations at SPS and below



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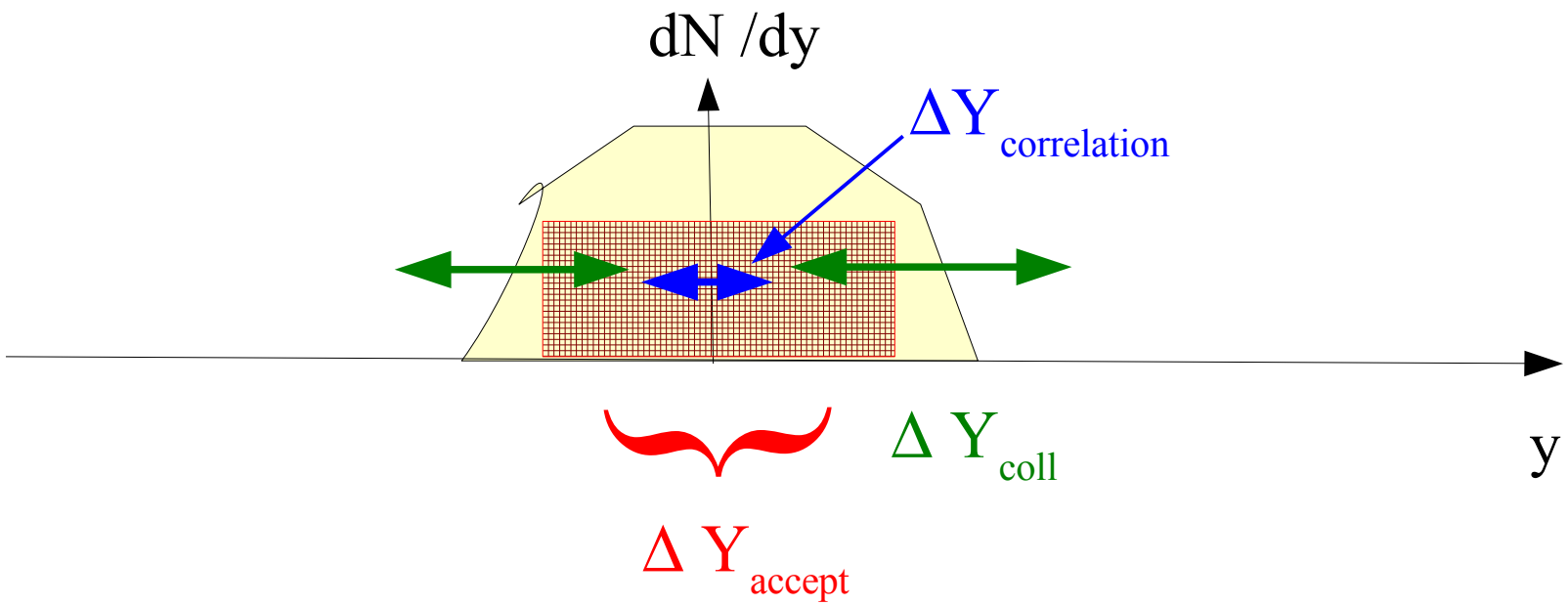

Condition for “charge” fluctuations:

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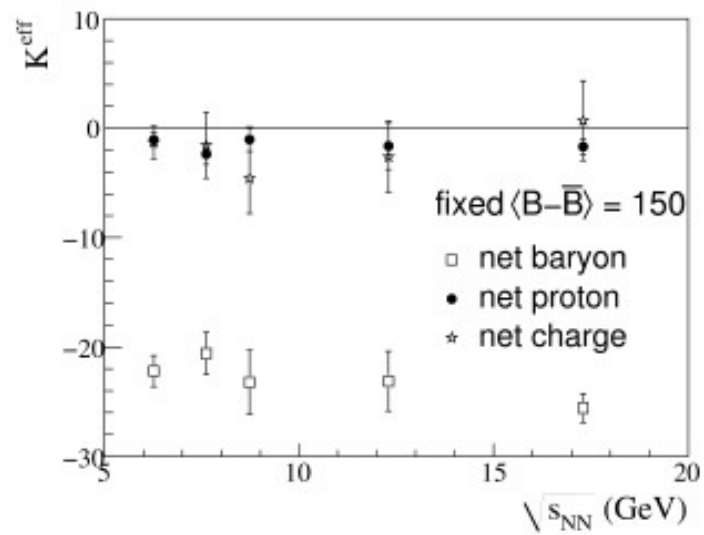
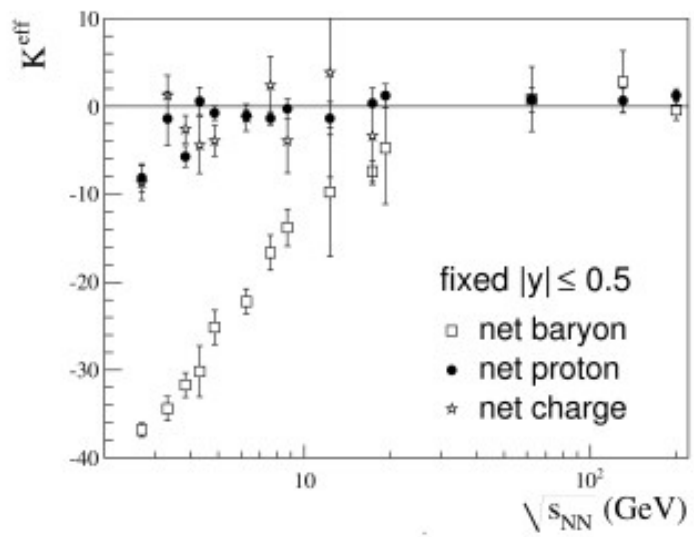
Energy dependence of the effective kurtosis

- adapting the rapidity window to fix the mean net-baryon number
- net-baryon effective kurtosis does not show an energy dependence



- fixed rapidity cut
- the net-baryon number varies with $\sqrt{s}$
- for lower $\sqrt{s}$ K^{eff} becomes increasingly negative
- at $E_{\mathrm{lab}} = 2A\mathrm{GeV}$: $\langle N_{B-\bar{B}} \rangle \simeq 240$

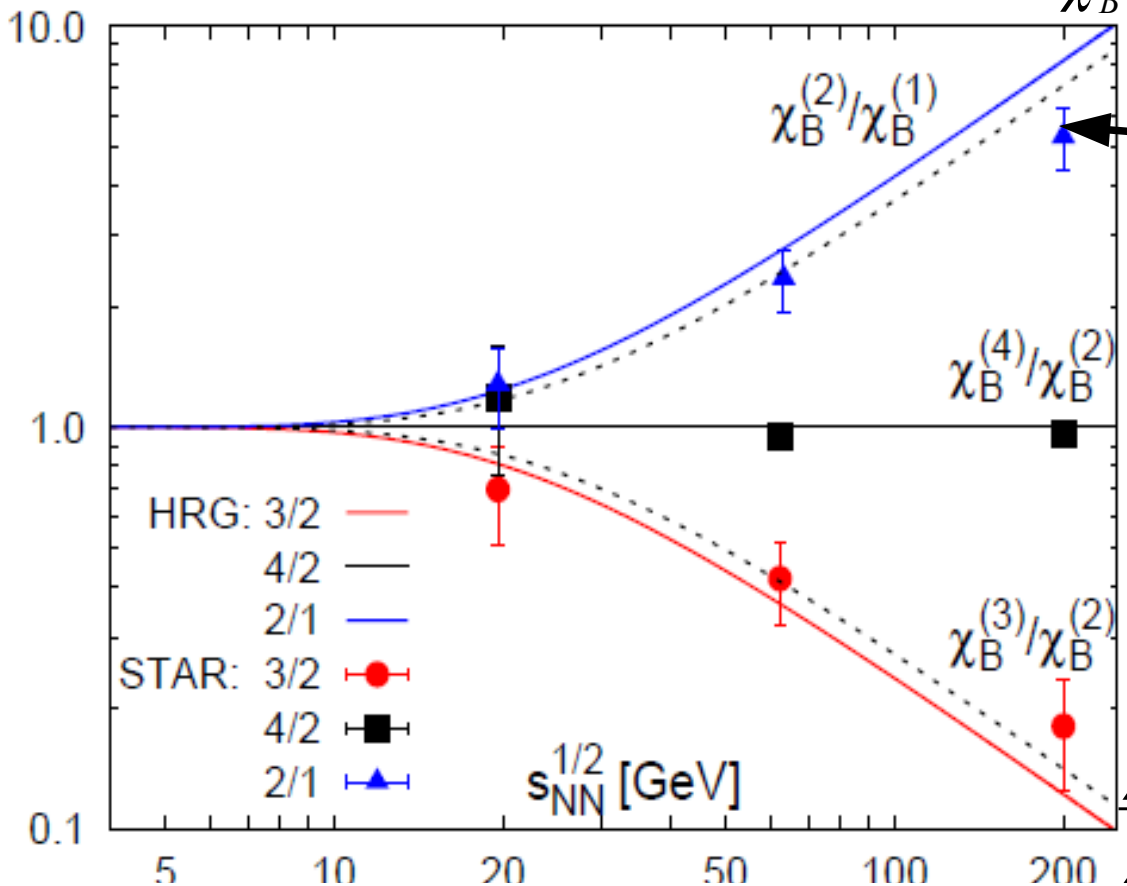


T. Schuster, MN, M. Mitrovski, R. Stock, M. Bleicher, [arXiv:0903.2911 [hep-ph]

Coparison of the Hadron Resonance Gas Model with STAR data

- Frithjof Karsch & K.R.

$$\frac{\chi_B^{(2)}}{\chi_B^{(1)}} \approx \coth(\mu_B / T)$$



Deviation due to Baryon Number conservation???

$$\frac{\chi_B^{(4)}}{\chi_B^{(2)}} = 1$$

$$\frac{\chi_B^{(3)}}{\chi_B^{(2)}} \approx \tanh(\mu_B / T)$$

deviations between HRG model and data for the variance ($\chi_B^{(2)}$)?

Error Estimation for Moments Analysis: Xiaofeng Luo arXiv:1109.0593v

Baryon Number conservation?

Only p and p-bar (zero net protons):

$$\langle(\delta N_p)^2\rangle = \frac{1}{2}\langle N_p\rangle$$

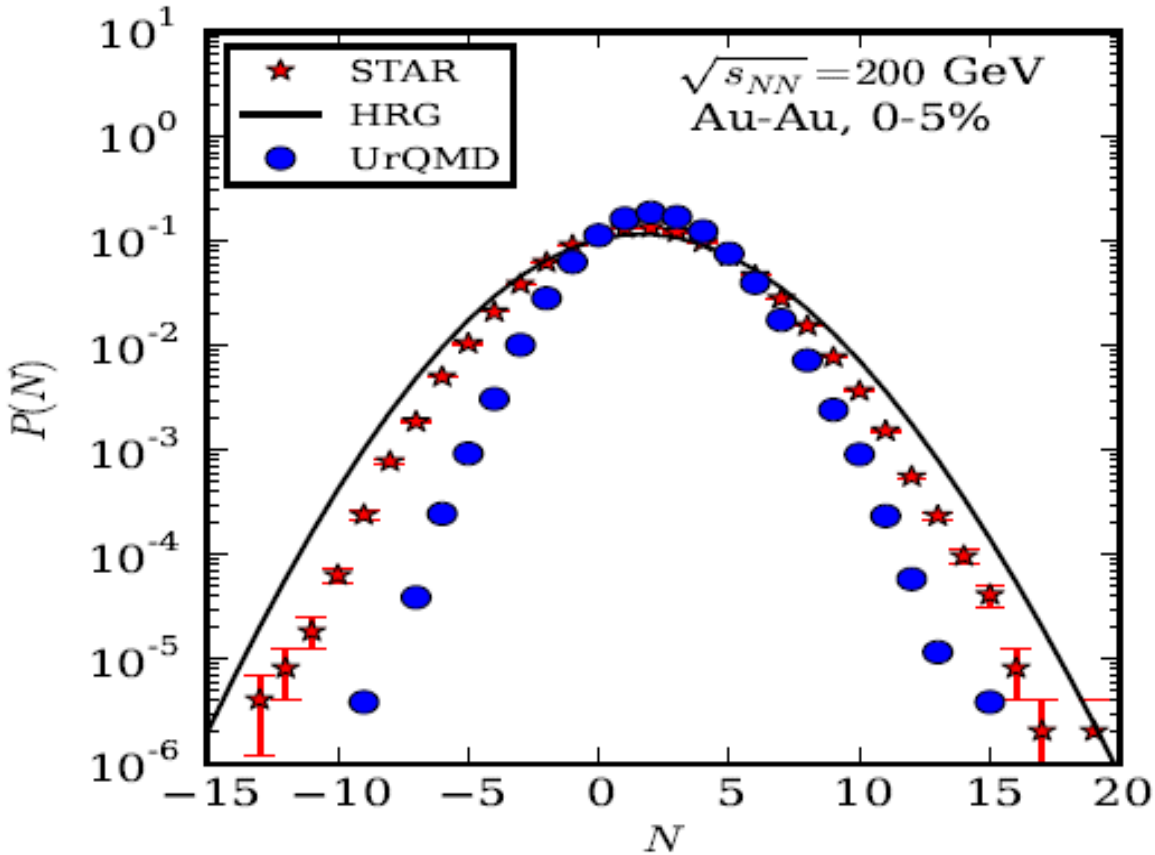
Binomial (finite acceptance, efficiency etc):

$$\langle(\delta N_p)^2\rangle_{binomial} = \langle N_p\rangle_{full}\, p(1-p) + p^2\langle(\delta N_p)^2\rangle_{full} = \langle N_p\rangle_{binomial}(1-\frac{1}{2}p)$$

$$\frac{\langle(\delta N_p)^2\rangle_{binomial}}{\langle N_p\rangle_{binomial}} = (1-\frac{1}{2}p)$$

Baryon Number conservation?

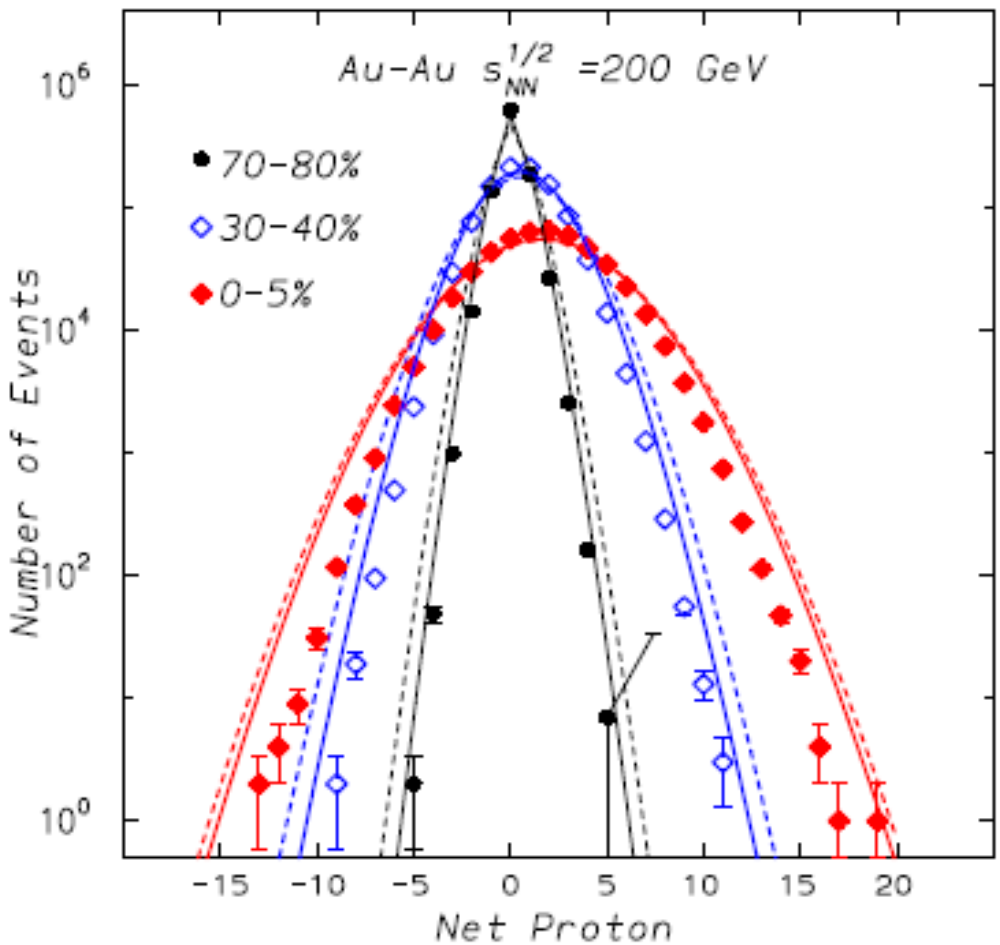
V. Skokov et al..



$$\frac{\langle (\delta N_p)^2 \rangle_{binomial}}{\langle N_p \rangle_{binomial}} = (1 - \frac{1}{2} p)$$

Peripheral collisions:
p <<1

Expect close to Poisson



$$\frac{\langle (\delta N_p)^2 \rangle_{binomial}}{\langle N_p \rangle_{binomial}} = (1 - \frac{1}{2} p)$$

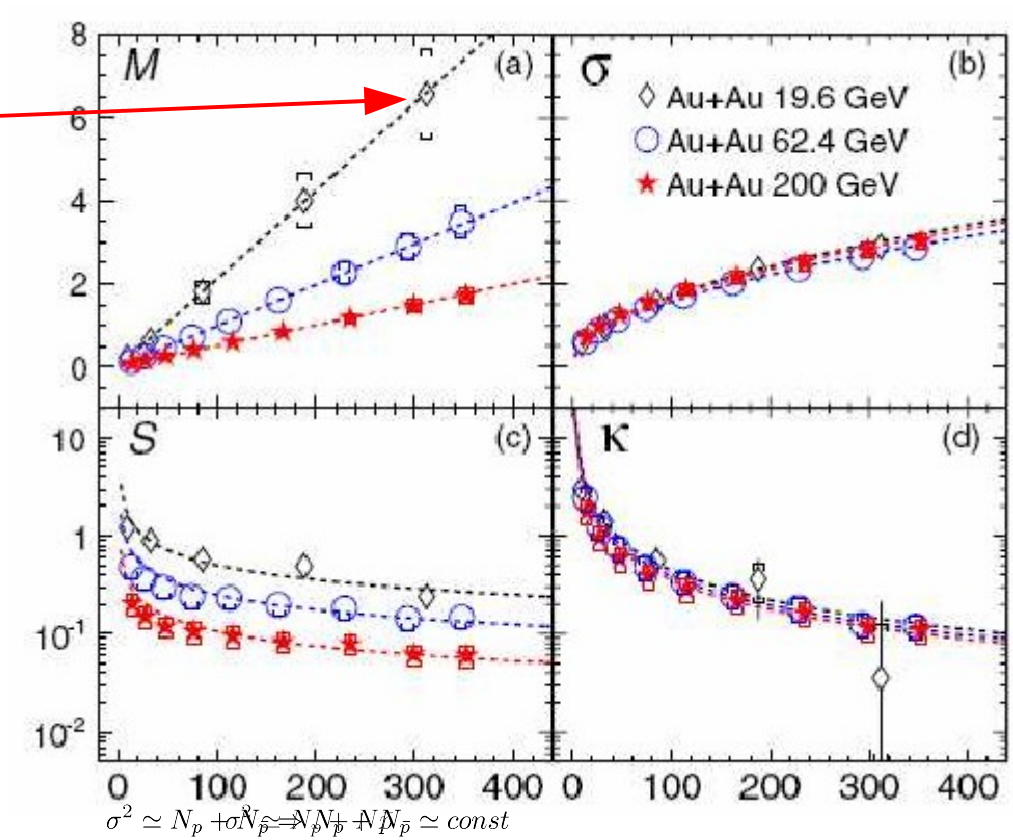
Peripheral collisions:
p <<1

Expect close to Poisson

Needs to be checked more quantitatively

Finite acceptance needs to be taken into account

8 protons!!!



$$\sigma^2 \approx N_p + N_{\bar{p}}$$
$$\Rightarrow N_p + N_{\bar{p}} \approx const$$

$$K_{binomial}[4] = p\,k[1] + p^2\,(-7\,k[1] + 7\,k[2]) + p^3\,(12\,k[1] - 18\,k[2] + 6\,k[3]) + p^4\,(-6\,k[1] + 11\,k[2] - 6\,k[3] + k[4])$$

Volume Fluctuations

$$N = \rho V$$
$$\delta N = V\,\delta\rho + \rho\,\delta V$$

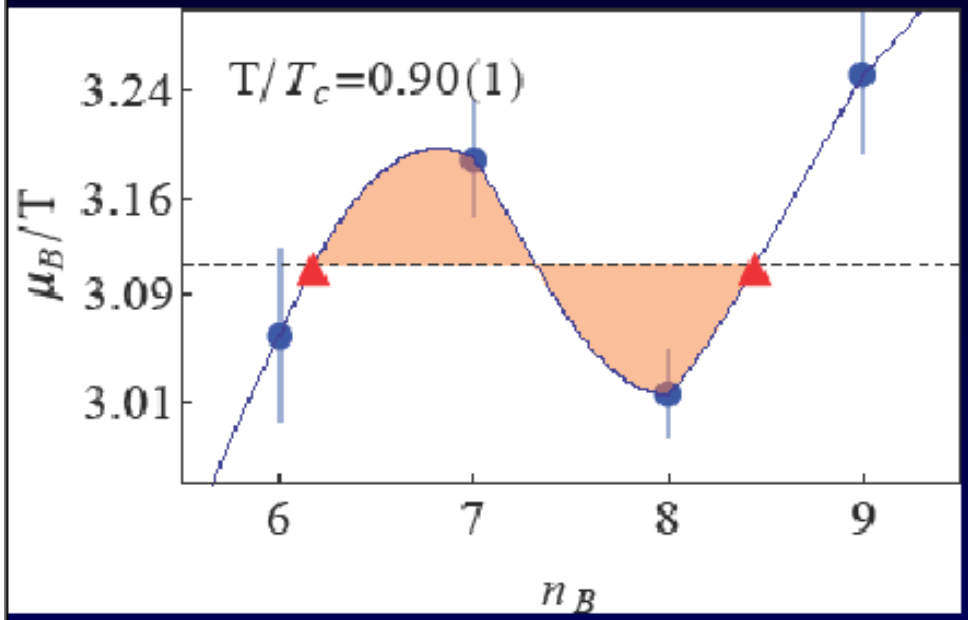
$$\langle(\delta N)^2\rangle = \langle V\rangle^2\langle(\delta\rho)^2\rangle + \langle\rho\rangle^2\langle(\delta V)^2\rangle$$
$$\langle(\delta N)^4\rangle = \langle V\rangle^4\langle(\delta\rho)^4\rangle + 6\langle V\rangle^2\langle\rho\rangle^2\langle(\delta V)^2\rangle\langle(\delta\rho)^2\rangle + \langle\rho\rangle^4\langle(\delta V)^4\rangle$$

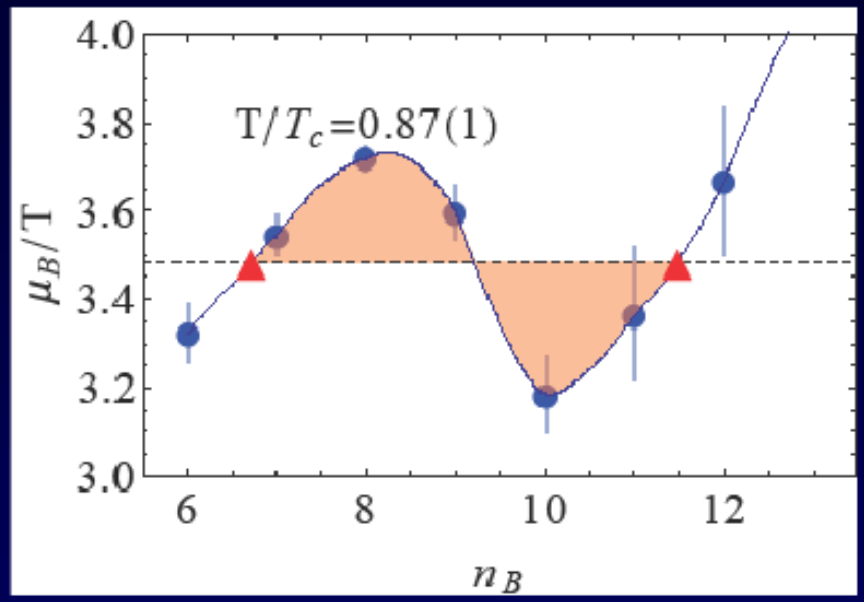
$K\sigma^2 = \frac{(\delta N)^4}{(\delta N)^2} - 3(\delta N)^2$ is independent of the volume but not independent of volume fluctuations

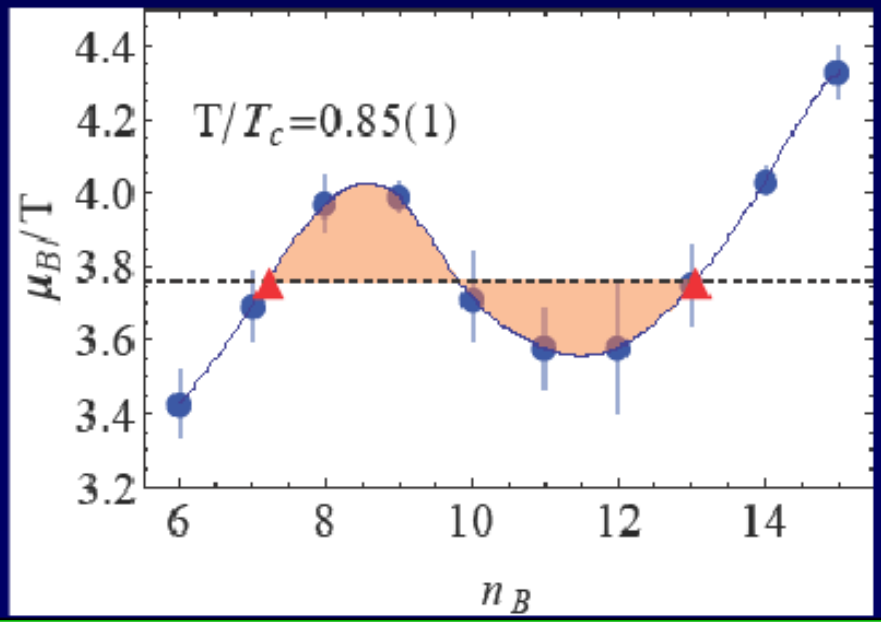
OK lets go slow and keep the middle man in the loop

Three flavor case
(m_π ~ 0.7 GeV, a~ 0.3 fm)

6^3 x 4 lattice,
Clover fermion

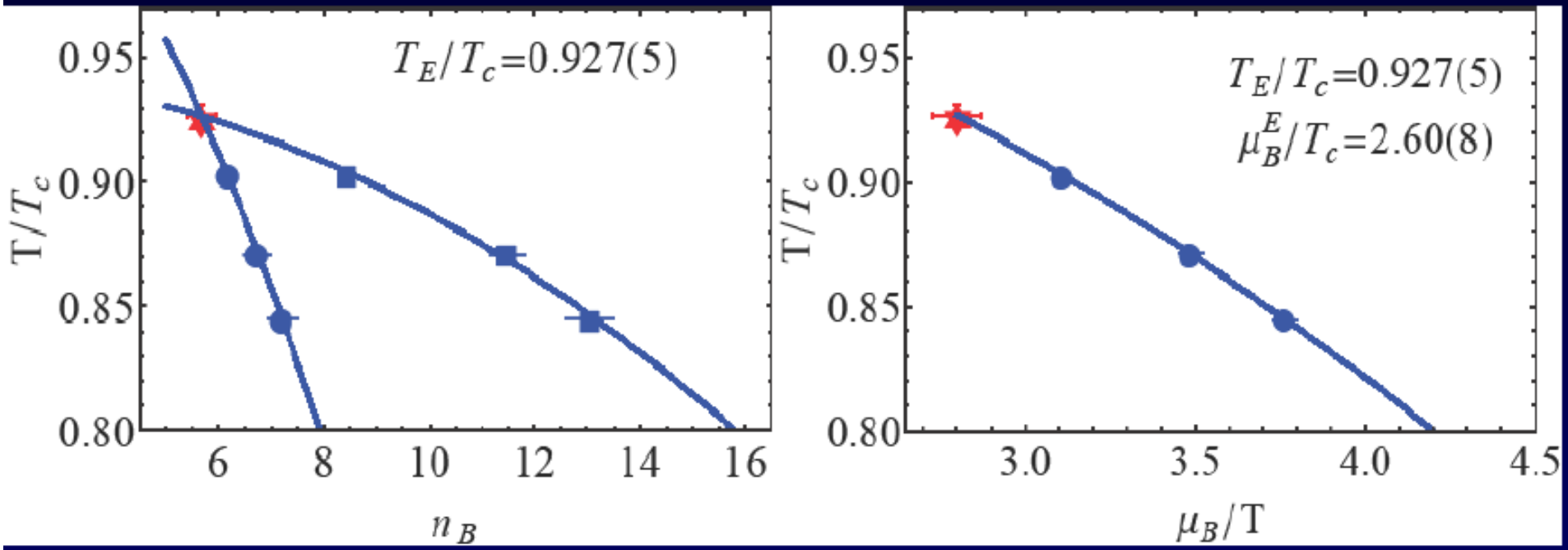






Critial Point of N_f = 3 Case

m_π ~ 0.7 GeV, 6^3 x 4 lattice, a ~ 0.3 fm



T_{CP} = 0.927(5) T_C (~ 157 MeV)

μ_{CP} = 2.60(8) T_C (~ 441 MeV)

Transition density ~ 5-8 ρ_{NM}

Nature of phase transition

Avoiding the sign problem

(SE, Phys.Rev.D77,014508(2008), WHOT-QCD, Phys.Rev.D82, 014508(2010))

θ: complex phase $\theta \equiv \mathrm{Im}\ln\det M$

- Sign problem: If $e^{i\theta}$ changes its sign,

$$\left\langle e^{i\theta}\right\rangle_{P,F\ \mathrm{fixed}} \ll \text{(statistical error)}$$

- Cumulant expansion

<..>$_{P,F}$: expectation values fixed F and P.

$$\left\langle e^{i\theta}\right\rangle_{P,F} = \exp\left[\underbrace{\cancel{i\langle\theta\rangle_C}}_{\to 0} - \frac{1}{2}\langle\theta^2\rangle_C - \underbrace{\cancel{\frac{i}{3!}\langle\theta^3\rangle_C}}_{\to 0} + \frac{1}{4!}\langle\theta^4\rangle_C + \cancel{\cdots}\right]$$

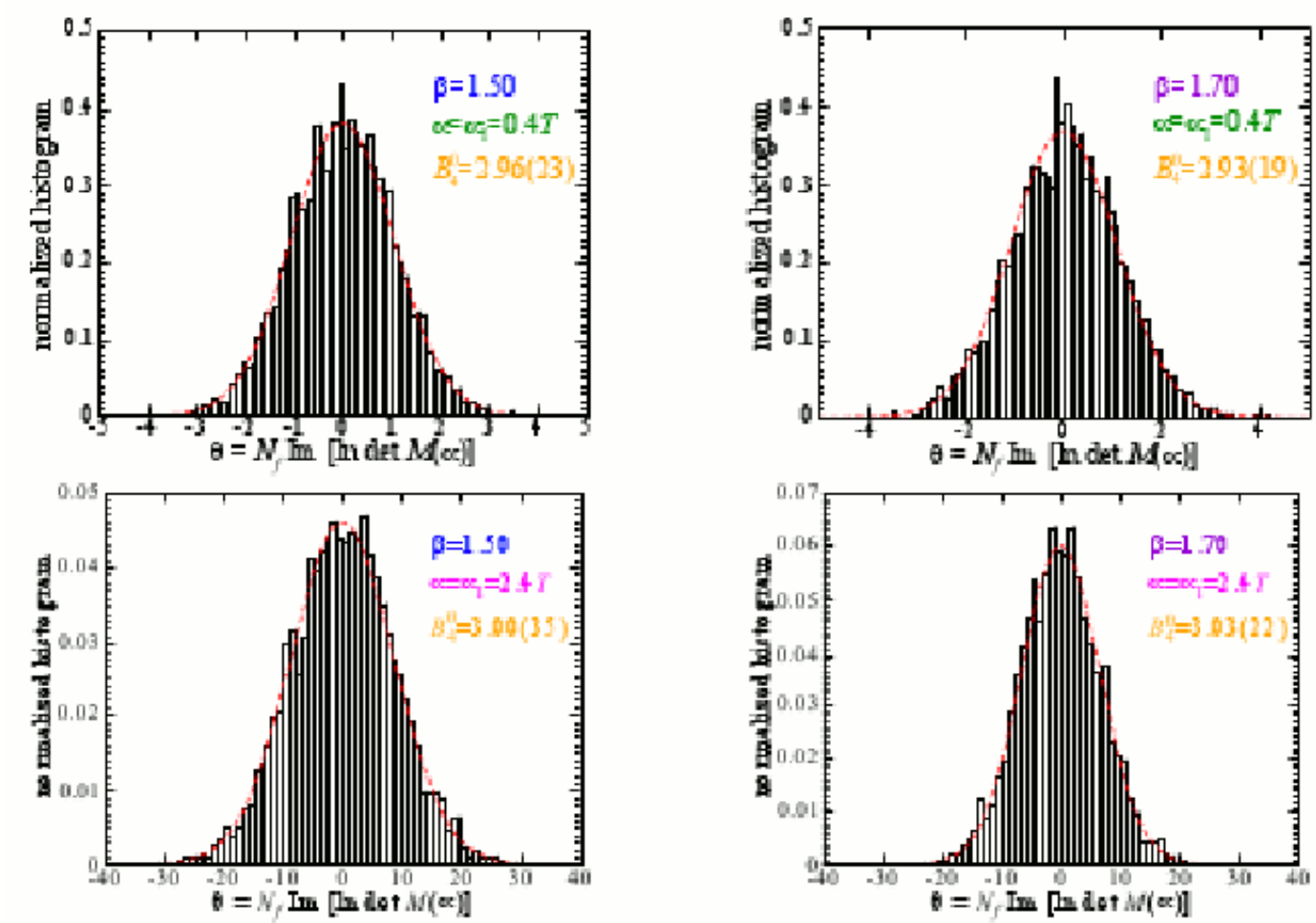
cumulants

$$\langle\theta\rangle_C = \langle\theta\rangle_{P,F},\quad \langle\theta^2\rangle_C = \langle\theta^2\rangle_{P,F} - \langle\theta\rangle_{P,F}^2,\quad \langle\theta^3\rangle_C = \langle\theta^3\rangle_{P,F} - 3\langle\theta^2\rangle_{P,F}\langle\theta\rangle_{P,F} + 2\langle\theta\rangle_{P,F}^3,\quad \langle\theta^4\rangle_C = \cdots$$

– Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ ($\theta \leftrightarrow -\theta$)
Source of the complex phase

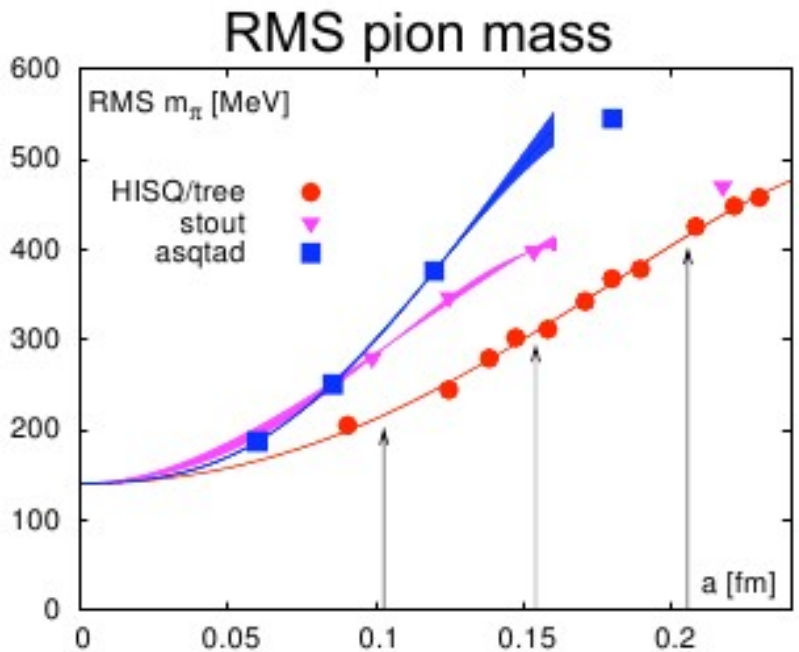
If the cumulant expansion converges, No sign problem.

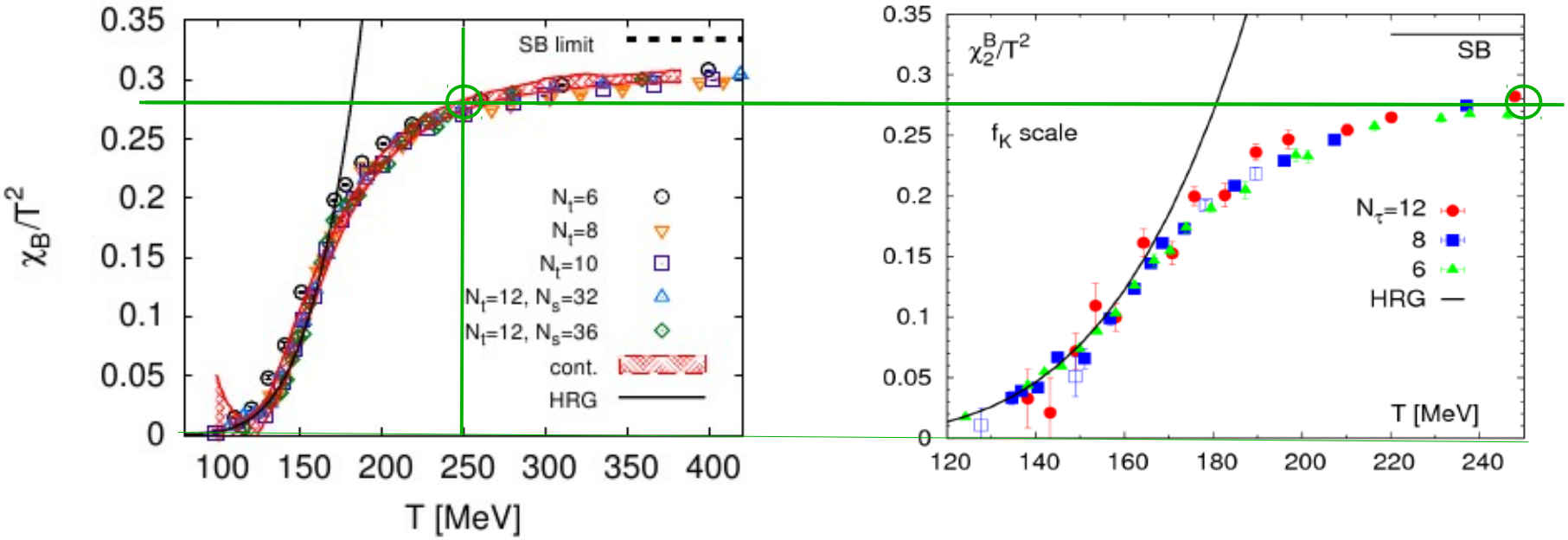
Distribution of the complex phase



- Well approximated by a Gaussian function. What does this mean?
- Convergence of the cumulant expansion: good.

Taste Violation





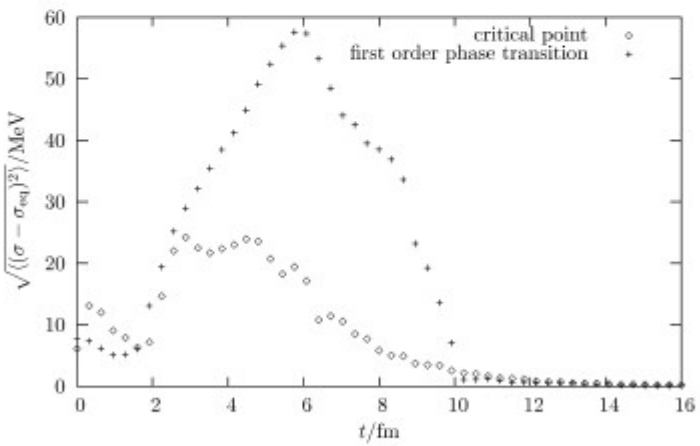
Not such a bad agreement

Intensity of sigma fluctuations

$$\frac{\mathrm{d}N_\sigma}{\mathrm{d}^3k} = \frac{(\omega_k^2|\sigma_k|^2 + |\partial_t\sigma_k|^2)}{(2\pi)^3 2\omega_k}$$

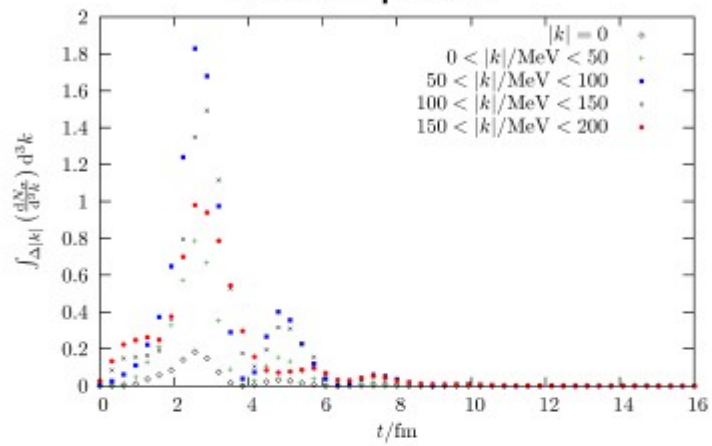
$$\omega_k = \sqrt{|k|^2 + m_\sigma^2}$$

$$m_\sigma = \sqrt{\partial^2 V_{\mathrm{eff}}/\partial\sigma^2|_{\sigma=\sigma_{\mathrm{eq}}}}$$

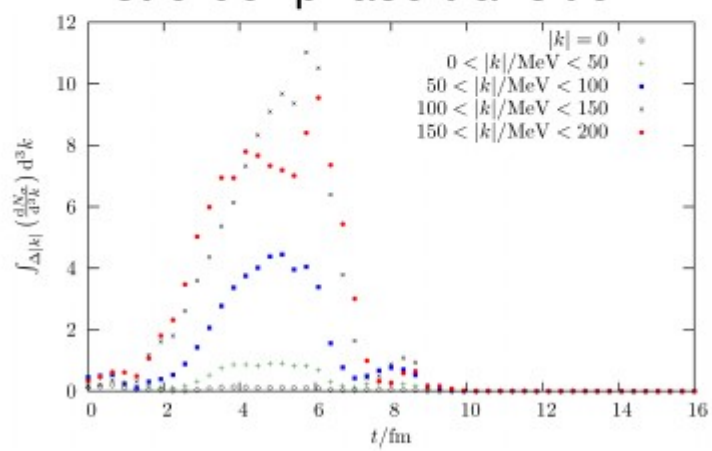


deviation from equilibrium

critical point

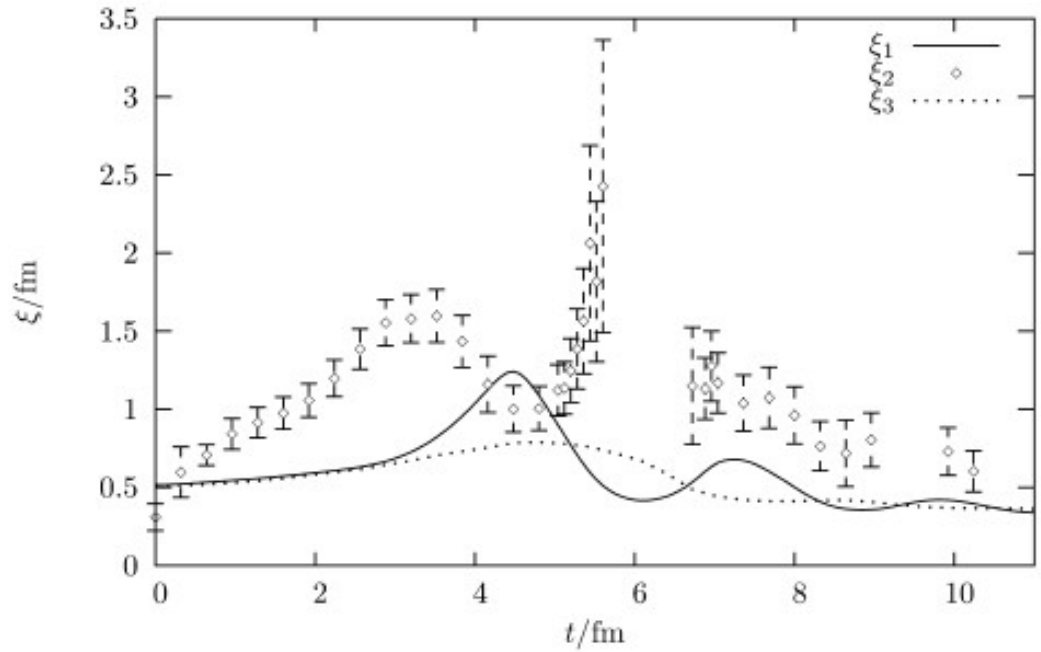


first order phase transition



MN, M. Bleicher, S. Leupold, I. Mishustin, arXiv:1105.1962

Correlation length



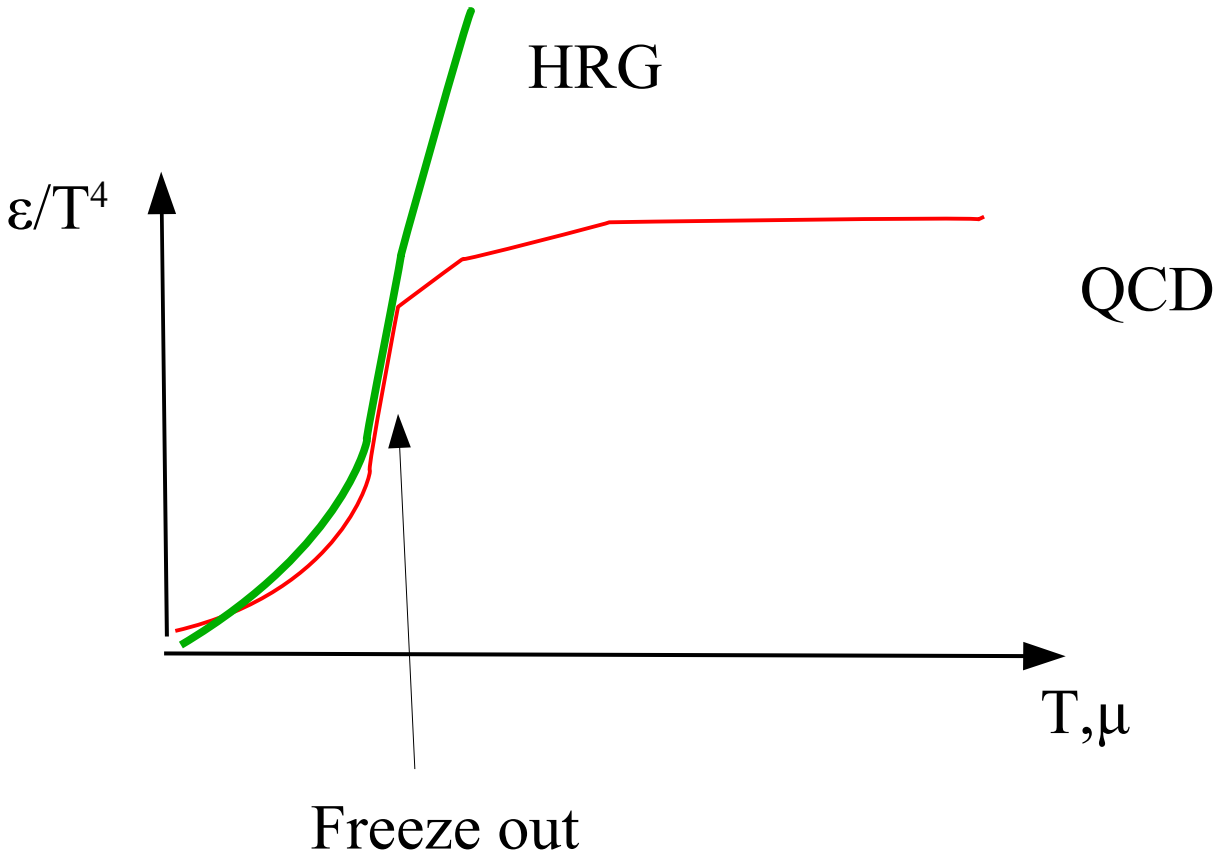
ξ_1: averaged correlation length from $\xi^{-1} = \sqrt{\frac{\partial^2\Omega}{\partial\sigma^2}\Big|_{\sigma=\sigma(x)}}$

ξ_2: correlation length obtained from fits to $G(r) = \sigma_{\rm eq}^2 + \frac{1}{r}\exp(-\frac{r}{\xi})$

ξ_3: averaged correlation length from $\xi^{-1} = \sqrt{\frac{\partial^2\Omega}{\partial\sigma^2}\Big|_{\sigma=\sigma_{\rm eq}}}$

A complementary strategy

- Study higher moments at large sqrt(s)
- Establish that EOS bends over



Hadronic fluctuations at $\mu_q = 0$

- expect 2nd order transition in 3-d, O(4) symmetry class;

scaling field: $t = \left|\frac{T - T_c}{T_c}\right| + A\left(\frac{\mu_q}{T_c}\right)^2$, $\mu_{crit} = 0$

singular part: $f_s(T, \mu_u, \mu_d) = b^{-1} f_s(t b^{1/(2-\alpha)}) \sim t^{2-\alpha}$

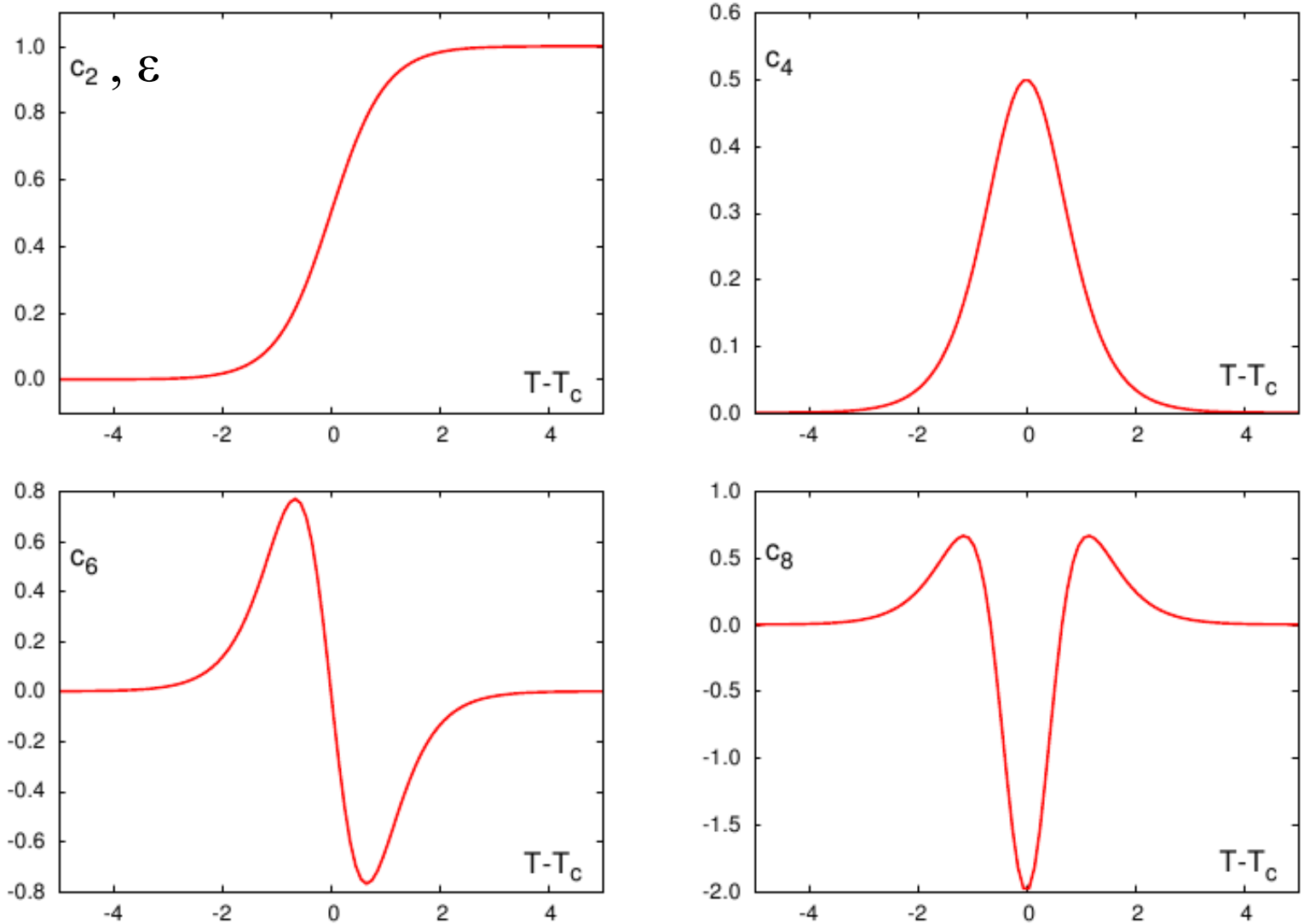
$$c_2 \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial \mu_q^2} \sim t^{1-\alpha} \quad , \quad c_4 \sim \frac{\partial^4 \ln \mathcal{Z}}{\partial \mu_q^4} \sim t^{-\alpha} \quad (\mu = 0)$$

$$\epsilon \sim \frac{\partial \ln \mathcal{Z}}{\partial T} \sim t^{1-\alpha} \quad , \quad C_V \sim \frac{\partial^2 \ln \mathcal{Z}}{\partial T^2} \sim t^{-\alpha} \quad (\mu = 0)$$

$\Rightarrow$ 2nd derivative w.r.t μ_q "looks like energy density"

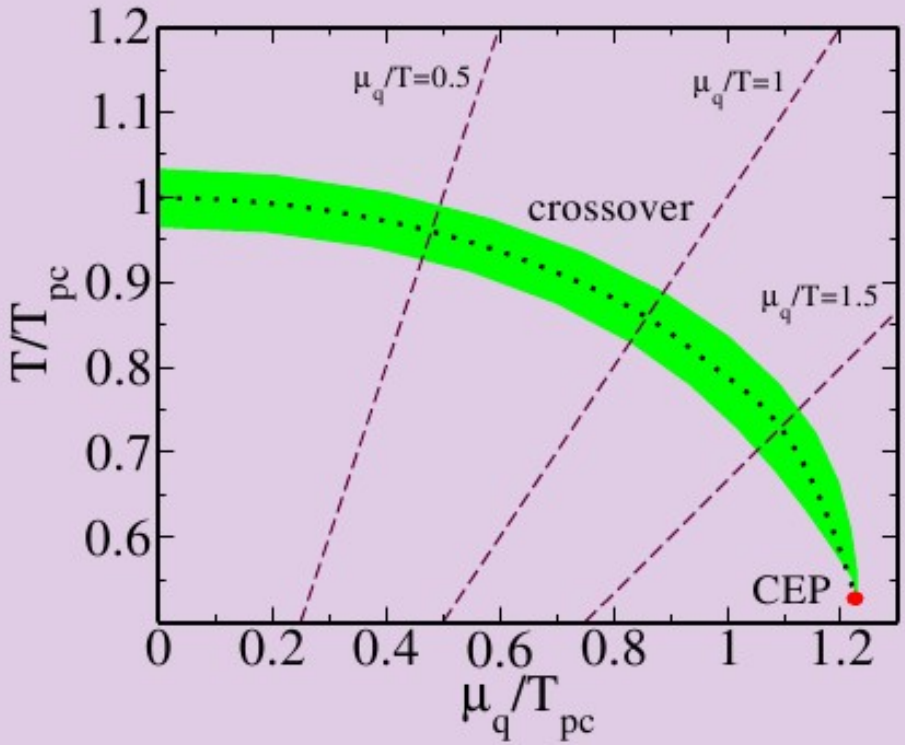
$\Rightarrow$ 4th derivative w.r.t μ_q "looks like specific heat"

Generic expansion coefficients



similar in PNJL model: S. Roessner et al, PR D75 (2007) 034007

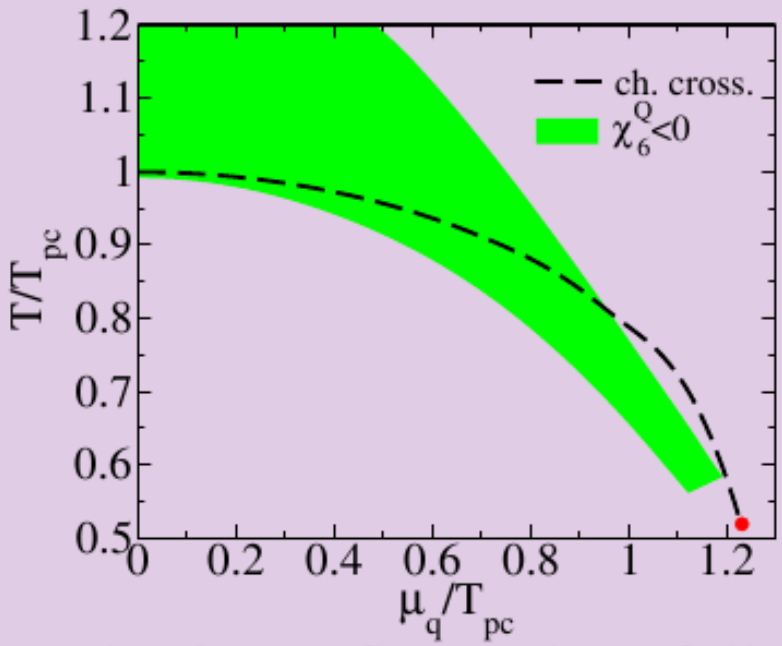
Phase diagram



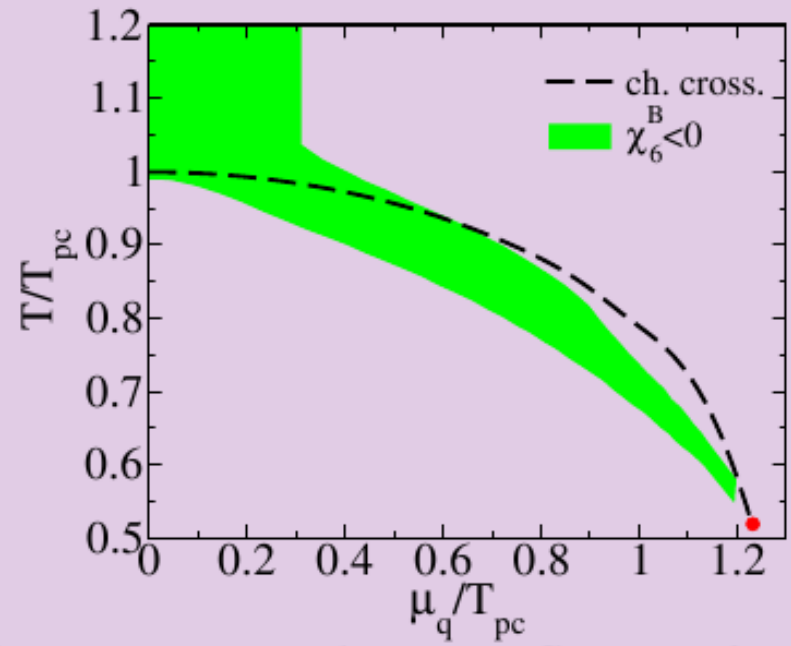
Crossover: $|\partial\sigma/\partial T| > 0.95 \cdot \max(|\partial\sigma/\partial T|)$

Electric charge fluctuations

Electric charge:



Baryon charge:



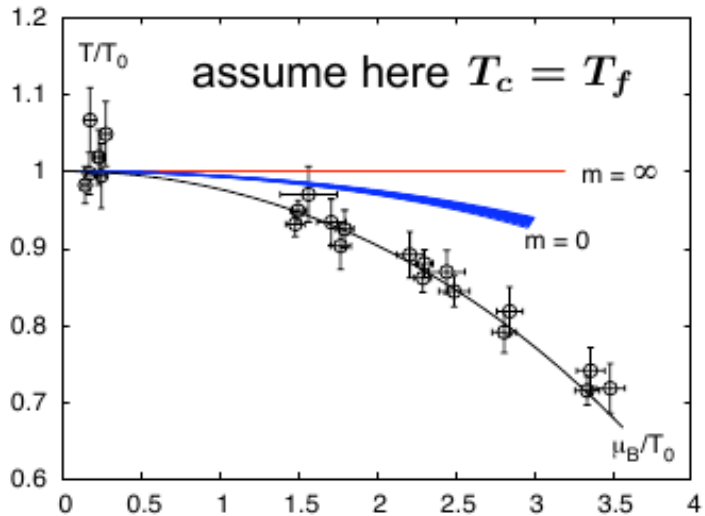
Electric charge fluctuations follow similar pattern as baryon fluctuations

Maybe not …… Or only at μ=0

2) Analyzing QCD critical behavior

Critical line vs. freeze-out line:

- Statistical models are very successful in describing particle abundances observed in heavy ion collision; use a parametrization of the freeze-out curve



statistical model:

$$\frac{T_c}{T} = 1 - 0.023\left(\frac{\mu_B}{T}\right)^2 - d\left(\frac{\mu_B}{T}\right)^4$$

Cleymans, et al., PRC 73 (2006) 034905

lattice:

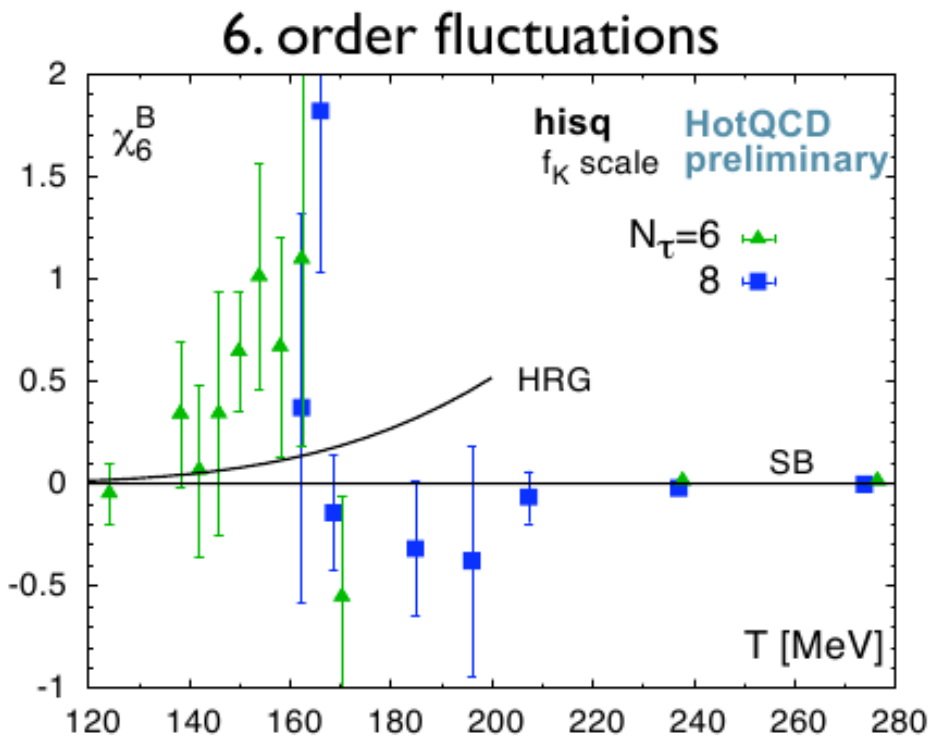
$$\frac{T_c}{T} = 1 - 0.0066(7)\left(\frac{\mu_B}{T}\right)^2$$

Kaczmarek *et al.*, PRD 83 (2001) 014504

$\Rightarrow$ **curvature of the freeze-out curve seems to be larger**

- **open issues**: continuum limit, strangeness conservation, nonzero electric charge

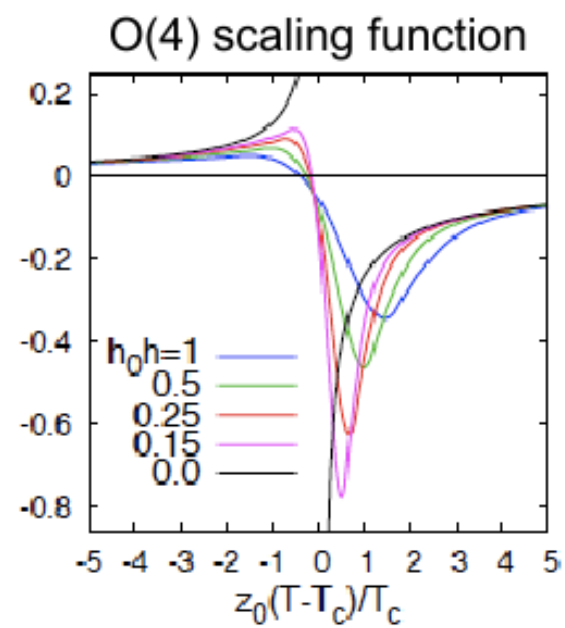
3) Baryon number fluctuations: lattice vs. HRG



notice for $T < T_c$:

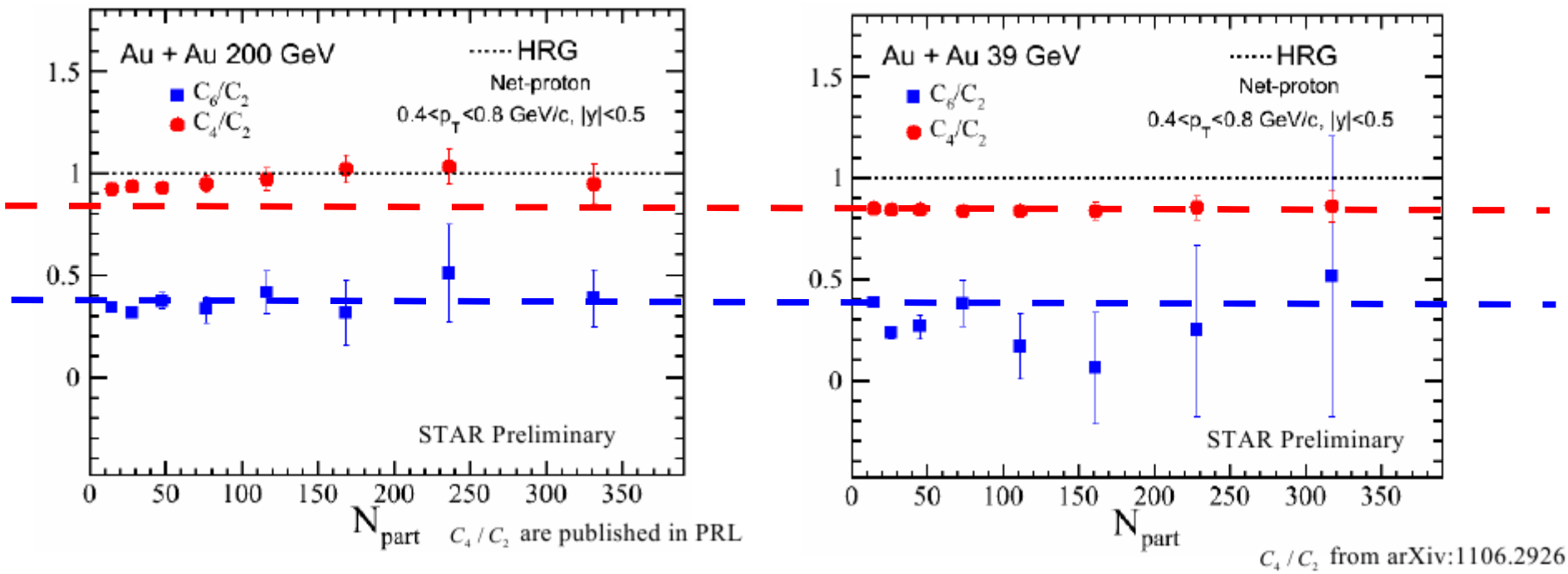
$\Rightarrow$ fluctuations increase over HRG

$\Rightarrow$ fluctuations stay positive



Friman *et al.*, EPJ C 71 (2011) 1694.

Let's be patient ….



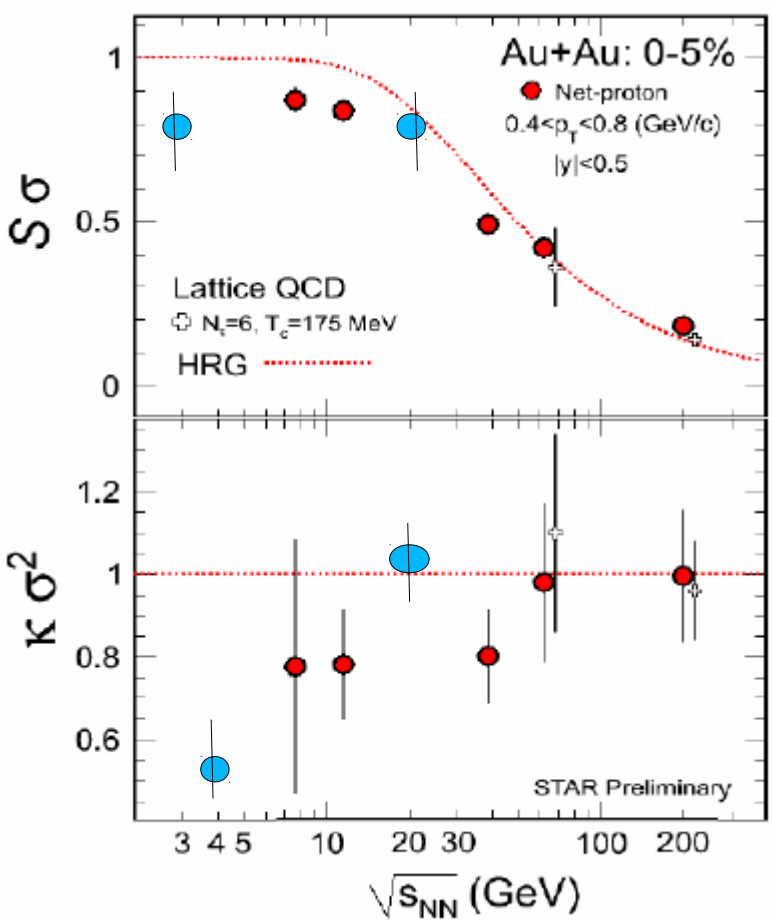
Centrality dependence ???

Things to do

- Account for conservation of Baryon number etc
- Understand proton vs Baryon
- Understand effect of multiplicity selection on cumulants
- Understand the relevant length scales in MOMENTUM space
  - Is there an optimal system size???
- Understand initial state fluctuations

- Can we access the density fluctuations directly???

Next time we meet....



- added by random thought

Thanks for a great workshop

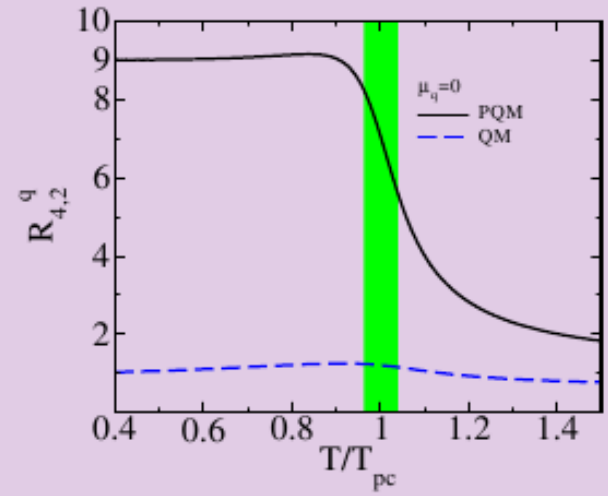
Kurtosis of net-quark number density

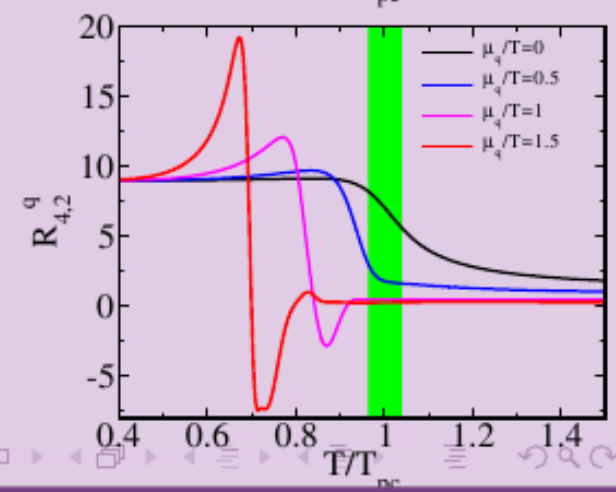
Kurtosis $R^q_{4,2} = \frac{\chi^q_4}{\chi^q_2} = \frac{\langle(\delta N_q)^4\rangle}{\langle(\delta N_q)^2\rangle} - 3\langle(\delta N_q)^2\rangle$

(S. Ejiri, F. Karsch and K. Redlich '05):

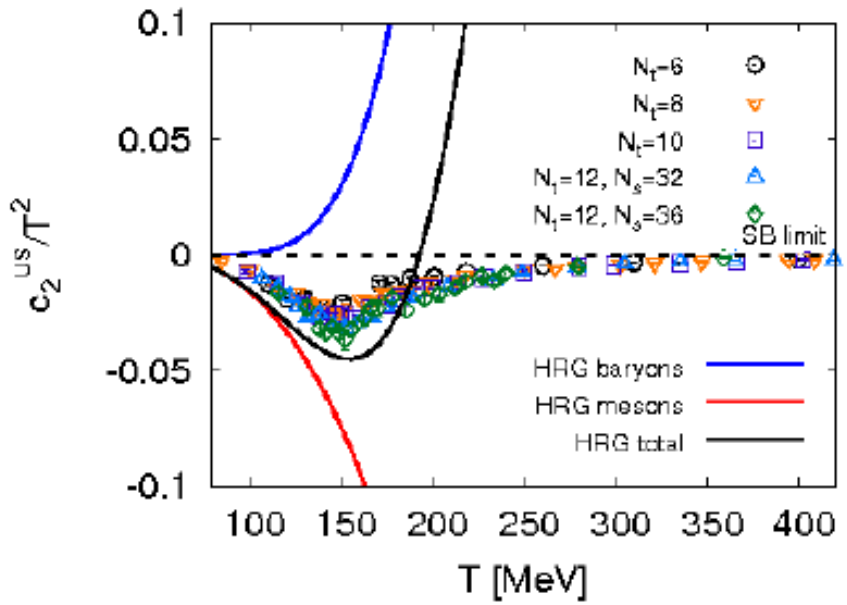
quark content of effective degrees of freedom that carry baryon number

- **Low temperature phase**: dominance of effective three-quark states:
  $P_{\text{baryons}}/T^4 \approx \sum_i F(m_i/T)\cosh(3\mu_q/T)$
  $\rightsquigarrow R^q_{4,2} = 9$
- **High-temperature phase**:
  $P_{q\bar{q}}/T^4 \approx N_f N_c \left[\frac{1}{12\pi^2}\left(\frac{\mu_q}{T}\right)^4 + \frac{1}{6}\left(\frac{\mu_q}{T}\right)^2 + \frac{7\pi^2}{180}\right]$
  $\rightsquigarrow R^q_{4,2} = (6/\pi^2) \approx 1$
- *PQM: statistical confinement*
- $m_\pi = 0, \mu_q \neq 0$: kurtosis diverges
  $R^q_{4,2} \sim \left(\frac{\mu_q}{T}\right)^4 / t^{2+\alpha}$ ($t \propto$ distance to chiral critical line)





Baryon-meson dependence in correlator

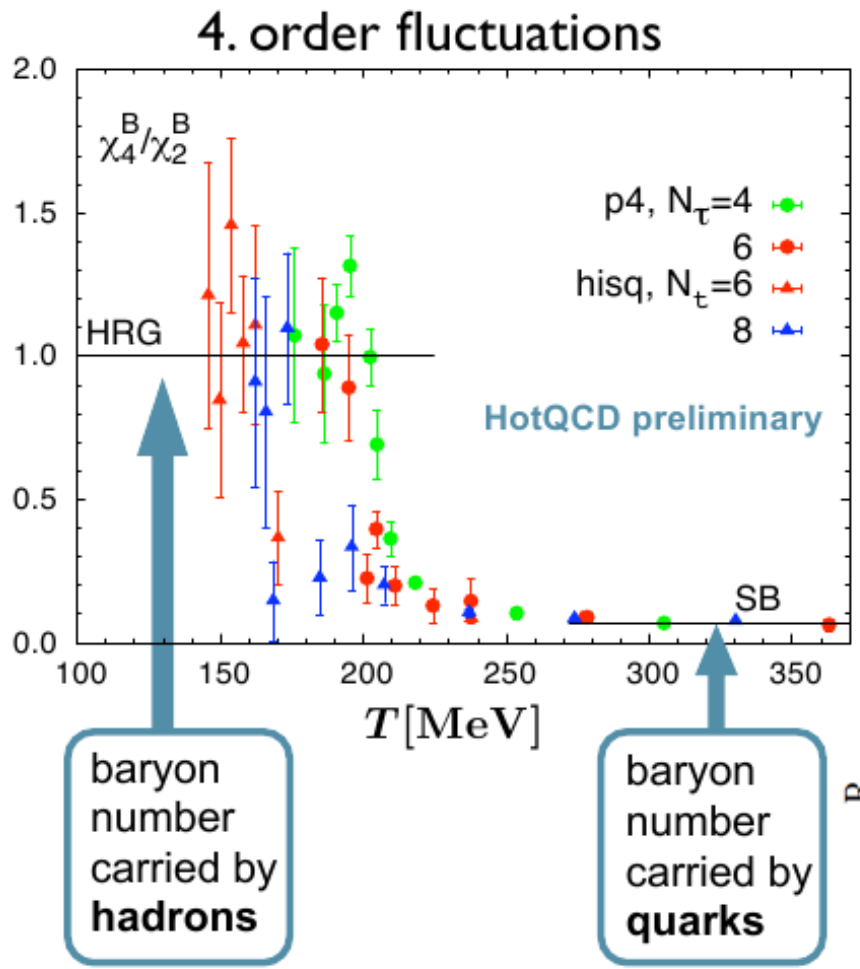


- Baryons dominate in HRG at $T > 190 MeV$
- The lattice correlator never turns positive
  - bound states above T_c are predominantly of mesonic nature
- The upswing in the lattice data shows that baryon contribution increases with T

C.R., R. Bellwied, M. Cristoforetti, M. Barbaro, arXiv:1109.6243

Possibly consistent with observations from baryon sector

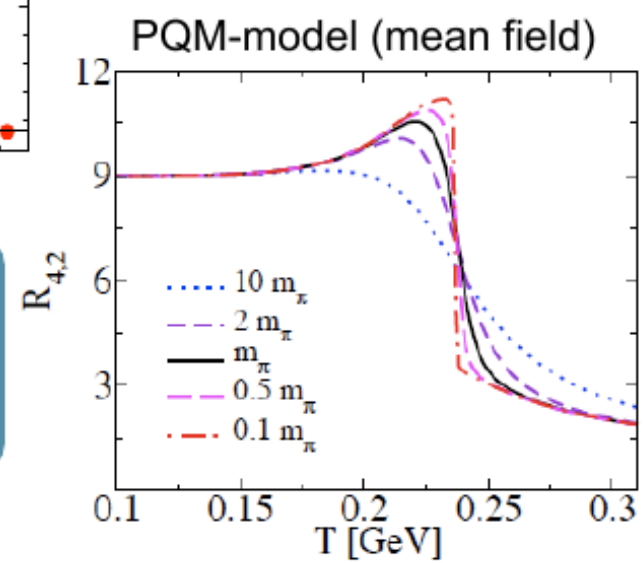
3) Baryon number fluctuations: lattice vs. HRG



⇒ cutoff dependence of Tc

⇒ pronounced step function indicating deconfinement

kink at Tc from critical behavior?

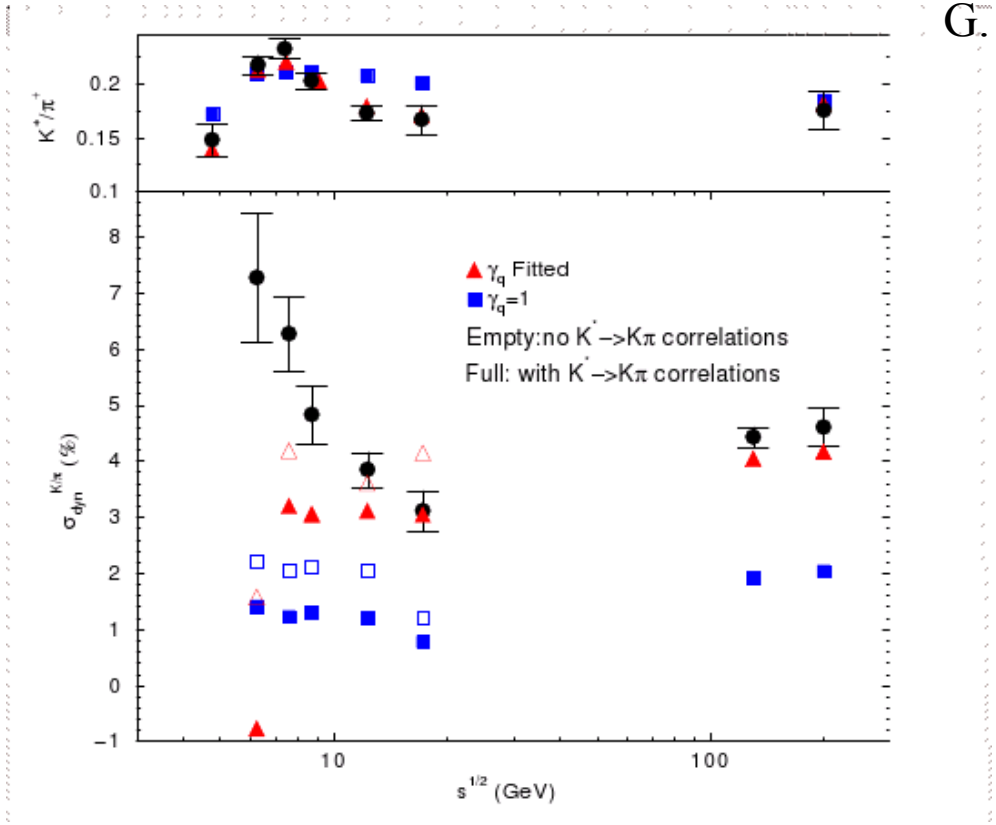


Skokov *et al.*, PRD 82 (2010) 034029.

Hadron gas predictions

$\langle \frac{K}{\pi} \rangle$

$\langle \delta(\frac{K}{\pi})^2 \rangle$



G. Torrieri, QM2006

Some trivial effects...

w. T.Schuster

$$\sigma^2_{dyn} = \frac{\langle \delta K^2 \rangle - \langle K \rangle}{\langle K \rangle^2} + \frac{\langle \delta \pi^2 \rangle - \langle \pi \rangle}{\langle \pi \rangle^2} - 2\frac{\langle \delta K \delta \pi \rangle}{\langle K \rangle \langle \pi \rangle}$$

$$= \frac{w_{KK} - 1}{\langle K \rangle} + \frac{w_{\pi\pi} - 1}{\langle \pi \rangle} - 2\frac{w_{K\pi}}{\sqrt{\langle K \rangle \langle \pi \rangle}}$$

$$\sim 1 \,/\, (\text{accepted Multiplicity})$$

$$w_{AB} \equiv \frac{\langle \delta A \delta B \rangle}{\sqrt{\langle A \rangle \langle B \rangle}}$$

Scaled correlation
independent of multiplicity

Scaling prescriptions

Poisson scaling:

$$\sigma_{\rm dyn}\left(\sqrt{s}\right) = \sigma_{\rm dyn}\left(200\,{\rm GeV}\right) \frac{\sqrt{\frac{1}{\langle K\rangle}+\frac{1}{\langle\pi\rangle}}\Big|_{\sqrt{s}}}{\sqrt{\frac{1}{\langle K\rangle}+\frac{1}{\langle\pi\rangle}}\Big|_{200\,{\rm GeV}}}$$

Part. Num. scaling:

$$\sigma_{\rm dyn}\left(\sqrt{s}\right) = \sigma_{\rm dyn}\left(200\,{\rm GeV}\right) \frac{\sqrt{\langle K\rangle+\langle\pi\rangle}\Big|_{200\,{\rm GeV}}}{\sqrt{\langle K\rangle+\langle\pi\rangle}\Big|_{\sqrt{s}}}$$

Kaon Num. scaling:

$$\sigma_{\rm dyn}\left(\sqrt{s}\right) = \sigma_{\rm dyn}\left(200\,{\rm GeV}\right) \frac{\sqrt{\langle K\rangle}\Big|_{200\,{\rm GeV}}}{\sqrt{\langle K\rangle}\Big|_{\sqrt{s}}}$$

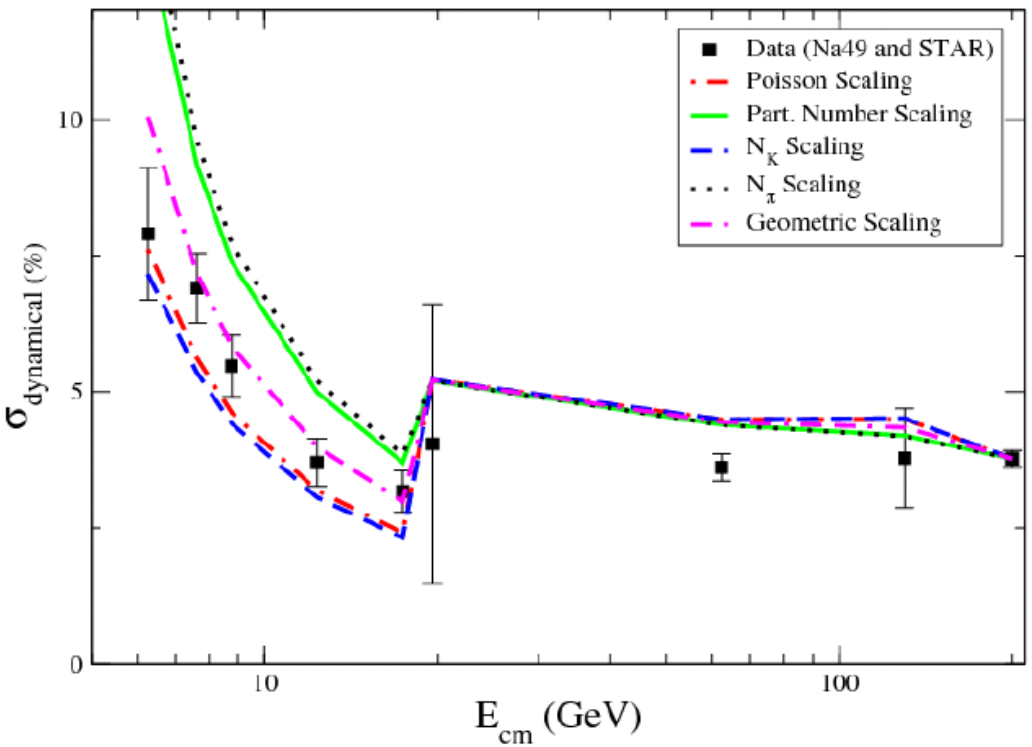
Pion Num. scaling:

$$\sigma_{\rm dyn}\left(\sqrt{s}\right) = \sigma_{\rm dyn}\left(200\,{\rm GeV}\right) \frac{\sqrt{\langle \pi\rangle}\Big|_{200\,{\rm GeV}}}{\sqrt{\langle \pi\rangle}\Big|_{\sqrt{s}}}$$

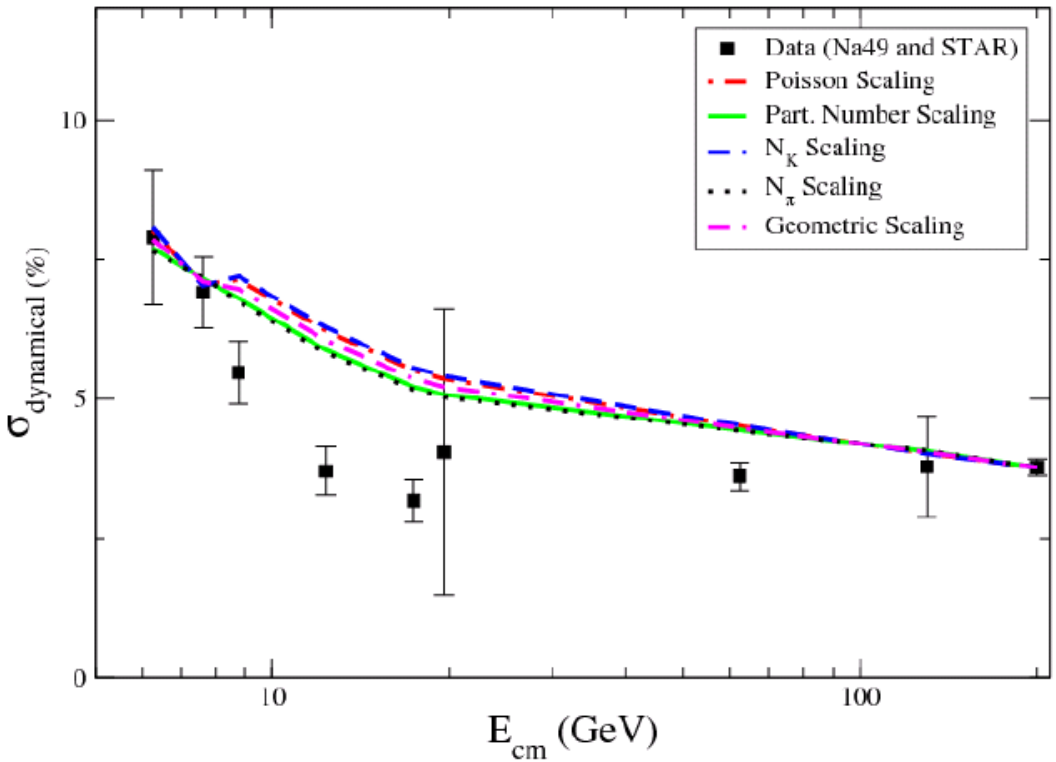
Geometric scaling:

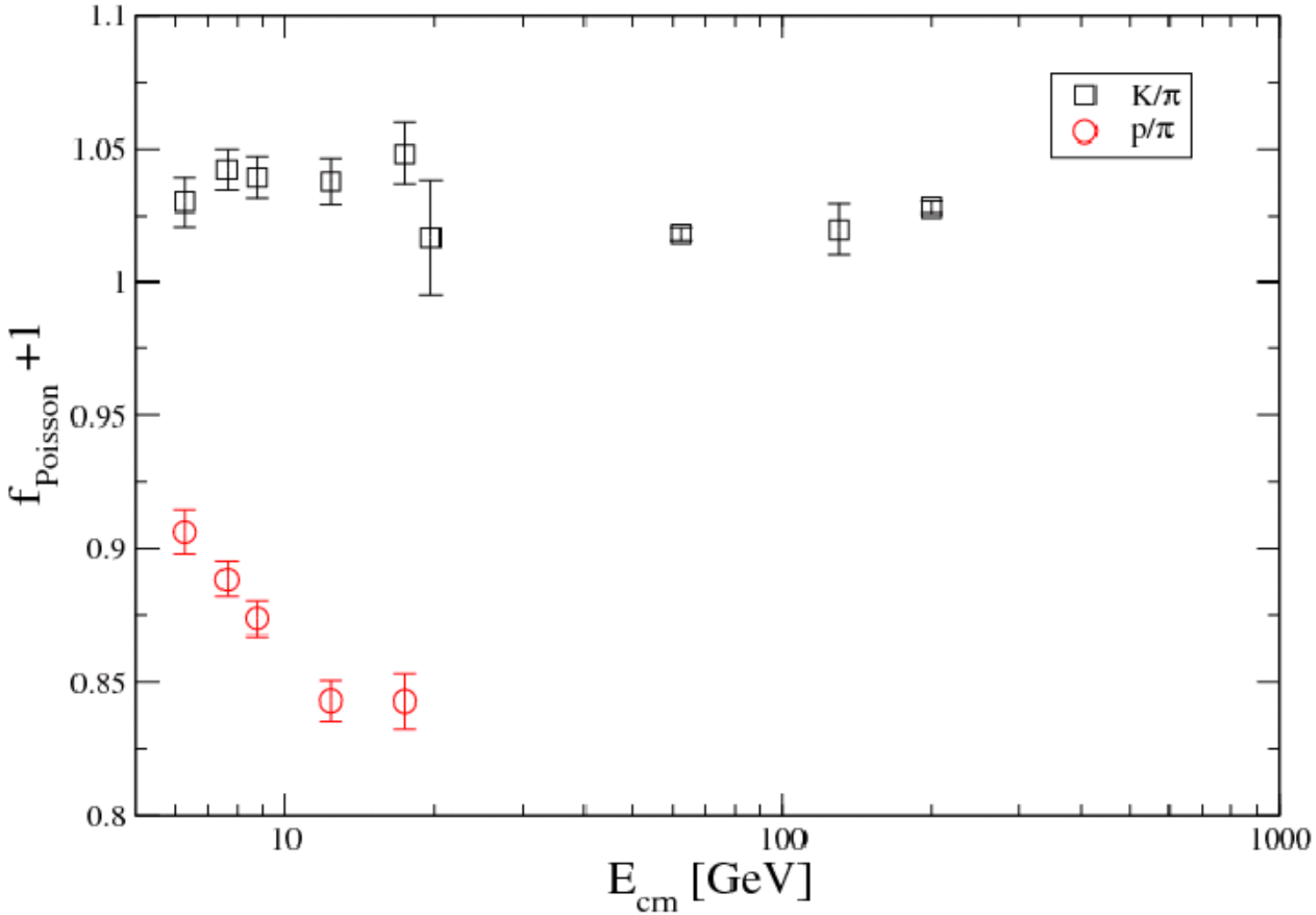
$$\sigma_{\rm dyn}\left(\sqrt{s}\right) = \sigma_{\rm dyn}\left(200\,{\rm GeV}\right) \frac{\left(\langle K\rangle\langle\pi\rangle\right)^{1/4}\Big|_{200\,{\rm GeV}}}{\left(\langle K\rangle\langle\pi\rangle\right)^{1/4}\Big|_{\sqrt{s}}}$$

Scaled with accepted Particles



Scaled with dN/dy





$$f_{Poisson} = \frac{\sigma^2_{dyn}}{\sigma^2_{Poisson}}$$