Gauge invariance of the fermion dispersion relation beyond hard thermal loops

with E. Mottola

Medium effects
$\longrightarrow$ collective modes or quasi-particles

In general:
behaviour from analytic structure of propagator

Our case:
$q_0 \sim eT$ and $\vec{q} = 0$
CM's appear as poles in the propagator

Real part: dispersion law ω
Imag part: damping rate γ

In gauge theories: propagator is gauge dependent
Question: is the pole gauge invariant?

Kobes, Kunstatter and Rebhan: from general principles the singularity structure of certain gauge and matter propagators is gauge independent
R. Kobes, G. Kunstatter and A. Rebhan, Phys. Rev. Lett **64** *(1990) 2992; R. Kobes, G. Kunstatter and A. Rebhan, Nucl. Phys.* **B355** *(1991) 1.*

HTL resummation $\longrightarrow$ gauge independent results.

Can we go beyond leading order HTL?

Motivation:
transport equation / transport coefficients

Method:

fermion propagator: $S(Q) = 1/(\not{Q} - \Sigma(Q))$

dispersion relation from poles: $S^{-1}(q_0, 0) = 0$

look at zero velocity fermions: $Q_\mu = (q_0, \vec{q} = 0)$

$$q_0 - \Sigma^0(q_0, 0) = 0$$

$$\Sigma^0(q_0, 0) = \frac{1}{4}\mathrm{Tr}[\gamma^0 \Sigma(q_0, 0)]$$

expand $\Sigma^0(q_0, 0)$ in a power series in q_0/T

Solve iteratively:

$q_0 = \bar{\omega} - i\bar{\gamma} = (\bar{\omega}_{(0)} + \bar{\omega}_{(1)} + \cdots) - i(\bar{\gamma}_{(0)} + \bar{\gamma}_{(1)} + \cdots)$

lo contribution to $\Sigma^0(q_0, 0)$ is the HTL contribution from the 1-loop diagram:

$$\mathrm{Re}\,\Sigma^0(q_0, 0) = e^2 T^2/8q_0\,; \quad \mathrm{Im}\,\Sigma^0(q_0, 0) = 0$$

dispersion relation gives:

$$\bar{\omega}_{(0)} = \frac{eT}{\sqrt{8}} := e\,\omega_{(0)}$$

$$\bar{\gamma}_{(0)} = 0$$

nlo contribution from subleading temperature corrections to the HTL part of the 1-loop diagram:

use $D_{\mu\nu}(K) = \left(g_{\mu\nu} - \alpha\frac{K_\mu K_\nu}{K^2}\right)\frac{1}{K^2}$

Result:

$$\mathrm{Re}\,\Sigma_{(1)} = c_1 e^2 q_0 \ln(\frac{T}{q_0}) \;\rightarrow\; \bar{\omega}_{(1)} = e^3 \ln(1/e) T \frac{c_1}{\sqrt{8}}$$

$$\mathrm{Im}\,\Sigma_{(1)} = d_1 e^2 T \qquad \rightarrow \quad \bar{\gamma}_{(1)} = d_1 e^2 T$$

$$c_1 = (1-\alpha)\frac{1}{8\pi^2}$$
$$d_1 = -\frac{1}{16\pi}(2-3\alpha)$$

I. Mitra, Phys. Rev. D **62**, *045023 (2000).*
S-Y Wang, Phys. Rev. D **20** *065011 (2004).*

Note: c_1 and d_1 are gauge dependent

nlo contribution from leading temperature 2-loop

Proposal:

$$\mathrm{Re}\ \Sigma_{(2)} = c_2\ \frac{e^4 T^2}{q_0}\ \ln(\frac{T}{q_0})$$

$$\mathrm{Im}\ \Sigma_{(2)} = d_2\ \frac{e^4 T^3}{q_0^2}$$

Dispersion relation:

$$\bar{\omega}_{(1)} = e^3 \ln\,(1/e) T\ (c_1 + 8c_2)\frac{1}{\sqrt{8}}$$

$$\bar{\gamma}_{(1)} = (d_1 + 8d_2)\ e^2 T$$

Note: lo 2-loop $\sim$ nlo 1-loop

Goal:

show $c_1 + 8c_2$ and $d_1 + 8d_2$ are gauge independent

Outline of the calculation of c_2.

Real time finite temperature field theory

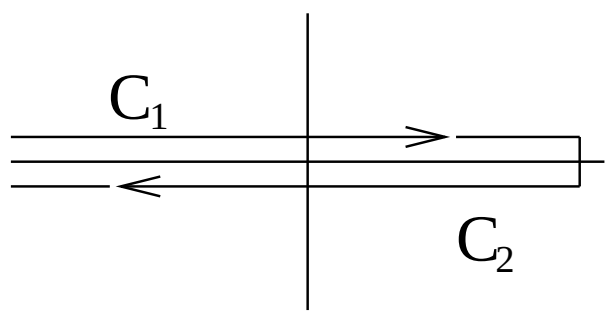
CTP formulation

Keldysh representation

Describe formalism for bosons:

Green functions are defined on a contour:

$$iD_C(x,y) = \langle \tilde{T}_c \phi(x)\phi(y) \rangle$$



$\tilde{T}_c$ indicates time ordering along the contour

Propagator w/ real time arguments is a 2×2 matrix:

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

$$iD_{11}(x,y) = \langle T\phi(x)\phi(y)\rangle$$
$$iD_{12}(x,y) = \langle \phi(y)\phi(x)\rangle$$
$$iD_{21}(x,y) = \langle \phi(x)\phi(y)\rangle$$
$$iD_{22}(x,y) = \langle T^*\phi(x)\phi(y)\rangle$$

Keldysh representation:

$$r_k = D_{11} - D_{12} = \frac{1}{K^2 - m^2 + i\mathrm{sgn}(k_0)\epsilon}$$

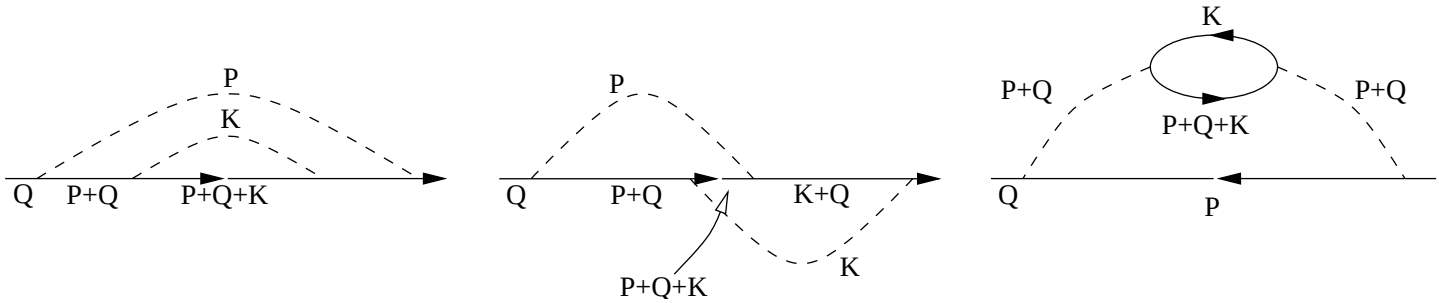
$$a_k = D_{11} - D_{21} = \frac{1}{K^2 - m^2 - i\mathrm{sgn}(k_0)\epsilon}$$

$$f_k = D_{11} + D_{22} = (1 + 2n(k_0))(D_R - D_A)$$
$$= -2\pi i N(k_0)\mathrm{sgn}(k_0)\,\delta(K^2 - m^2)$$

Vertex is a tensor with non-zero components:

$$\Gamma^{111} = -\Gamma^{222} = -ie$$

Two loop contributions:



Use methods developed in

M.E. Carrington, Hou Defu, A. Hachkowski, D. Pickering, J.C. Sowiak, Phys. Rev. D **61** *(2000) 25011;*

M.E. Carrington, Hou Defu, R. Kobes, Phys. Rev. D **67** *(2003) 025021.*

sum over internal indices

chose combination of external indices:

$$\Sigma_R = \Sigma_{11} + \Sigma_{12}\,; \quad \Sigma_A = \Sigma_{11} + \Sigma_{21}$$

construct real and imaginary parts:

$$\mathrm{Re}\Sigma = \frac{1}{2}\left(\Sigma_R + \Sigma_A\right)\,; \quad \mathrm{Im}\Sigma = \frac{1}{2i}\left(\Sigma_R - \Sigma_A\right)$$

Step 1: sum over internal indices

An example:

$$\Sigma^{\alpha=0}_{R\ rain}$$

$$= \frac{2e^4}{(2\pi)^6} \int dp\, p^2 \int dk\ k^2 \int dx\ \int dp_0 \int dk_0$$

$$[((P^2 + Q^2 + 2\,\mathrm{QP}^2)\, k_0$$

$$-(2\,\mathrm{KP} + 2\,\mathrm{KQ} + P^2 + Q^2 + 2\,\mathrm{QP})\,(p_0 + q_0))$$

$$(f_p\, {r_{p+q}}^2\, (a_k\, f_{k+p+q} + f_k\, r_{k+p+q})$$

$$+a_p\, (f_{p+q}\, r_{p+q}\, (a_k\, f_{k+p+q} + f_k\, r_{k+p+q})$$

$$+a_{p+q}\, (f_{p+q}\, f_{k+p+q}\, r_k + a_{k+p+q}\, (f_k\, f_{p+q} + r_k\, r_{p+q})$$

$$+r_{p+q}\, (f_k\, f_{k+p+q} + a_k\, r_{k+p+q}))))]$$

Step 2: collect delta functions

[a] Each graph has 5 propagators

[b] Each term has 2 symmetric props = 2 δ fcns

[c] Shift variables so each term $\sim \delta(K^2)\delta(P^2)$

Problems:
[1] pinch terms (give 0 by the KMS conditions)
[2] divergent terms of the form $\delta(P^2)\mathcal{P}(1/P^2)$.
(i) rainbow and bubble have factors $D(P)^2$
(ii) α dependent part of the on-shell gauge prop

[d] regulate these terms:

$$\begin{aligned}
\delta(P^2)\mathcal{P}(1/P^2) &\sim (r_p^2 - a_p^2) \\
&= \frac{d}{dm^2}\,(r_{p,m} - a_{p,m})|_{m^2=0} \\
&= \frac{d}{dm^2}\,d_{p,m}|_{m^2=0} \\
\text{with} \quad r_{p,m} &= 1/(P^2 - m^2 + i\epsilon\,\mathrm{sgn}(p_0))
\end{aligned}$$

Step 3: Gather terms with the same structure:

$$\Sigma_R^{\alpha=0}$$
$$= -\frac{e^4}{(2\pi)^6}\int dp\, p^2 \int dk\; k^2 \int dx \;\int dp_0 \int dk_0$$
$$(d_p\; d_k\; A + d_{p,m_1}\; d_k\; B + d_p\; d_{k,m_3}\; D + d_{p,m_1}\; d_{k,m_3}\; E)$$

$$A_{rain}^{\alpha=0} = -4\mathcal{P}_{k+p+q} N_B(p)$$
$$(((K^2 - P^2)k_0 + 2(K^2 + \mathrm{KP})p_0) N_F(k) \mathcal{P}_{k+p}^2$$
$$+\mathcal{P}_{p+q}^2((P^2 + Q^2 + 2\mathrm{QP})k_0(N_B(k) - N_F(k))$$
$$-(p_0 + q_0)((P^2 + Q^2 + 2\mathrm{KP} + 2\mathrm{KQ} + 2\mathrm{QP})N_B(k)$$
$$-2(\mathrm{KP} + \mathrm{KQ})N_F(k))))$$

$$B_{rain}^{\alpha=0} = -4\mathcal{P}_{k+p}\mathcal{P}_{p+q}(k_0(N_F(k) - N_B(k))P^2$$
$$+p_0((P^2 + 2\mathrm{KP})N_B(k) - 2\mathrm{KP}N_F(k)))N_F(p)$$

Step 4: Use delta functions to do the k_0 and p_0 integrals

$$d_{p,m_1} = -\frac{i\pi}{\sqrt{p^2+m_1^2}} \sum_{j=\pm 1} j\ \delta(p_0 - j\sqrt{p^2-m_1^2})$$

$$d_{k,m_3} = -\frac{i\pi}{\sqrt{k^2+m_3^2}} \sum_{n=\pm 1} n\ \delta(k_0 - n\sqrt{k^2-m_3^2})$$

Result: an integral over p, k and x.

$$\Sigma_R^{\alpha=0}|_{A_{rain}} = \frac{e^4\pi^2}{(2\pi)^6}\int dp \int dk \int dx \sum_{\{j,n\}}$$
$$[4kp\mathcal{P}(q_0^2+2knq_0+2jpq_0+2jknp-2kpx)N_B(p)$$
$$(q_0(2p^2+3jq_0p+q_0^2)N_B(k)\mathcal{P}(q_0(2jp+q_0))^2$$
$$+k(n(2p^2+2jq_0p+q_0^2)-2px(jp+q_0))$$
$$(N_B(k)-N_F(k))\mathcal{P}(q_0(2jp+q_0))^2)$$

$$-8k^2p^3(n-jx)\mathcal{P}(4k^2p^2(x-jn)^2)$$
$$\mathcal{P}(q_0^2+2knq_0+2jpq_0+2jknp-2kpx)N_B(p)N_F(k)]$$

Step 5: Find pieces that correspond to the regions of phase space that give the dominant contribution in the high temperature limit.

One momentum must be soft
$\longrightarrow$ 1/soft enhancement in the denominator.

One momentum must be hard
$\longrightarrow$ result isn't phase space supressed.

basic structure: hard/soft in the integrand.

We need: $\{k$-hard, p-soft$\}$ and $\{p$-hard, k-soft$\}$

$$\Sigma_R^{\alpha=0}|_{A_{rain}} = \frac{e^4\pi^2}{(2\pi)^6}\int dp \int dk \int dx$$

$$8k\,p\,n_B(p)(n_B(k)+n_F(k))$$

$$[(-\frac{2(x-1)p^2+2(x-1)q_0p-q_0^2}{q_0^2(2p+q_0)^2(-xp+p+q_0)}$$

$$+\frac{-2(x-1)p^2+2(x-1)q_0p+q_0^2}{q_0^2(q_0-2p)^2(p(x-1)+q_0)}$$

$$-\frac{2(x+1)p^2-2(x+1)q_0p+q_0^2}{(xp+p-q_0)q_0^2(q_0-2p)^2}$$

$$+\frac{2(x+1)p^2+2(x+1)q_0p+q_0^2}{q_0^2(2p+q_0)^2(xp+p+q_0)})]$$

Step 6: extract the leading high T piece.

Expand in q_0/T small
Since leading term is logarithmic
$\rightarrow$ be careful with denominators

We rewrite terms in the form:

$$\int dp \int dk \int dx \; f(p,k,x)\mathcal{P}\left(\frac{1}{p^2 - q_0^2/4}\right)$$

$$\int dp \int dk \int dx \; f'(p,k,x)\mathcal{P}\left(\frac{1}{k^2 - q_0^2/4}\right)$$

1st form for terms dominated by k-hard and p-soft
2nd form for terms dominated by p-hard and k-soft

For the A_{rain} term the result is:

$$\Sigma_R^{\alpha=0}|_{A_{rain}} = \frac{e^4\pi^2}{(2\pi)^6}\int dp \int dk \int dx$$

$$\frac{16\,k\,p\,n_B(p)\,(n_B(k)+n_F(k))}{\left(x^2-1\right)q_0}\mathcal{P}\left(\frac{1}{p^2 - q_0^2/4}\right)$$

Step 7: Combine and do the remaining integrals

Colinear divergences cancel.

Int over x and hard momentum can be done directly.

Int over soft momentum can be done using

$$n_F(p) = n_B(p) - 2n_B(2p)$$

and the formula

$$\int_0^\infty dp\, p\, n_B(p)\mathcal{P}(\frac{1}{p^2 - x^2}) \sim -\frac{1}{2}\ln(\frac{T}{x})$$

Final Result:

$$\mathrm{Re}\,\Sigma_{(2)} = \frac{e^4 T^2 (\alpha - 4) \log\left(\frac{T}{q_0}\right)}{64 q_0 \pi^2}$$

Compare: $\mathrm{Re}\,\Sigma_{(2)} = c_2 \, \frac{e^4 T^2}{q_0} \ln(\frac{T}{q_0})$

$$\Rightarrow c_2 = (\alpha - 4)\frac{1}{64\pi^2}$$

Next-to-leading order part of the 1-loop diagram:

$$\mathrm{Re}\,\Sigma_{(1)} \quad \rightarrow \quad c_1 = (1-\alpha)\frac{1}{8\pi^2}$$

Recall dispersion relation:

$$\bar{\omega}_{(1)} = e^3 \ln(1/e) T \, (c_1 + 8c_2) \frac{1}{\sqrt{8}}$$

$$c_1 + 8c_2 = -\frac{3}{8\pi^2} = \text{gauge independent}$$

Conclusion:

mass of a soft fermion $q_0 \sim eT$ and $\vec{q} = 0$
is gauge independent at next-to-leading-order
beyond the HTL approximation

calculation for the damping rate is in progress