Weak-coupling expansions in hot QCD

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SEWM06, BNL, 11 May 2006

Motivation

RHIC $\rightarrow$ QCD at $T \gtrsim$ (a few) 100 MeV

asymptotic freedom $\rightarrow$ weak coupling expansion

slow convergence, non-trivial structure

problematic dof's are identified

- soft modes $p \sim gT$ $\rightarrow$ odd powers in g
- ultrasoft modes $p \sim g^2T$ $\rightarrow$ non-pert coeffs

general picture

- perturbation theory OK for parametrically hard scales $p \sim 2\pi T$
- soft and ultrasoft scales need improved analytic schemes, or non-pert treatment
- starting point: dim red eff. theory, or HTL eff. theory

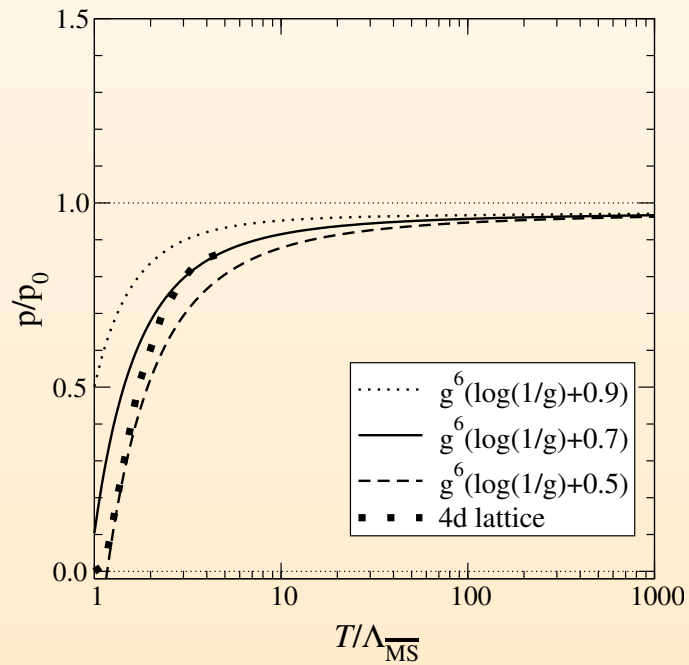
quantitative evidence:

- pick some simple observables
- compare 4d lattice vs soft/ultrasoft eff. theory
- e.g. static correlation lengths $\rightarrow$ agreement down to $T \sim 2T_c$

Thermal pressure $p(T)$: 4d vs 3d

$$p_{\mathsf{QCD}}(T) \equiv \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}[A^a_\mu, \psi, \bar{\psi}] \exp\Big(-\frac{1}{\hbar}\int_0^{\hbar/T} d\tau \int d^{3-2\epsilon}x\, \mathcal{L}_{\mathsf{QCD}}\Big)$$

$$\mathcal{L}_{\mathsf{QCD}} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi}\gamma_\mu D_\mu \psi + \mathcal{L}_{\mathsf{GF}} + \mathcal{L}_{\mathsf{FP}}$$



asymptotically, expect ideal gas: $p_{\mathsf{QCD}}(T\to\infty) \equiv p_0 = \left(16 + \frac{21}{2}N_f\right)\frac{\pi^2 T^4}{90}$

Thermal pressure $p(T)$: status

For $N_f = 0$, $N_c = 3$, write $p_{\mathsf{QCD}} = \frac{8\pi^2 T^4}{45}\{p_{\mathsf{h}} + p_{\mathsf{s}} + p_{\mathsf{us}}\}$, where

$$
\begin{aligned}
p_{\mathsf{h}} &= 1 - 0.2\,\tilde{g}^2 + 0.92\,\tilde{g}^4 + (\#_0 - 0.25)\;\tilde{g}^6 \\
p_{\mathsf{s}} &= 0.37\,\tilde{m}_{\mathsf{E}}{}^3 + 0.38\ln(0.11\,\tilde{m}_{\mathsf{E}})\;\tilde{g}_{\mathsf{E}}{}^2\,\tilde{m}_{\mathsf{E}}{}^2 - 0.54\,\tilde{g}_{\mathsf{E}}{}^4\,\tilde{m}_{\mathsf{E}} \\
&\quad - 0.21\ln(0.32\,\tilde{m}_{\mathsf{E}})\;\tilde{g}_{\mathsf{E}}{}^6 - 0.14\,\tilde{g}_{\mathsf{E}}{}^4\,\tilde{m}_{\mathsf{E}}{}^2 \\
p_{\mathsf{us}} &= \tilde{g}_{\mathsf{M}}{}^6\left[-0.072\ln\left(1.3\,\tilde{g}_{\mathsf{M}}{}^2\right) \pm 0.02^{nspt} \pm 0.02^{stat}\right]
\end{aligned}
$$

with couplings

$$
\begin{aligned}
\tilde{g} &\sim \frac{5\sqrt{3}}{4\pi}\,g \\
\tilde{m}_{\mathsf{E}}{}^2 &= \tilde{g}^2 + 0.24\,\tilde{g}^4 \\
\tilde{g}_{\mathsf{E}}{}^2 &= \tilde{g}^2 + 0.18\,\tilde{g}^4 + 0.083\,\tilde{g}^6 \\
\tilde{g}_{\mathsf{M}}{}^2 &= \tilde{g}_{\mathsf{E}}{}^2\left[1 - 0.029\frac{\tilde{g}_{\mathsf{E}}{}^2}{\tilde{m}_{\mathsf{E}}} + 0.0071\frac{\tilde{g}_{\mathsf{E}}{}^4}{\tilde{m}_{\mathsf{E}}{}^2}\right]
\end{aligned}
$$

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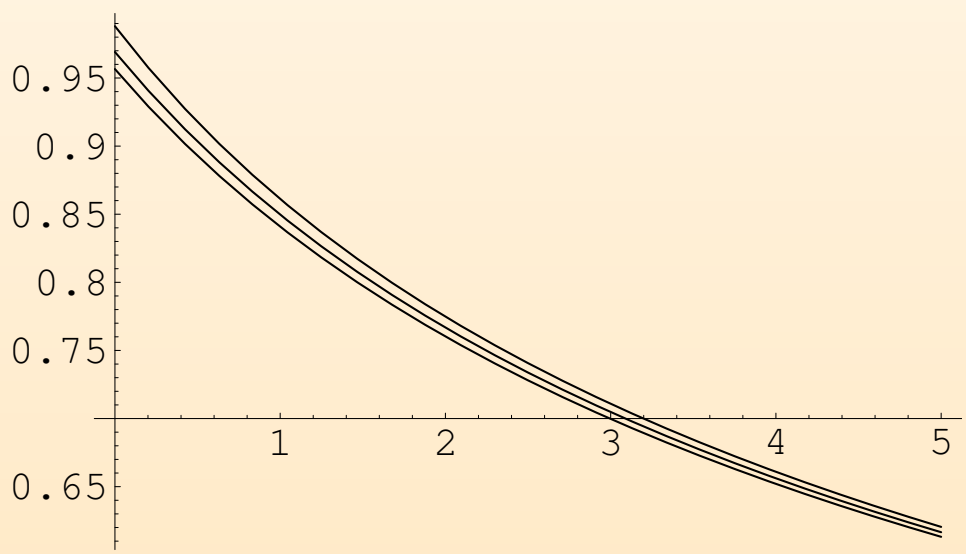
value of $\tilde{g}$ vs $\ln(T/T_c)$:



Thermal pressure $p(T)$: status

For $N_f = 0$, $N_c = 3$, write $p_{\rm QCD} = \frac{8\pi^2 T^4}{45}\{p_{\rm h} + p_{\rm s} + p_{\rm us}\}$, where

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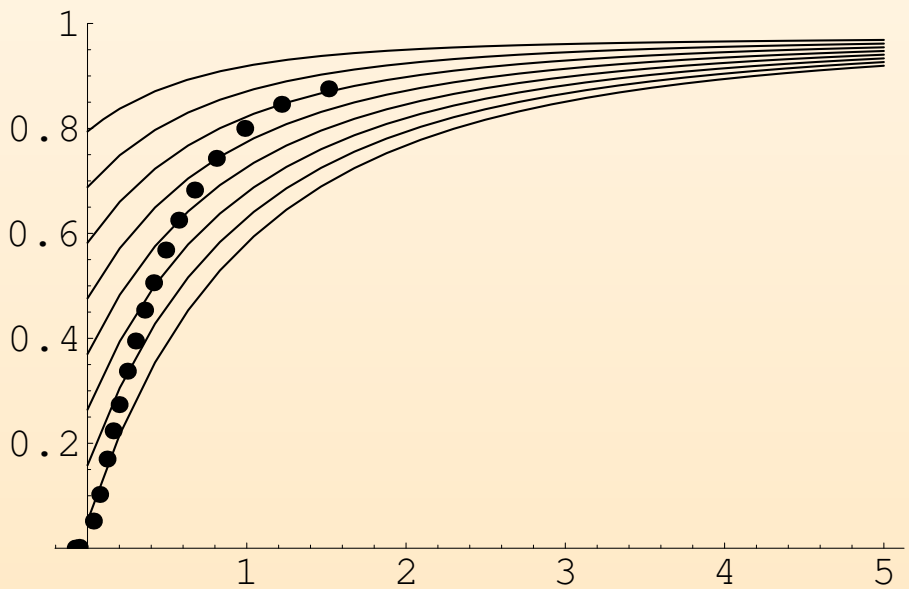
\end{aligned}
$$

$p_{\rm QCD}/p_{\rm SB}$ vs $\ln(T/T_c)$, for $\#_0 = -0.26, \ldots, 0.64$:



Spatial string tension σ_s

study an observable allowing an unambiguous comparison

take rectangular Wilson loop $W_s(R_1, R_2)$ in (x_1, x_2) plane

def potential $V_s(R_1) = -\lim_{R_2\to\infty} \frac{1}{R_2} \ln W_s(R_1, R_2)$

def spatial string tension $\sigma_s \equiv \lim_{R_1\to\infty} \frac{V_s(R_1)}{R_1}$

σ_s has been measured in SU(3) on the (4d) lattice

as e.g. $\frac{\sqrt{\sigma_s}}{T} = \phi\left(\frac{T}{T_c}\right)$ [Boyd et al, 96]

aim: get the eff. theory prediction for σ_s

- effective theory setup
- σ_s from 3d lattice
- perturbative matching to 4d
- $\Lambda_{\overline{\text{MS}}}$ vs T_c

Effective theory setup: QCD $\rightarrow$ EQCD

high T: QCD dynamics contained in 3d EQCD

$$\mathcal{L}_{\mathsf{E}} = \frac{1}{2} Tr\, F_{kl}^2 + Tr\, [D_k, A_0]^2 + m_{\mathsf{E}}^2 Tr\, A_0^2 + \lambda_{\mathsf{E}}^{(1)} (Tr\, A_0^2)^2 + \lambda_{\mathsf{E}}^{(2)} Tr\, A_0^4 + \ldots$$

matching coefficients [E. Braaten, A. Nieto, 95; M. Laine, YS, 05]

$$\begin{aligned} m_{\mathsf{E}}^2 &= T^2 \left\{ \#g^2 + \#g^4 + \ldots \right\} \\ \lambda_{\mathsf{E}}^{(1/2)} &= T \left\{ \#g^4 + \#g^6 + \ldots \right\} \\ g_{\mathsf{E}}^2 &= T \left\{ g^2 + \#g^4 + \#g^6 + \ldots \right\} \end{aligned}$$

higher order operators do not (yet) contribute [S. Chapman, 94; Kajantie et al, 97, 02]

$$\delta \mathcal{L}_{\mathsf{E}} \sim g^2 \frac{D_k D_l}{(2\pi T)^2} \mathcal{L}_{\mathsf{E}} \sim g^2 \frac{(g^2 T)^2}{(2\pi T)^2} \mathcal{L}_{\mathsf{E}} \qquad \text{(i.e. } g^8 \text{ for } g_{\mathsf{E}}^2\text{)}$$

Digression: g_{E}^2 numerically

in practice, need to renormalize: $g^2 = g^2(\bar{\mu})$ is $\overline{\mathsf{MS}}$ coupling

from soln of RGE at 2-loop level, define as usual

$$\Lambda_{\overline{\mathsf{MS}}} \equiv \lim_{\bar{\mu}\to\infty} \bar{\mu}\Big[b_0 g^2(\bar{\mu})\Big]^{-b_1/2b_0^2} \exp\Big[-\frac{1}{2b_0 g^2(\bar{\mu})}\Big]$$

hence $g_{\mathsf{E}}^2 = g_{\mathsf{E}}^2\left(\bar{\mu}, \Lambda_{\overline{\mathsf{MS}}}, T\right) = T\,\phi\left(\frac{\bar{\mu}}{T}, \frac{T}{\Lambda_{\overline{\mathsf{MS}}}}\right)$

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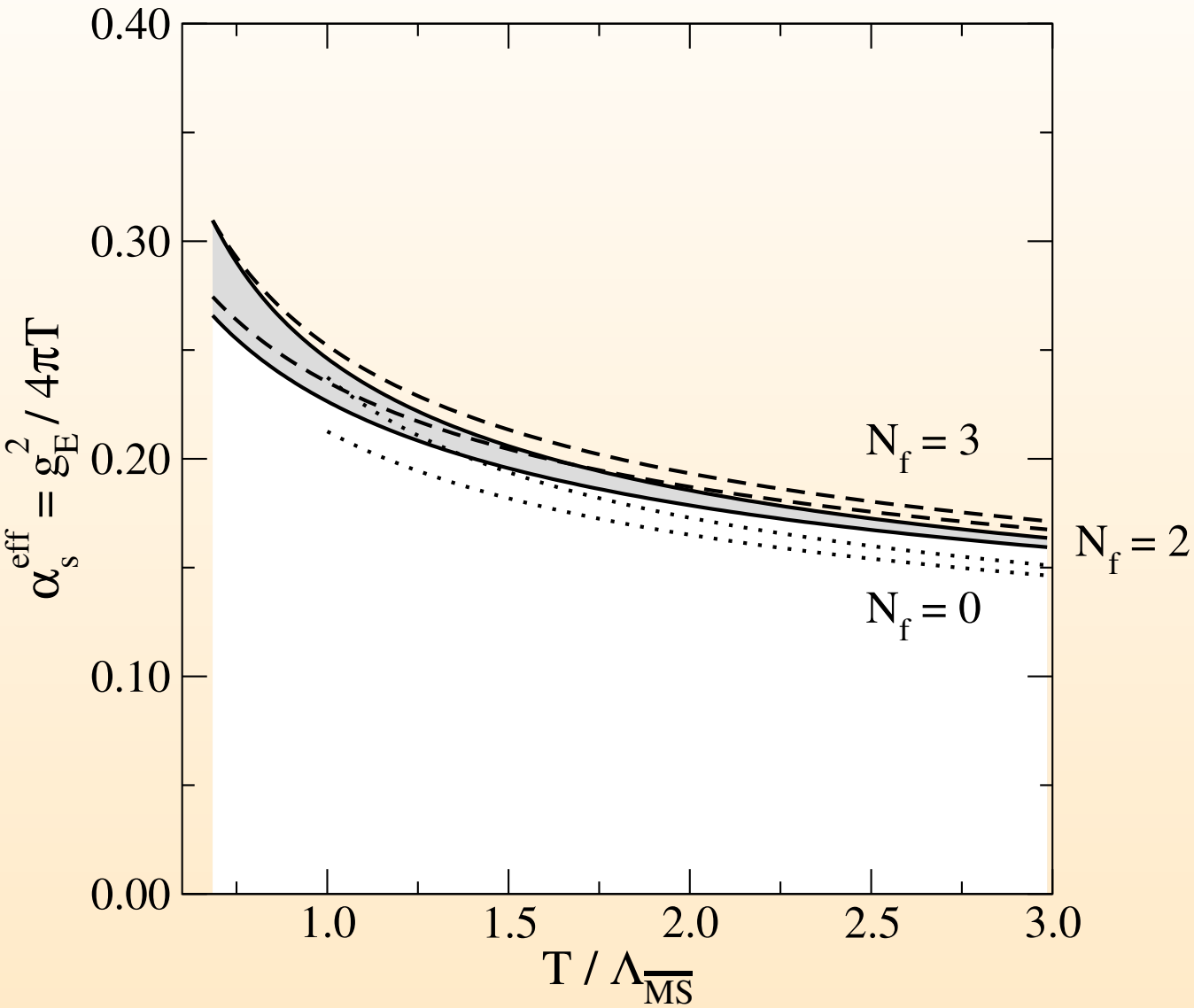
ren. scale dependence

- formally, $\bar{\mu}$ dependence is of higher order
- numerically, there is $\bar{\mu}$ dependence
- free to choose some optimisation procedure, e.g. PMS
  - ▷ *choose $\bar{\mu}_{opt}$ at extremum of 1-loop*
  - ▷ *vary scale within $\bar{\mu} = (0.5 \ldots 2.0) \times \bar{\mu}_{opt}$*

Digression: g_{E}^2 numerically

[Laine, Schröder, hep-ph/0303061]



Effective theory setup: QCD $\to$ EQCD $\to$ MQCD

3d EQCD is contained in 3d MQCD

$$\mathcal{L}_{\mathsf{M}} = \frac{1}{2} Tr\, F_{kl}^2 + \ldots$$

matching coefficient

[P. Giovannangeli, 04; M. Laine, YS, 05]

$$g_{\mathsf{M}}^2 \quad = \quad g_{\mathsf{E}}^2 \left\{ 1 + \# \frac{g_{\mathsf{E}}^2}{m_{\mathsf{E}}} + \# \frac{g_{\mathsf{E}}^4}{m_{\mathsf{E}}^2} + \# \frac{g_{\mathsf{E}}^2 \lambda_{\mathsf{E}}^{(1/2)}}{m_{\mathsf{E}}^2} + \ldots \right\}$$

expansion converges extremely well, even close to T_c

can safely ignore higher loop corrections for g_{M}^2

higher order operators could contribute

$$\delta \mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2 \frac{D_k D_l}{m_{\mathsf{E}}^3} \mathcal{L}_{\mathsf{M}} \sim g_{\mathsf{E}}^2 \frac{(g^2 T)^2}{m_{\mathsf{E}}^3} \mathcal{L}_{\mathsf{M}}$$

Effective theory prediction for σ_s

observable σ_s exists in 3d SU(3) gauge theory (MQCD)

- dimensionful gauge coupling $\rightarrow$ $\sigma_s = \# g_{\mathsf{M}}^4$
- most recent lattice data $\frac{\sqrt{\sigma_s}}{g_{\mathsf{M}}^2} = 0.553(1)$ [M. Teper, B. Lucini, 02]

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- $\frac{\sqrt{\sigma_s}}{T} = 0.553(1) \, \frac{g_{\mathsf{M}}^2}{g_{\mathsf{E}}^2} \, \frac{g_{\mathsf{E}}^2}{T} = \phi\left(\frac{T}{\Lambda_{\overline{\mathsf{MS}}}}\right)$

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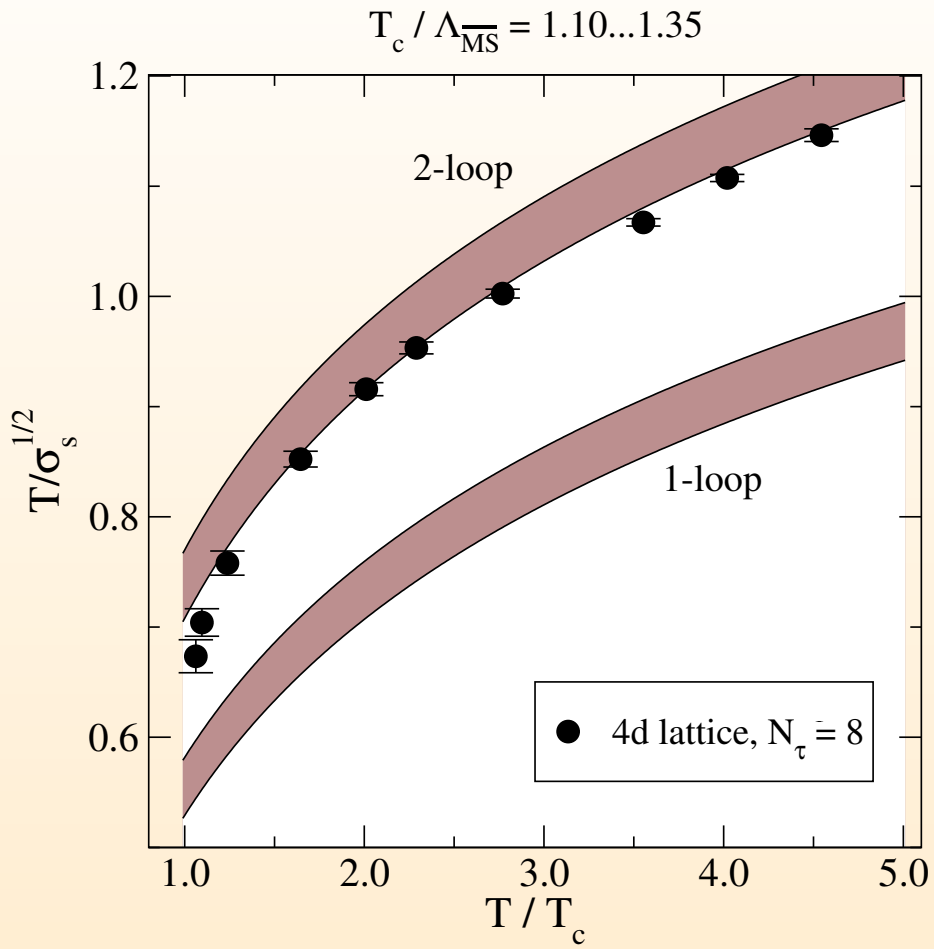
finally, need to relate $\Lambda_{\overline{\mathsf{MS}}}$ and T_c

- e.g. via $T = 0$ string tension $\frac{T_c}{\Lambda_{\overline{\mathsf{MS}}}} = \frac{T_c/\sqrt{\sigma}}{\Lambda_{\overline{\mathsf{MS}}}/\sqrt{\sigma}} = 1.16(4)$ [Teper et al, 03; Bali, Schilling, 92]
- e.g. via Sommer scale $\frac{T_c}{\Lambda_{\overline{\mathsf{MS}}}} = \frac{r_0 T_c}{r_0 \Lambda_{\overline{\mathsf{MS}}}} = 1.25(10)$ [S. Necco, 03; ALPHA coll, 98]
- e.g. via scaling at crit. point $\frac{T_c}{\Lambda_{\overline{\mathsf{MS}}}} = 1.15(5)$ [S. Gupta, 00]
- to be conservative, consider the interval 1.10 ...1.35

Spatial string tension σ_s: 4d vs 3d



[4d lattice data from Boyd et al, 96] (cave: $32^3 \times 8$, no cont. extrapolation: $N_\tau = 8$, $T = 1/aN_\tau$)

parameter-free comparison!

support for hard/soft+ultrasoft picture of thermal QCD

Quark mass effects in $p(T)$

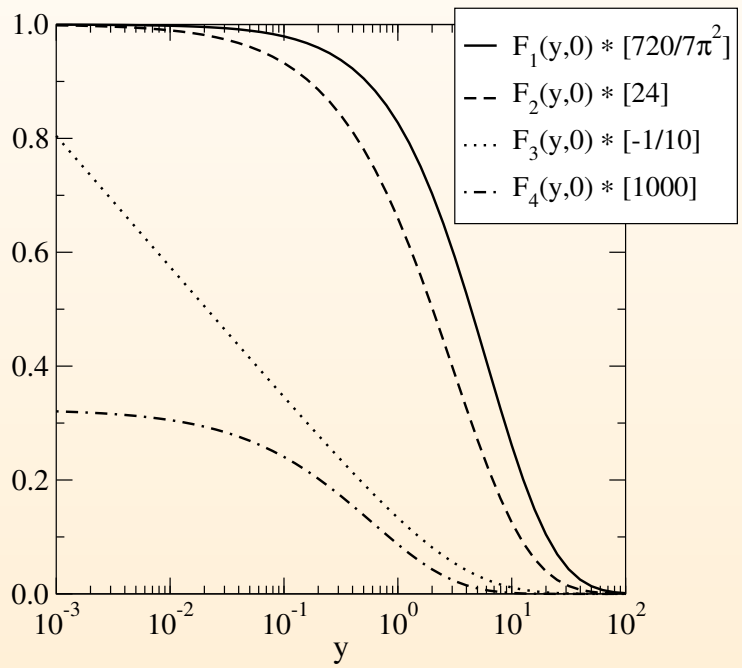
- at $m_q = 0$: know N_f dependence to high order
- $\Rightarrow$ interpolate between integer N_f ?
- better: take gluonic contributions to highest available order
  consider effects of quarks with physical masses at NLO
  - ▷ *''correction factor''* $\frac{T^4\left[\alpha_{E1}+g^2\alpha_{E2}\right](N_f)}{T^4\left[\alpha_{E1}+g^2\alpha_{E2}\right](0)}$

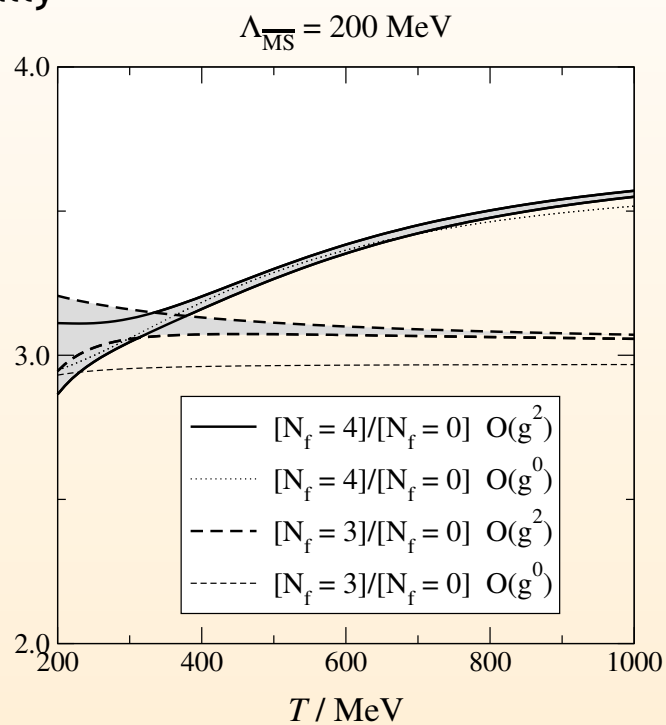
$$\alpha_{\mathsf{E1}} = d_A\frac{\pi^2}{45} + 4C_A\sum_{i=1}^{N_f} F_1\left(\frac{m_i^2}{T^2},\frac{\mu_i}{T}\right)$$

$$\alpha_{\mathsf{E2}} = -\frac{d_A C_A}{144} - d_A\sum_{i=1}^{N_f}\left\{\frac{1}{6}F_2\left(\frac{m_i^2}{T^2},\frac{\mu_i}{T}\right)\left[1+6F_2\left(\frac{m_i^2}{T^2},\frac{\mu_i}{T}\right)\right] + \right.$$
$$\left. +\frac{m_i^2}{4\pi^2T^2}\left(3\ln\frac{\bar{\mu}}{m_i}+2\right)F_2\left(\frac{m_i^2}{T^2},\frac{\mu_i}{T}\right) - \frac{2m_i^2}{T^2}F_4\left(\frac{m_i^2}{T^2},\frac{\mu_i}{T}\right)\right\}$$

Quark mass effects in $p(T)$

- the F_i are integrals involving $n_{\mathsf{F}}(x) = \frac{1}{e^x+1}$ limits $m \to 0$, $T \to 0$ known analytically



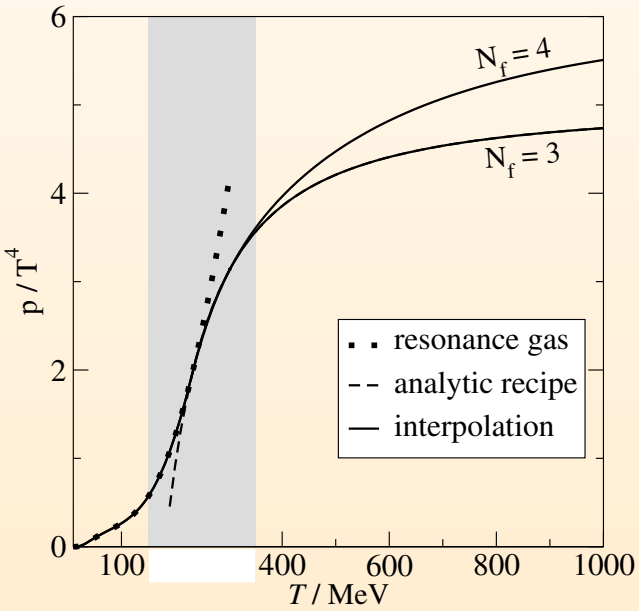


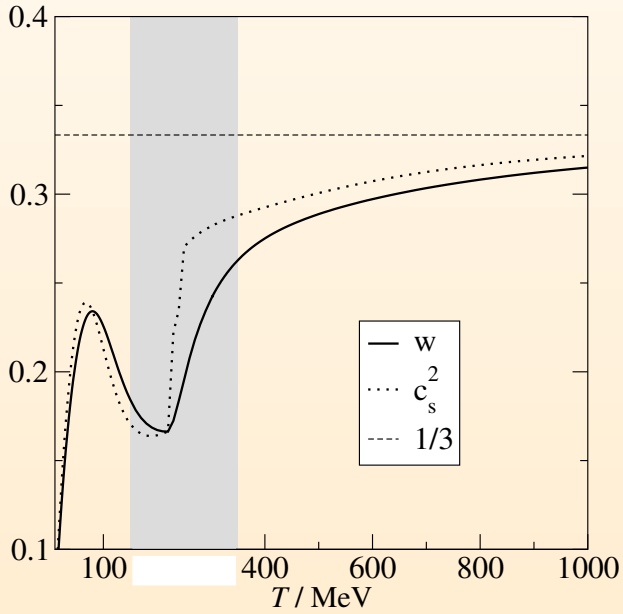
- estimate numerical importance of correction factor: take running masses
- $\Rightarrow$ $\mathcal{O}(g^2)$ correction factors are a few % only
- IF charm quark thermalizes in heavy ion coll., THEN it has effects at relatively low temperatures

Quark mass effects in $p(T)$

- for phenomenological applications, re-write derivatives of pressure as equation of state $w(T) = \frac{p(T)}{e(T)} = \frac{p(T)}{Tp'(T)-p(T)}$
  sound speed squared $c_s^2(T) = \frac{p'(T)}{e'(T)} = \frac{p'(T)}{Tp''(T)}$

- then, e.g. from Einstein eqs, $\frac{1}{T}\,\partial_t T = -\frac{\sqrt{24\pi}}{m_{\mathsf{Pl}}}\sqrt{\frac{p(T)}{w(T)}}\,c_s^2(T)$





- for phenomenology, tune $\Lambda_{\overline{\mathsf{MS}}}$ by fitting to hadron resonance gas [Karsch et al 03]
- in shaded region, only lattice can give a quantitative result

Conclusions

- thermodynamic quantities of QCD are relevant for cosmology and heavy ion collisions
- these quantities can be determined numerically at $T \sim 200$ MeV, and via effective field theory at $T \gg 200$ MeV
- effective field theory opens up tremendous opportunities: analytic treatment of fermions (incl. physical m, μ), universality, superrenormalizabilty
- for precise results, sometimes need very deep expansions
- setup is systematic, and testable