Axial anomaly and magnetism of nuclear and quark matter

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QCD vacuum in the magnetic field

Typically, critical $B$ which modifies QCD vacuum $eB \sim m_\rho^2, f_\pi^2$.
(Shushpanov-Smilga, Kabat et al, Miransky et al, Cohen et al)

In (or near) the chiral limit, the response is governed by chiral Lagrangian.

$$\mathcal{L} = \frac{f_\pi^2}{4} \text{tr} \left[ \partial_\mu \Sigma \partial^\mu \Sigma^\dagger + 2m_\pi^2 \Sigma \right]; \quad \Sigma = \exp \left( \frac{i\tau_a \varphi_a}{f_\pi} \right);$$

We shall look at a nontrivial solution — $\pi^0$ domain wall:

$$\pi^0 \equiv \varphi_3 = 4f_\pi \arctan e^{m_\pi z}; \quad \varphi_1 = \varphi_2 = 0;$$

which is unstable (“unwinding”). The spectrum of excitations has tachyonic branch:

$$E^2 = k_x^2 + k_y^2 - 3m_\pi^2.$$

This solution becomes *metastable* in the magnetic field $B > B_0$

$$E^2 = (2n + 1)eB - 3m_\pi^2, \quad n = 0, 1, \ldots$$

$$B_0 = \frac{3m_\pi^2}{e} \approx 1.0 \times 10^{19} \text{ G},$$

Can it become *globally* stable?

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The wall carries energy per unit area:
\[
\frac{\mathcal{E}}{S} = 8 f^2 \pi m. 
\]

But, for \( B \neq 0 \), it also carries baryon number!

The *gauge invariant* baryon current is given by (Goldstone-Wilczek, Witten)
\[
J^\mu_B = -\frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2} \text{tr} \left\{ (L_\nu L_\alpha L_\beta) - 3i e \partial\nu [A_\alpha Q (L_\beta + R_\beta)] \right\},
\]
where \( L_\mu = \Sigma \partial_\mu \Sigma^\dagger \), \( R_\mu = \partial_\mu \Sigma^\dagger \Sigma \) and \( Q = \tau_3/2 + 1/6 \).

For the wall, \( \nabla \pi^0 \neq 0 \), in the magnetic field \( B \):
\[
J^0_B = \frac{e}{4\pi^2 f_\pi} B \cdot \nabla \pi^0; \quad \Rightarrow \quad \frac{N_B}{S} = \frac{eB}{2\pi}.
\]

I.e., the wall is stable towards decay into vacuum when \( \mu_B > \frac{\mathcal{E}}{N_B} = \frac{16\pi f^2 \pi m_\pi}{eB} \).

And if
\[
B > B_1 = \frac{16\pi f^2 \pi m_\pi}{em_N} \approx 1.1 \times 10^{19} \text{ G}
\]
the wall wins over nuclear matter \( (\mu_B \approx m_N) \) in terms of \( \mathcal{E}/N_B \).

\( B_1 \sim m_\pi \Rightarrow B_0 \sim m_\pi^2 \) and both vanish in the chiral limit.
Large $\mu$ and color superconductivity

- Asymptotic freedom $\Rightarrow \alpha_s(\mu) \to 0$.

- Quarks of “different color” (color antisymmetric state) attract. Fermi sphere is unstable towards condensation of quark pairs (Cooper).

- For 2 flavors – 2SC: (Rapp et al, Alford et al, '97)

\[
\langle u_R d_R \rangle = \langle u_L d_L \rangle \neq 0
\]

— flavor singlet $\Rightarrow$ breaks $\text{U}(1)_A$ (not $\text{SU}(2)_A$).

- For 3 flavors — CFL: (Alford, Rajagopal, Wilczek)

\[
\langle u_R d_R \rangle = \langle d_R s_R \rangle = \langle s_R u_R \rangle = (R \to L) \neq 0
\]

— flavor triplet and color triplet

\[
\begin{align*}
\text{SU}(3)_R &\times \text{SU}(3)_{\text{color}} \to \text{SU}(3)_{R+\text{color}} \\
\text{SU}(3)_L &\times \text{SU}(3)_{\text{color}} \to \text{SU}(3)_{L+\text{color}}
\end{align*}
\]

$\Rightarrow$ $\text{SU}(3)_R \times \text{SU}(3)_L \to \text{SU}(3)_{L+R}$

Breaks both $\text{U}(1)_A$ and $\text{SU}(3)_A$. 
Domain walls in 2SC and CFL

- Spontaneously broken $U(1)_A$ and SU(3)$_A$ $\Rightarrow$ Goldstone bosons.

- Note: U(1)$_A$ violation by QCD anomaly is suppressed at large $\mu$.

- In 2SC and CFL: there are neutral axial Goldstone bosons.

The lightest is $\eta$ in 2SC and $\eta - \eta'$ mixture ($s\bar{s}$) in CFL.

- Domain wall is energetically favorable state when $\mathcal{E}/N_B < \mu$. 

\[
\mathcal{E} \sim f_\eta^2 m_\eta \sim \mu^2 m_\eta, \quad N_B \sim eB;
\]

\[
B_c \sim \frac{\mu m_\eta}{e} \sim 10^{17} \text{ - } 10^{18} \text{ G}
\]

\[
\text{CFL} \quad \text{2SC}
\]
Magnetism of the wall

Consider coupling of the Goldstone-Wilczek baryon current to the source $A_\nu = (\mu, 0)$:

$$\mathcal{L}_{GW} = -A_\nu^B J_B^\nu = \frac{e}{4\pi^2 f_\pi} \mu \mathbf{B} \cdot \nabla \pi^0$$

This means the wall is magnetized with magnetization density (Son, Zhitnitsky, '04)

$$M = \frac{e}{4\pi^2 f_\pi} \mu \nabla \pi^0.$$ 

If the wall is generated spontaneously, it will be ferromagnetic.
Goldstone gradient (supercurrent)

Since it costs energy, how can a nonzero $\nabla \pi$ be spontaneously generated?

There is a coupling, $\nabla \pi \cdot N^\dagger \gamma_5 \gamma N$, between the Goldstone gradient (axial supercurrent) and nucleon axial current. If $N^\dagger \gamma_5 \gamma N$ was present, this could offset the cost from $f_\pi^2 (\nabla \pi)^2$.

In vacuum we would have to pay $m_N$ to create the requisite nucleons.

In CFL quark excitations are also gapped. By $\Delta$.

But finite $m_s$ lowers the energy cost of exciting a fermion.
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- In CFL quark excitations are also gapped. By $\Delta$.
- But finite $m_s$ lowers the energy cost of exciting a fermion.

- When the excitation is close to being gapless, one can lower the energy by creating supercurrent $j \sim \nabla \phi$:
- Modes with $\varepsilon(p) = \varepsilon_0(p) - j \cdot p < 0$ are occupied and contribute negatively to the energy due to the supercurrent–normal current coupling. (In cold atoms: cond-mat/0507586).
Narrow window of $\mu$

As a function of $j$, the effective potential develops the second minimum:

$$\left[ \Omega(j) - \Omega(j = 0) \right] / (\mu^2 \Delta(0)^2)$$

It is lower than $j = 0$ minimum only in a small interval of $\mu_s$:

$$1.605 \Delta < \mu_s < 1.615 \Delta,$$

But in a neutron star $\mu$ changes and all one needs is this small interval to be present somewhere in the full range of $\mu$ from surface to center.
Ferromagnetism of CFL quark matter

If $\nabla \phi$ is spontaneously generated at finite $\mu$, the term $e \mu B \cdot \nabla \phi$ means there is spontaneous magnetization.

$$M \sim e \mu \nabla \phi$$

Such a $\nabla \phi \sim \Delta$ does occur in the “Goldstone current” (or “meson supercurrent”) state in CFL. I.e., such a state is ferromagnetic.

$$M \sim \frac{e}{3\pi^2} \mu \Delta \approx 2.4 \cdot 10^{16} \text{ G} \times \left( \frac{\mu}{1.5 \text{ GeV}} \right) \left( \frac{\Delta}{30 \text{ MeV}} \right)$$

Only a narrow window of $\mu$ (Gerhold-Schäfer-Kryjevski):

$$\frac{m_s^2}{2\Delta} (1.615)^{-1} < \frac{\mu}{3} < \frac{m_s^2}{2\Delta} (1.605)^{-1},$$

i.e., a shell in a star of width $d \sim 2\% R$. Then

$$B \sim M \frac{d}{R} \sim 10^{14} - 10^{15} \text{ G}.$$ 

Could account for the field of a magnetar.
At $B_0 = 3m_\pi^2/e$ the $\pi^0$ domain wall becomes metastable.

The wall carries baryon number $N_B/S = eB/(2\pi)$ and competes with the nuclear matter.

The wall wins at $B > B_1 \approx \frac{16\pi f_\pi^2 m_\pi}{em_N} \sim 10^{19}$ G.

Both $B_1$ and $B_0$ vanish in the chiral limit.

In color superconducting quark matter $\eta/\eta'$ domain wall wins for $B \sim 10^{17} - 10^{18}$ G.

The “meson supercurrent” state in CFL is ferromagnetic and capable of producing $B \sim 10^{14} - 10^{15}$ G in a typical compact star.