

Matrix product variational formulation for lattice gauge theory

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Introduction

Non-perturbative physics in QCD

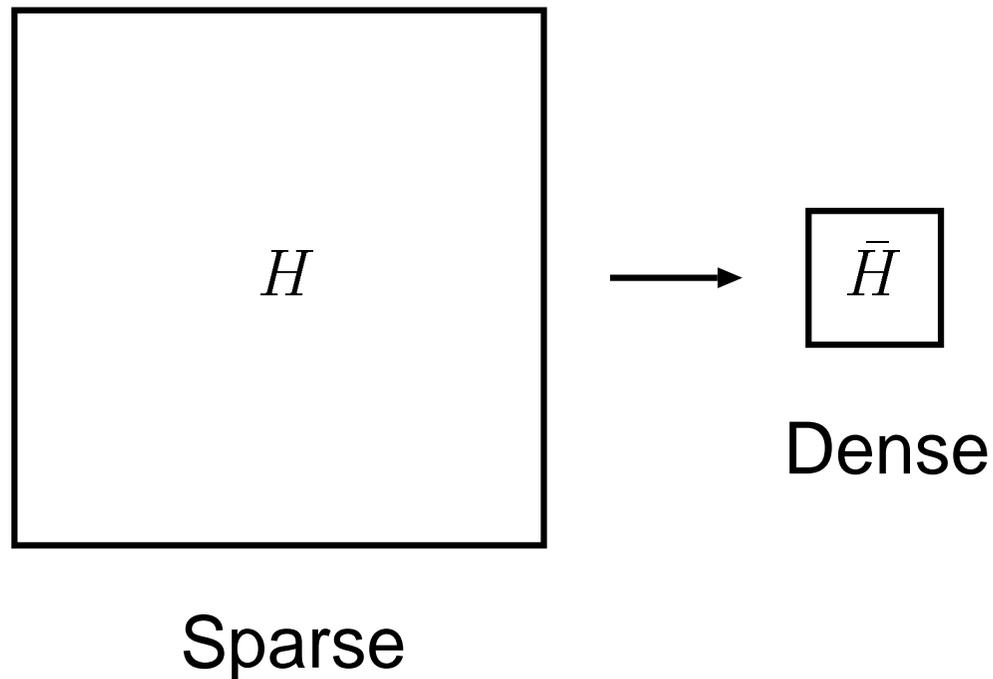
- RHIC experiment
- Confinement \leftrightarrow Vacuum wavefunction
- QCD at finite density
- Non-equilibrium quantum states

Variational methods with Hamiltonian

- Direct calculation of vacuum
- Time evolution of quantum states

How do we compress Hamiltonian?

What are the important degrees of freedom?



Target the low-lying states!

Density Matrix RG (DMRG)

Calculation load $\propto L$

- 1d quantum models at $T = 0$ and $T \neq 0$
(Heisenberg, Hubbard, t-J, Kondo,...)
- 2d quantum models at $T = 0$
(Heisenberg, Hubbard,...)
- Non-equilibrium quantum states
- Quantum information theory
- QED₁₊₁ with θ term

Fermion and boson

Does DMRG work in gauge theories?

On a finite lattice

- Fermion: **finite** dimensional
- Boson: **infinite** dimensional

Essential difference.

DMRG needs to be tested in a bosonic model.

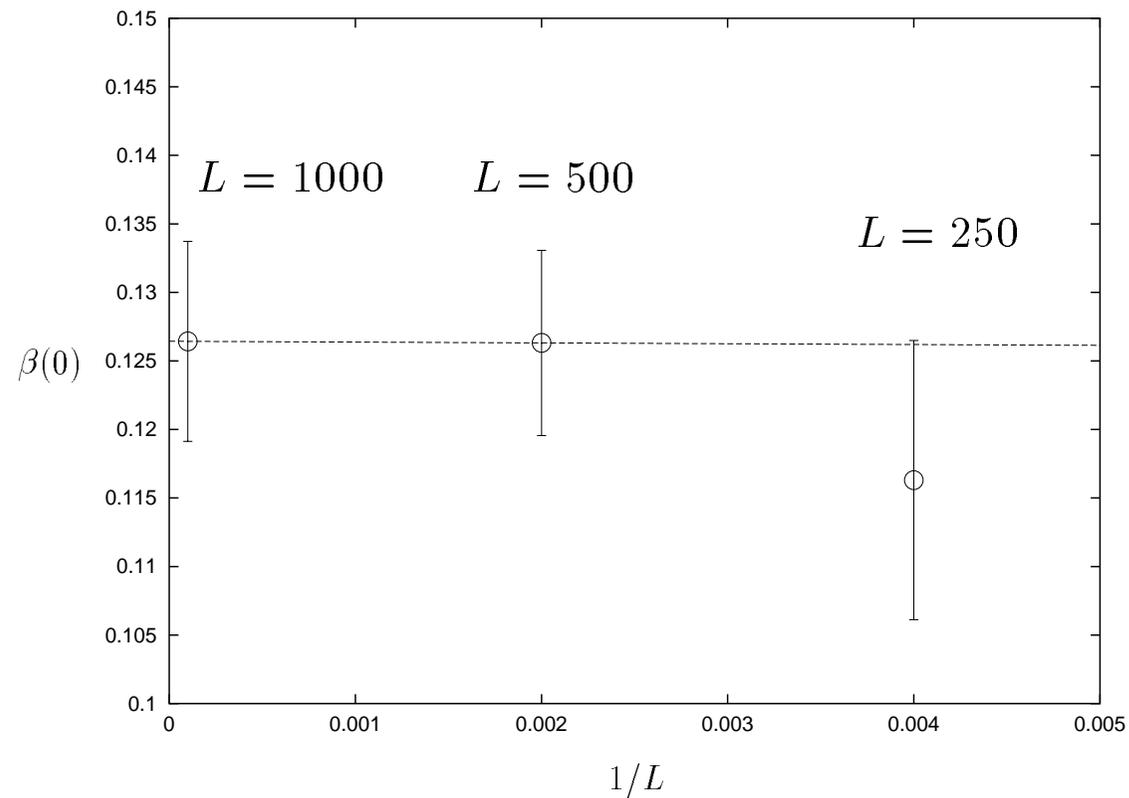
(1+1)-dim $\lambda\phi^4$ model: Critical coupling and exponent.

Critical coupling

TS, JHEP 0405, 007 (2004)

Method	Result
DMRG ($L = 1000$)	59.89 ± 0.01
Monte Carlo ($L = 512$)	$61.56^{+0.48}_{-0.24}$
Gaussian effective potential	61.266
Gaussian effective potential	61.632
Connected Green function	58.704
Coupled cluster expansion	$22.8 < (\lambda/\mu^2)_c < 51.6$
Non-Gaussian variational	41.28
Discretized light cone 1	43.896, 33.000
Discretized light cone 2	42.948, 46.26

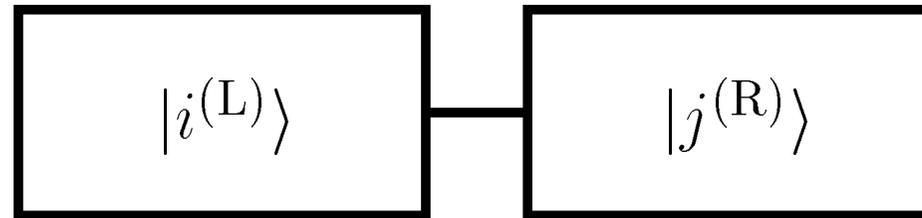
Critical exponent β



DMRG: $\beta = 0.1264 \pm 0.0073$

Exact: $\beta = 0.125$

Singular value decomposition



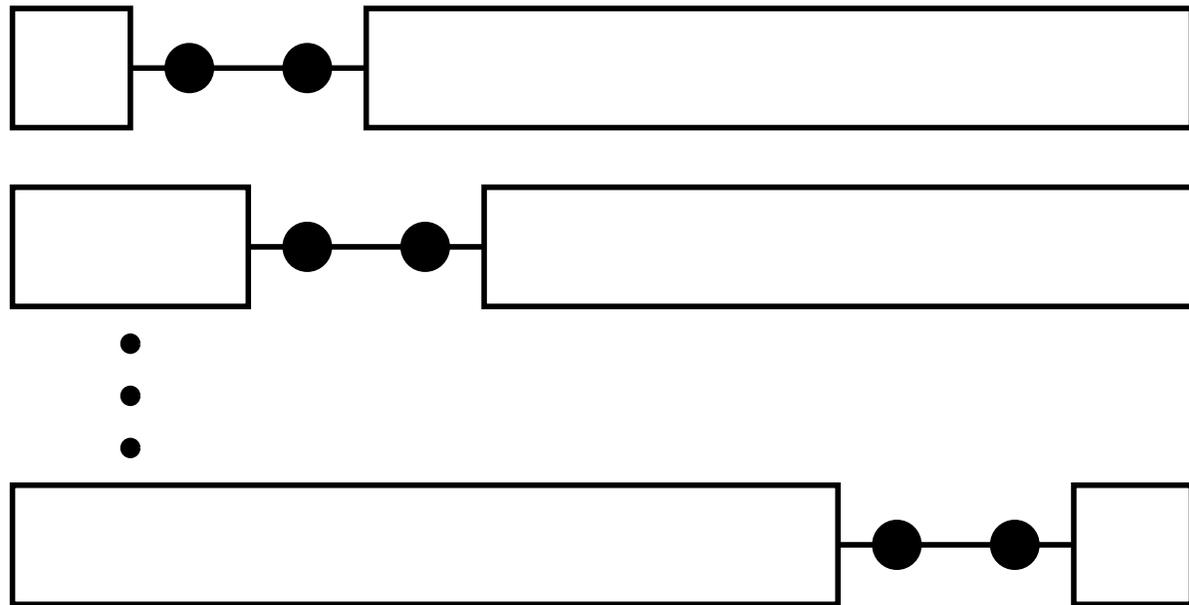
$$\begin{aligned} |\Psi\rangle &= \sum_{i=1}^{N_L} \sum_{j=1}^{N_R} \Psi_{ij} |i^{(L)}\rangle |j^{(R)}\rangle \\ &= \sum_{k=1}^{N_L} D_k |u_k^{(L)}\rangle |v_k^{(R)}\rangle. \end{aligned}$$

D_k are singular values of Ψ_{ij}

$$\Psi_{ij} = \sum_{k=1}^{N_L} U_{ik} D_k V_{kj}$$

Finite system algorithm of DMRG

A method for finite L .



A pair of sites is swept from end to end.

DMRG in higher dimensions

- DMRG is accurate
- DMRG works in bosonic models
- DMRG works in higher dim models

But

- Lattice size is small
- Large memory is necessary
- Programming is complicated

I want only the power of DMRG but not complexity.

Matrix product states

Östlund and Rommer, PRL75, 3537 (1995)

Add a site $|s\rangle$

$$|\alpha_{n+1}\rangle = \sum_{\alpha_n, s} A_{\alpha_{n+1}, \alpha_n} [s] |s\rangle |\alpha_n\rangle$$

Add many sites

$$|\alpha_{n+L}\rangle = A^{(L)} [s_L] \dots A^{(1)} [s_1] |s_L\rangle \dots |s_1\rangle |\alpha_n\rangle$$

In $L \rightarrow \infty$, assume $A^{(n)} = A$ and periodicity

$$|\Psi\rangle = \text{tr}[A[s_L] \dots A[s_1]] |s_L\rangle \dots |s_1\rangle$$

$S = 1/2$ Heisenberg chain

Energy function

$$E[A] = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{1}{\text{tr}(\hat{1}^L)} \text{tr}(\hat{S}^a \hat{S}^a \hat{1}^{L-2})$$

where

$$\hat{S}^a = \sum_{s,s'} \langle s | S_1^a | s' \rangle A^*[s] \otimes A[s'], \quad \hat{1} = \sum_s A^*[s] \otimes A[s]$$

Diagonalization of $\hat{1}$ simplifies $\hat{1}^{L-2}$ and $\hat{1}^L$.

The Powell method is used for minimization.

$S = 1/2$ Heisenberg chain

Ground state energy per site

M : matrix dimension

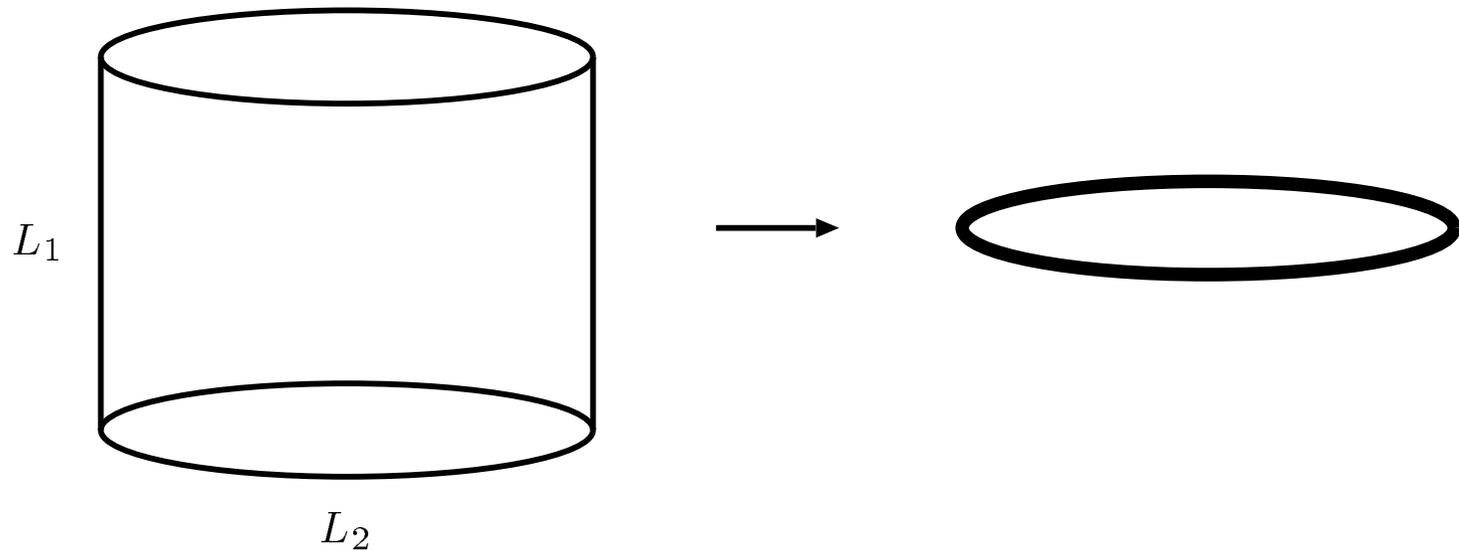
$M \setminus L$	10	100	1000	10000
6	-0.4092	-0.4372	-0.4371	-0.4368
12	-0.4092	-0.4427	-0.4425	-0.4425
Exact	-0.4515	-0.4438	-0.4431	-0.4431

The method works well for large L , but not for small L .

The error is less than 1% for $L = 10000$.

$2d \mathcal{S} = 1/2$ Heisenberg model

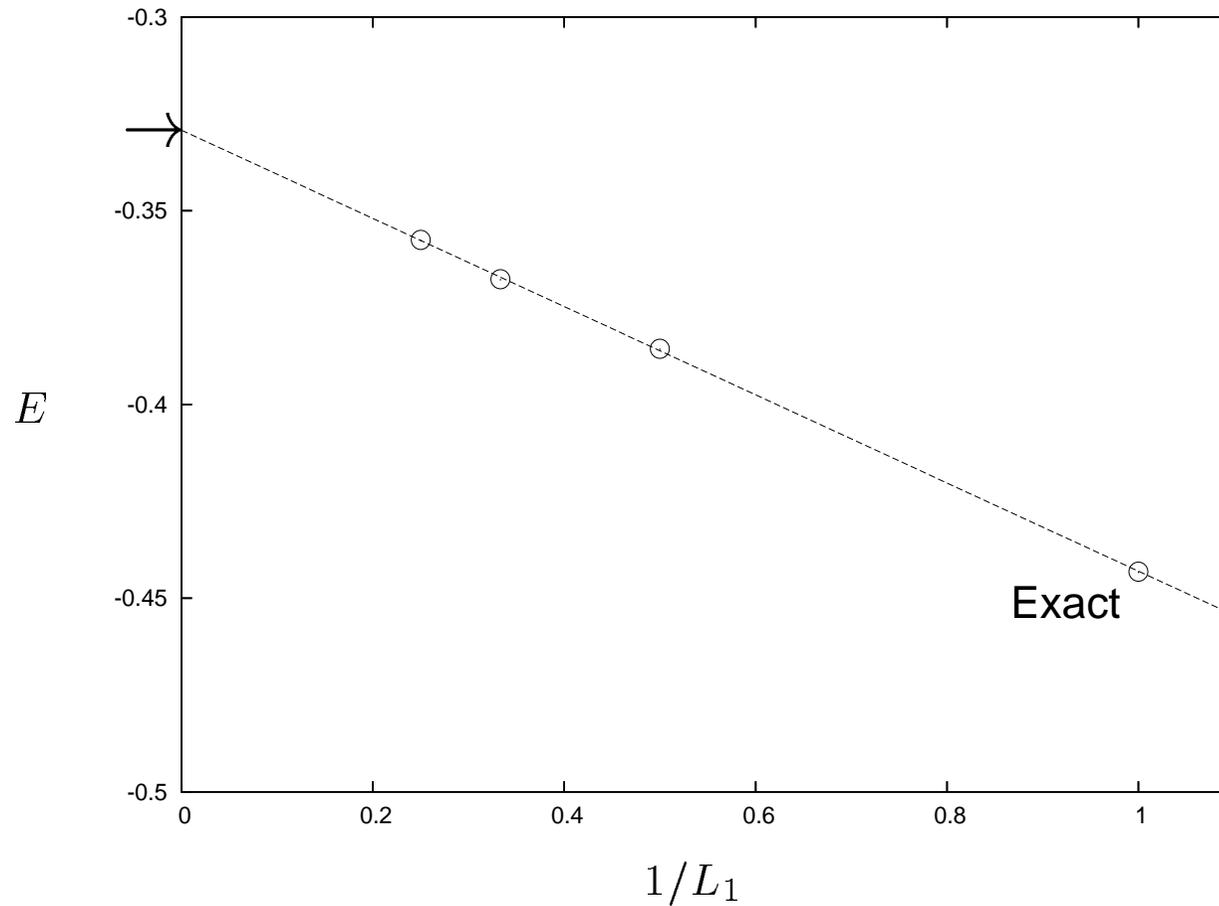
Need to find a one-dimensional structure.



Consider the tube as a ring.

Vacuum energy per bond

Ring size: $L_2 = 10000$



Comparison

2d $S = 1/2$ Heisenberg

Vacuum energy per bond in $L \rightarrow \infty$

Method	E	Lattice size	Year
Monte Carlo	-0.3347	16^2	1999
DMRG	-0.3347	12^2	2001
TPVA	-0.3272	∞	2004
This work	-0.328 ± 0.001	10000×4	2004

There is no exact solution.

$U(1)$ plaquette chain model

Kogut-Susskind Hamiltonian

$$H = \sum_{n,i} E_l^2 - x \sum_p (U_p + U_p^\dagger).$$

Quantization

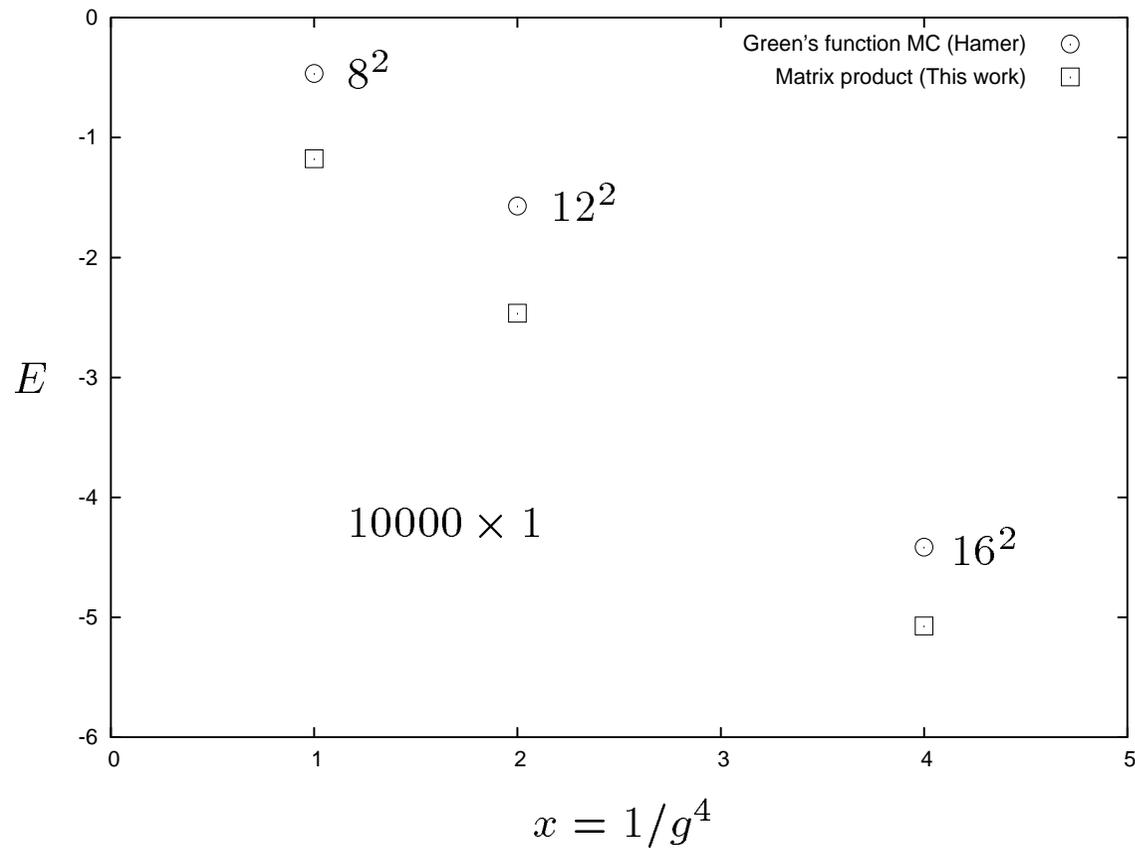
$$[E_l, U_l] = U_l, \quad [E_l, U_l^\dagger] = -U_l^\dagger$$

Quantum mechanics on S^1 (Ohnuki and Kitakado)

$$E|\alpha\rangle = \alpha|\alpha\rangle, \quad U^n|\alpha\rangle = |n + \alpha\rangle, \quad 0 \leq \alpha < 1$$

Set $\alpha = 0$ for numerical calculation.

Vacuum energy per site



This work: $N_{\text{basis}} = 9$, $M_{\text{matrix}} = 2$.

Gauge invariance

VEV of electric field E_l

$$\langle \Psi | E_l | \Psi \rangle \sim 10^{-3}$$

for $x = 1.0, 2.0,$ and 4.0 .

The obtained vacuum is almost gauge invariant because the Gauss law

$$\nabla \cdot \mathbf{E} = 0$$

is approximately satisfied.

Summary

- DMRG works well in the $\lambda\phi_{1+1}^4$ model
- 1d $S = 1/2$ Heisenberg model
 - Numerical implementation of matrix product
 - Two digits of the exact values
 - No gap between $S_T = 0$ and $S_T = 1$
- 2d $S = 1/2$ Heisenberg model
 - Consistent with the existing results
- U(1) plaquette chain
 - Gauge-singlet state is energetically favorable

Future works

- Larger lattices for refinement
- SU(2) gauge theory
- Finite temperature and density
- Non-equilibrium system
- Model mapping like bosonization
- Connection with matrix models
- Protein as a many-body quantum state