Gauge invariance in a Z_2 hamiltonian lattice gauge theory

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Trinity College

Lattice 2005 talk

Introduction

Lattice gauge theory for RHIC and LHC

- Temperature (it works!)
- Density (the sign problem)
- Non-equilibrium physics (how?)

I choose real time $\&$ variational method

In this talk, T and μ will not appear,

but this work will be useful for those purposes.

Matrix product ansatz

A variational method that originates in DMRG (density matrix renormalization group)

- Diagonalization of hamiltonian and transfer matrices
- Very accurate in (1+1)-d models at $T = 0$ and $T \neq 0$
- Free from the sign problem
- Non-equilibrium quantum physics
- Quantum information theory

Does the ansatz work in lattice gauge theory?

Lattice gauge hamiltonian

The hamiltonian version of LGT is not popular because

1. No systematic method like Monte Carlo
2. Difficult to maintain gauge invariance

These problems can be solved by introducing the matrix product ansatz.

Matrix product ansatz in lattice gauge theory.

TS, hep-lat/0411017, U(1) lattice gauge and Heisenberg

TS, JHEP07(2005)022, Z_2 lattice gauge theory

Z_2 lattice gauge theory

Hamiltonian is constructed from partition function with the transfer matrix formalism.

$$H = -\sum_{n,i} \sigma_x(n,i) - \lambda \sum_{\text{plaquette}} \sigma_z \sigma_z \sigma_z \sigma_z$$

Hamiltonian is gauge invariant

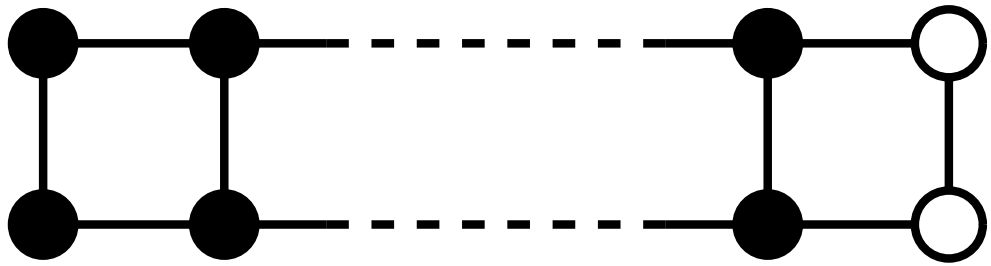
$$G(n)^{-1} H G(n) = H, \quad G(n) = \prod_{\pm i} \sigma_x(n,i).$$

Gauss law needs to be imposed.

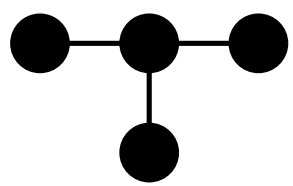
$$G(n)|\Psi\rangle = |\Psi\rangle.$$

Z_2 lattice gauge on a ladder chain

Consider a ladder chain for simplicity



- The links are dynamical.
- Periodicity is assumed in the horizontal direction.
- The Gauss law operator is a product of three σ_x's.

$G =$ 

Exact solutions

$\lambda = 0$: $\langle H \rangle / L = -3$ and $\langle P \rangle = 0$

$$|\mathrm{vac}\rangle = \prod_{n,i} \frac{1}{\sqrt{2}} (|+\rangle_{n,i} + |-\rangle_{n,i})$$

$\lambda = \infty$: $\langle H \rangle / L = -\lambda$ and $\langle P \rangle = 1$

$$|\mathrm{vac}\rangle = \frac{1}{\sqrt{2^{2L+1}}} \sum_{m=0}^{2L} G^{(m)} \prod_{n,i} |\pm\rangle_{n,i}$$

The both states are gauge invariant.

$$G(n)|\mathrm{vac}\rangle = |\mathrm{vac}\rangle.$$

Matrix product ansatz

$$\boxed{|\alpha_n\rangle}$$

Add a site $|s\rangle$ to a renormalized block $|\alpha_n\rangle$.

$$|\alpha_{n+1}\rangle = \sum_{\alpha_n, s} A_{\alpha_{n+1},\alpha_n}[s]|s\rangle|\alpha_n\rangle$$

Add many sites

$$|\alpha_{n+L}\rangle = A^{(L)}[s_L]\ldots A^{(1)}[s_1]|s_L\rangle\ldots|s_1\rangle|\alpha_n\rangle$$

In $L \to \infty$, assume $A^{(n)} = A$ and periodicity

$$|\Psi\rangle = \mathrm{tr}[A[s_L]\ldots A[s_1]]|s_L\rangle\ldots|s_1\rangle$$

Matrix product ansatz

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Energy function

Energy function has a matrix product form.
Example: $S = 1/2$ Heisenberg chain

$$E[A] = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\mathrm{tr}(\hat{S}^a \hat{S}^a \hat{1}^{L-2})}{\mathrm{tr}(\hat{1}^L)}$$

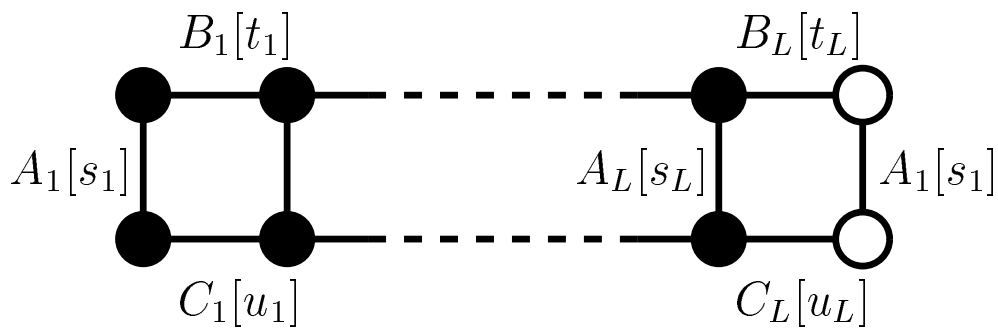
where

$$\hat{S}^a = \sum_{s,s'} \langle s | S_1^a | s' \rangle A^*[s] \otimes A[s'], \quad \hat{1} = \sum_s A^*[s] \otimes A[s]$$

Evaluation of the energy function is easy.

Extension to Z_2 lattice gauge

Each link is assigned a different set of matrices.



Matrix product state

$$|\Psi\rangle = \mathrm{tr}\left(\prod_{n=1}^{L}\sum_{s_n=\pm}\sum_{t_n=\pm}\sum_{u_n=\pm} A_n[s_n]B_n[t_n]C_n[u_n]\ |s_n\rangle_n|t_n\rangle_n|u_n\rangle_n\right)$$

where

$$\sigma_z|\pm\rangle_n = \pm|\pm\rangle_n.$$

Minimization $\rightarrow$ diagonalization

Energy is a function of the matrices A, B, and C.

$$E[A,B,C] = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

Minimize this by varying $A_n[s]$ with other matrices fixed.

$$\begin{aligned}\langle\Psi|H|\Psi\rangle &= \textstyle\sum_{i,j,k,l}\sum_{s,t}(A_n^*[s])_{ij}\bar{H}_{(i,j,s),(k,l,t)}(A_n[t])_{kl},\\ \langle\Psi|\Psi\rangle &= \textstyle\sum_{i,j,k,l}\sum_{s,t}(A_n^*[s])_{ij}N_{(i,j,s),(k,l,t)}(A_n[t])_{kl},\end{aligned}$$

If $A_n[s]$ is regarded as v, the minimization problem reduces to a generalized eigenvalue problem

$$v^\dagger\bar{H}v = Ev^\dagger Nv.$$

Sweep process

The norm matrix may have very small eigenvalues if the matrices $A_n[s]$, $B_n[s]$, and $C_n[s]$ are varied freely.

For numerical stability, the obtained matrices are transformed to keep orthonormality.

$$\sum_{j=1}^{M} \sum_{s=\pm} (X_n[s])_{ij} (X_n[s])_{i'j} = \delta_{ii'},$$

where X stands for A, B, and C.

The eigenvalue problem is solved iteratively

$$C_L[u_L] \to B_L[t_L] \to A_L[s_L] \to \cdots \to C_1[u_1] \to B_1[t_1] \to A_1[s_1].$$

One sweep = $3L$ sets of diagonalization

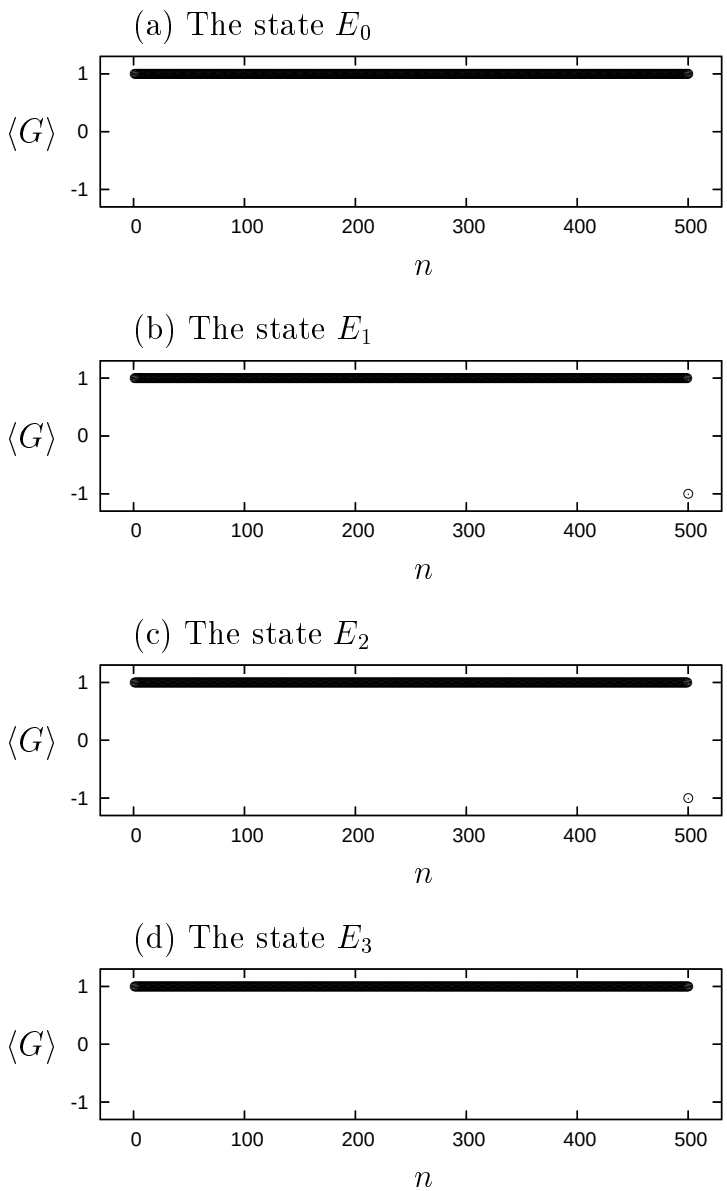
Convergence of energy

Lattice size $L = 500$

M	E_0/L	E_1/L	E_2/L	E_3/L	E_4/L	E_5/L
			$\lambda = 0.1$			
2	$\underline{-3.001}$	-2.997	-2.997	-2.997	$\underline{-2.993}$	$\underline{-2.993}$
3	$\underline{-3.001}$	-2.997	-2.997	-2.997	$\underline{-2.994}$	$\underline{-2.993}$
4	$\underline{-3.001}$	-2.997	-2.997	-2.997	$\underline{-2.997}$	$\underline{-2.995}$
			$\lambda = 1$			
2	$\underline{-3.124}$	-3.121	-3.121	-3.118	$\underline{-3.114}$	$\underline{-3.112}$
3	$\underline{-3.124}$	-3.121	-3.121	-3.118	$\underline{-3.114}$	$\underline{-3.112}$
4	$\underline{-3.124}$	-3.121	-3.121	-3.118	$\underline{-3.114}$	$\underline{-3.112}$

Good convergence with small M.

Gauge invariance



Gauss law operator

$\lambda = 10$, $L = 500$, $M = 4$

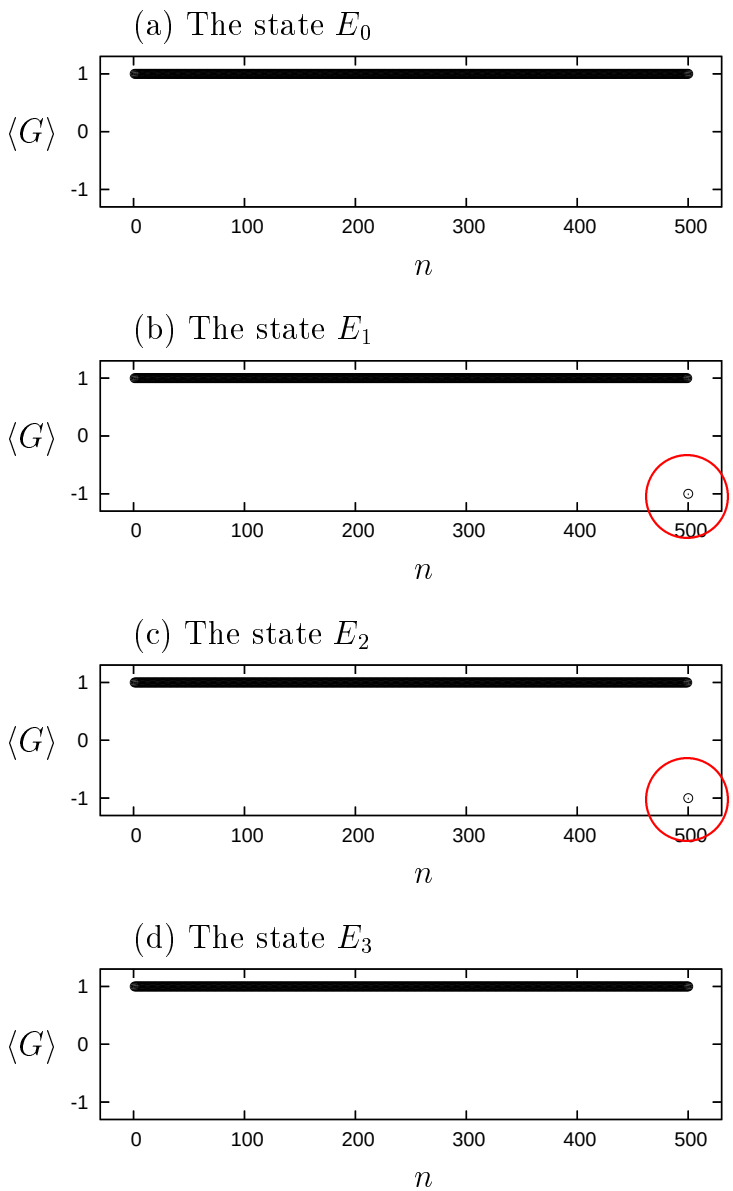
Gauge inv. states: E_0 and E_3

$\langle G \rangle = 1$

Gauge var. states: E_1 and E_2

$\langle G \rangle = -1$ at $n = 500$

Gauge invariance



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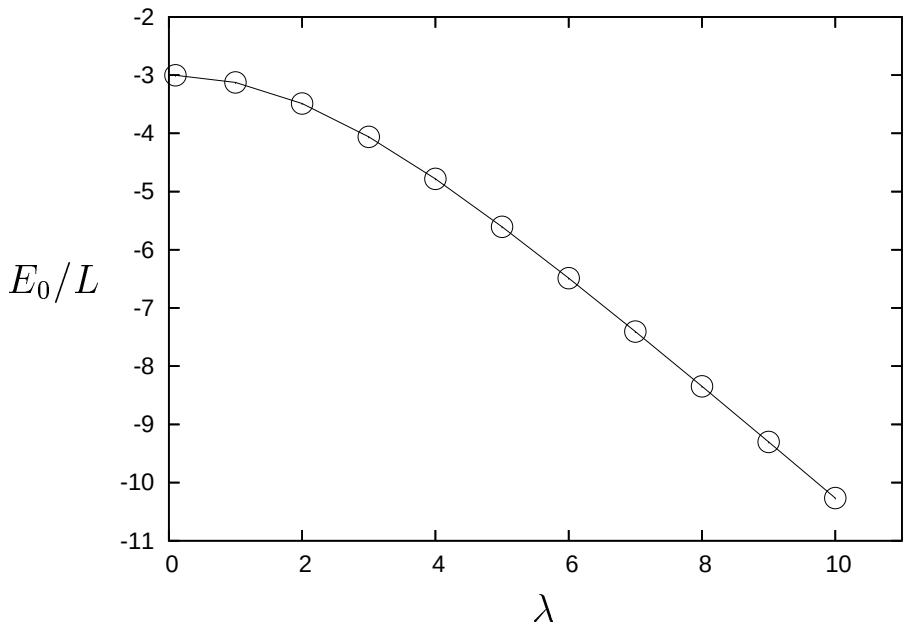
Vacuum energy vs λ

$L = 500$, $M = 4$

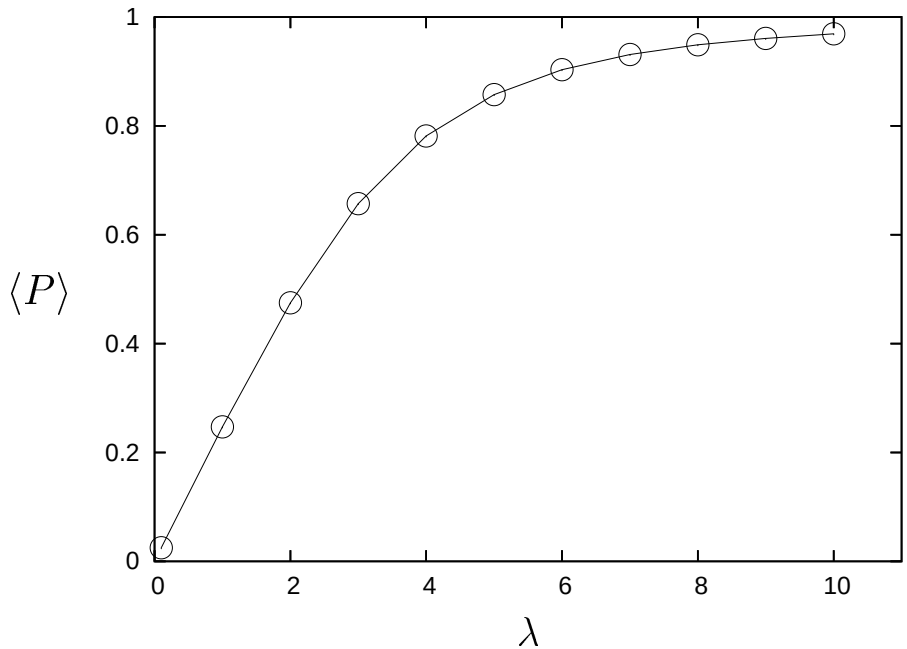


Consistent with the exact values

$$E_0/L = \begin{cases} -3 & (\lambda = 0) \\ -\lambda & (\lambda = \infty) \end{cases}$$

Plaquette vs λ

$L = 500$, $M = 4$



Consistent with the exact values

$$\langle P \rangle = \begin{cases} 0 & (\lambda = 0) \\ 1 & (\lambda = \infty) \end{cases}$$

Z_2 lattice gauge on a square lattice

- Square spatial lattice
- Second order phase transition

The Gauss law can be solved analytically.
The model is equivalent to transverse field Ising model.

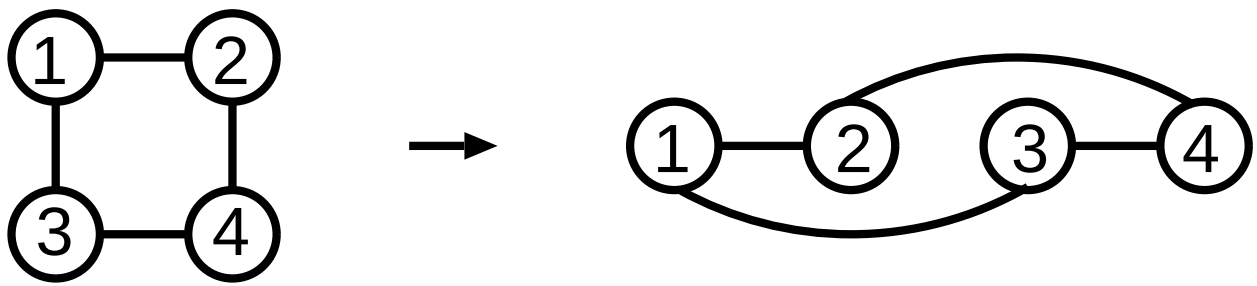
$$H = -\sum_{n,i} \sigma_z(n)\sigma_z(n+i) - \lambda \sum_n \sigma_x(n).$$

The matrix product ansatz describes the phase transition?

Matrix product ansatz for two-dim lattice

The ansatz is a method for one-dim spatial lattice.
Find one-dim structure on a two-dim lattice.

Example: 2×2 lattice

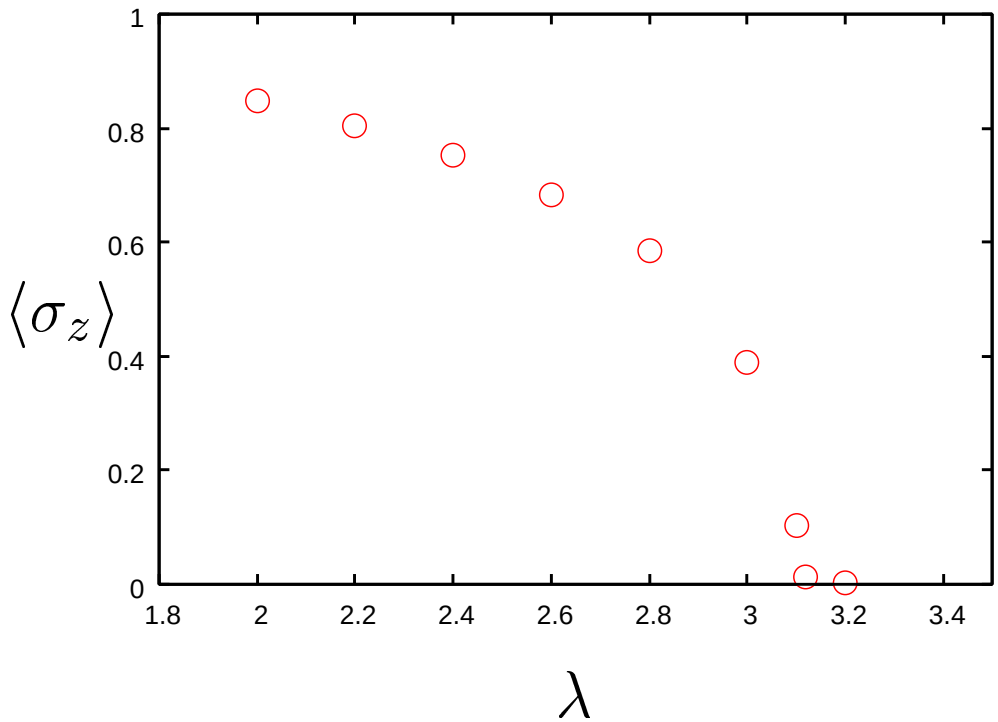


Matrix product state

$$|\Psi\rangle = \sum_{s_1,s_2,s_3,s_4} \mathrm{tr}\left(A_1[s_1]A_2[s_2]A_3[s_3]A_4[s_4]\right)|s_1\rangle|s_2\rangle|s_3\rangle|s_4\rangle$$

Magnetization $\langle \sigma_z \rangle$ vs λ

$L = 12$, $M = 30$



The second order phase transition is well described.

$$\lambda_c \sim 3.12$$

Critical coupling

(1+2)-dim transverse field Ising model

λ_c	L	Method	Year
3.07	26	Quantum Monte Carlo	1998
3.15 ± 0.05	20	Variational quantum Monte Carlo	2000
3.2	NA	Density Matrix RG	2001
3.12	12	Matrix product ansatz	2005

Consistent with the others.

The obtained wavefunction is sufficiently accurate.

Summary

Results

- The matrix product ansatz works in (1+2)-dimensional Z_2 lattice gauge theory
  - Ladder chain lattice - gauge invariant states
  - Square lattice - second order phase transition

Future study

- Larger lattice
- (1+3)-dim Z_2 lattice gauge theory
- SU(2) and SU(3) lattice gauge theory
- Fermion with $\mu \neq 0$
- Time evolution