

# On a Numerical Evaluation of Loop Integrals

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## 1 Introduction

The scalar one-loop  $n$ -point integral is given by

$$\mathcal{I} = \int \frac{d^4l}{(2\pi)^{4i}} \frac{1}{(l^2 - m_1^2 + i\varepsilon)((l + p_1)^2 - m_2^2 + i\varepsilon) \dots ((l + \sum_{j=1}^{n-1} p_j)^2 - m_n^2 + i\varepsilon)} \quad (1)$$

where

- $l$  is the loop momentum,
- $p_j$  the momentum of the  $j$ -th external particle,
- $m_j$  the mass carried by the  $j$ -th internal line.

$\varepsilon > 0$  is a real constant which is supplied to prevent the integral from diverging. A physical scattering amplitude contains this type of integrals and its value is defined at  $\varepsilon = 0$ .

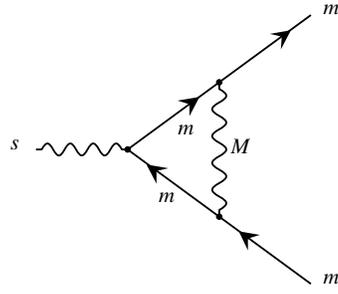
We consider scalar one-loop integrals in the form  $(-1)^n/(16\pi^2)I_n$  where

$$I_n = \int_{\mathcal{S}_{n-1}} \frac{1}{(D_n(\mathbf{x}) - i\varepsilon)^{n-2}} d\mathbf{x} \quad (2)$$

is obtained from (1) by introducing Feynman parameters and integrating over the loop momentum  $l$ .

- The integration region  $\mathcal{S}_{n-1}$  is the  $n - 1$  dimensional unit simplex.
- $D_n(\mathbf{x})$  is a quadratic;  $1/D_n(\mathbf{x})$  may have a non-integrable singularity if  $D_n(\mathbf{x})$  vanishes in the domain of integration.
- For the simplest cases, the results can be obtained analytically. So far, numerical techniques have been successful only after considerable analytic manipulation.

In this talk we present a method which has promise for an automatic calculation of loop integrals, relying on **multivariate integration** and **extrapolation**. We apply several variations of the method to sample cases of one-loop vertex ( $n = 3$ ), one-loop box ( $n = 4$ ) and two-loop planar vertex diagrams.



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Fig. 1. Vertex example

## 2 Numerical Extrapolation and Three-point Vertex Function

We consider the loop integral in the limit as  $\varepsilon \rightarrow 0$ . For example, for  $n = 3$ , the real part of (2) is given by

$$\lim_{\varepsilon \rightarrow 0} I(\varepsilon) = \lim_{\varepsilon \rightarrow 0} \int_{S_2} \frac{D_3(x, y)}{D_3(x, y)^2 + \varepsilon^2} dx dy$$

Example: fermion vertex with  $Z^0$  exchange (Figure 1) In this case,

$$D_3(x, y) = -xys + (x + y)^2 m^2 + (1 - x - y)M^2.$$

where  $s$  denotes the squared energy and  $m$  and  $M$  are particle masses corresponding to the fermion and  $Z^0$  boson, respectively.

We construct a sequence of  $I(\varepsilon_\iota)$ ,  $\iota = 0, 1, \dots$  and extrapolate to the limit  $I(0)$ .

## LINEAR EXTRAPOLATION

- Assume we obtain a sequence of approximations  $Q(\varepsilon_\iota)$  which satisfy an expansion of the form

$$Q(\varepsilon) \approx I(\varepsilon) = I(0) + \sum_{j=1}^{\nu} a_j \varphi_j(\varepsilon) + R_{\nu+1}(\varepsilon) \quad (3)$$

- We set  $a_0 = I(0)$ ,  $\varphi_0(\varepsilon) = 1$ , and consider the  $\varphi$  functions ordered so that  $\lim_{\varepsilon \rightarrow 0} \varphi_{j+1}(\varepsilon)/\varphi_j(\varepsilon) = 0$ .
- Denoting  $\beta_\iota = Q(\varepsilon_\iota)$  and disregarding the remainder term in (3), we solve a  $(\nu + 1) \times (\nu + 1)$  linear system of equations  $\Phi\alpha = \beta$  of the form

$$\sum_{j=0}^{\nu} \varphi_j(\varepsilon_\iota) \alpha_j^{(\nu)} = \beta_\iota, \quad \iota = 0, \dots, \nu. \quad (4)$$

- For successive  $\nu = 1, 2, \dots$ , this delivers  $\alpha_j^{(\nu)} \approx a_j$  and, in particular,  $\alpha_0^{(\nu)} \approx a_0 \approx I(0)$ .

Geometric sequence (G)						Harmonic sequence (H)			
nu	eps	Q(eps)	Extrapolated			eps	Q(eps)	Extrapolated	
	256	0.2120319070127095E-03				20	0.2421479625475858E-03		
1	128	0.2280362779518554E-03	0.2440406488910013E-03			10	0.2434730201795342E-03	0.2447980778114825E-03	
2	64	0.2363491905451424E-03	0.2448692545542390E-03			20/3	0.2439151979858242E-03	0.2448002914918651E-03	
3	32	0.2405610711989021E-03	0.2448014224054772E-03			5	0.2441363743331252E-03	0.2448002403715809E-03	
4	16	0.2426777113780660E-03	0.2448002064167953E-03			4	0.2442691075095314E-03	0.2448002403543701E-03	
5	8	0.2437382985741919E-03	0.2448002401471510E-03			20/6	0.2443576075549443E-03	0.2448002403553846E-03	
6	4	0.2442691075095314E-03	0.2448002403568659E-03			20/7	0.2444208273439749E-03	0.2448002403552964E-03	
7	2	0.2445346343652990E-03	0.2448002403554218E-03			2.5	0.2444682451604905E-03	0.2448002403558393E-03	
8	1	0.2446674275837987E-03	0.2448002403554184E-03			20/9	0.2445051274397318E-03	0.2448002403553095E-03	
		Analytic	0.244800240355414541E-03						
	nu	1	2	3	4	5	6	7	8
cond#	G	3	5	6.4	7.3	7.8	8.0	8.1	8.2
cond#	H	3	9	28	92	301	1007	3392	11506

Fig. 2. Three-point vertex diagram, extrapolated results for G and H

Considering an example (Oyanagi et al.) with  $m = 40$  GeV,  $M = 93$  GeV,  $s = 9000$  GeV<sup>2</sup>, and assuming an expansion (3) in integer powers of  $\varepsilon$ , i.e.,  $\varphi_j(\varepsilon) = \varepsilon^j$ , solving (4) using a geometric sequence (G) with  $\varepsilon_\nu = b^{-\nu}\varepsilon_0$ ,  $\nu = 0, 1, \dots$ ,  $b = 2$  and  $\varepsilon_0 = 256$  gives the results in the first half of the table in Figure 2.

The second half of Figure 2 shows the results obtained with a harmonic type progression (H) using  $\varepsilon_\nu = \varepsilon_0/\nu$ ,  $\nu = 1, 2, \dots$  and  $\varepsilon_0 = 20$ . This helps to construct sequences  $\{Q(\varepsilon_\nu)\}$  which may be easier to compute since  $\varepsilon_\nu$  decreases slowly.

The integral approximations  $Q(\varepsilon)$  were calculated (in double precision) to a relative tolerated error of  $10^{-12}$  using a multivariate integration routine for hyper-rectangular regions (DCUHRE by Genz et al., sequential predecessor of the adaptive algorithms in ParInt), after a transformation of the triangular domain.

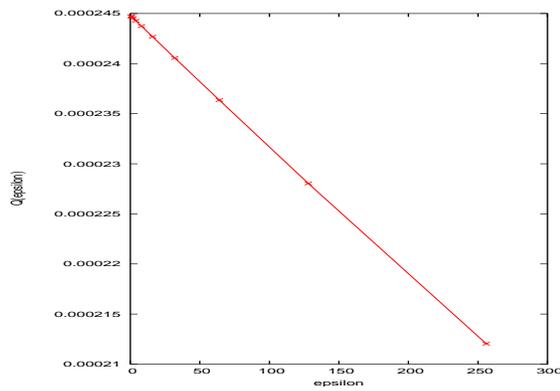


Fig. 3.  $Q(\varepsilon)$  vs.  $\varepsilon$  for  $m = 40$  GeV,  $M = 93$  GeV,  $s = 9000$  GeV<sup>2</sup>

Allowing for a sequence of integer powers of  $\varepsilon$  in (4) corresponds to assuming that  $I(\varepsilon)$  satisfies a **polynomial approximation**. Figure 3 displays  $Q(\varepsilon)$  as a function of  $\varepsilon$  for this example.

#### Notes:

- The stability of the procedure can be checked by computing a **condition number** obtained by solving the same system as in (4) but with the right hand side replaced by  $\beta_l = (-1)^l$ . It is expected that the process is significantly less stable using H (Lyness). One can also use sequences with properties in between those of G and H.
- It emerges that the final results can be obtained to high accuracy as long as the  $Q(\varepsilon)$  calculations can be performed to high accuracy.
- It is not necessary to solve a system for each  $\nu$ .
- The computation for  $\varphi_j(\varepsilon) = \varepsilon^j$  using G can also be carried out by a recursive procedure (cfr. **Richardson extrapolation**).

### 3 Extrapolation by the $\varepsilon$ -algorithm

For linear extrapolation it is required to know the nature of the functions  $\varphi_j(\varepsilon)$ . Extrapolation by the  $\varepsilon$ -algorithm does not require this specific information, as long as the extrapolation method is known to be valid for the class of expansions of interest. The  $\varepsilon$ -algorithm is a recursive implementation (by Wynn) of a nonlinear sequence to sequence transformation (by Shanks).

Given a sequence  $\{\beta_\iota\}$ ,  $\iota = 0, 1, \dots$  of real numbers, a triangular table is computed as depicted in Figure 4, according to

$$\begin{aligned}\tau_{\iota,-1} &= 0 \\ \tau_{\iota 0} &= \beta_\iota \\ \tau_{\iota,\kappa+1} &= \tau_{\iota+1,\kappa+1} + \frac{1}{\tau_{\iota+1,\kappa} - \tau_{\iota\kappa}}.\end{aligned}$$

Only the even-numbered columns have meaning; the odd-numbered ones are to store temporary values.

**Theorem 1 (Convergence)** *If the sequence  $\{\sigma_\iota\}$  satisfies a homogeneous linear difference equation of order  $\nu$  with constant coefficients,  $\sum_{\iota=0}^{\nu} c_\iota \sigma_\iota = 0$ , and if  $\beta_\iota = S + \sigma_\iota$ ,  $\iota = 0, 1, \dots$ , then  $\tau_{\iota,2\nu} = S$  (pending  $\tau_{\iota,2\nu}$  exists).*

	$\tau_{00}$		
0		$\tau_{01}$	
	$\tau_{10}$		$\tau_{02}$
0		$\tau_{11}$	...
		...	...
		...	...
0		$\tau_{\iota-1,1}$	...
	$\tau_{\iota 0}$		$\tau_{\iota-1,2}$
0		$\tau_{\iota 1}$	
	$\tau_{\iota+1,0}$		

Fig. 4.  $\varepsilon$ -algorithm table

**Theorem 2 (Sufficient condition)** *The sequence  $\sigma_\iota = u(\iota)\iota^\kappa\varepsilon^\iota$ ,  $\iota = 0, 1, \dots$ , where  $\kappa \geq 0$  integer,  $\varepsilon \in \mathfrak{R}$  and the  $u(\iota)$  are periodic functions with period  $\rho$ , satisfies a homogeneous linear difference equation with constant coefficients of order  $\nu = (\iota + 1)\rho$ .*

**Corollary 3 (Extension)** *The sequence*

$$\sigma_\iota = \sum_{\kappa=0}^{\mu} \sum_{j=1}^{\nu} u_{j\kappa}(\iota)\iota^\kappa\varepsilon_j^\iota, \quad \iota = 0, 1, \dots$$

*where  $\nu \geq 1$ ,  $\mu \geq 0$  integer,  $\varepsilon_j \in \mathfrak{R}$ , and  $u_{j\kappa}(\iota)$  are periodic functions with period  $\rho$ , satisfies a linear homogeneous difference equation with constant coefficients.*

b = 2 ( eps = 2^(9-p) )			b = 1.2 ( eps = 1.2^(41-p) )			
p	eps	Q(eps)	Extrapolated	p	Q(eps)	Extrapolated
1	256	0.2120319070127095E-03		4	0.1528743894317701E-03	0.4316691758746829E-03
2	128	0.2280362779518554E-03		8	0.1941968510705258E-03	0.2311914999988003E-03
3	64	0.2363491905451424E-03	0.2453337696296586E-03	12	0.2191889439432741E-03	0.2451361661323442E-03
4	32	0.2405610711989021E-03	0.2448867967487399E-03	16	0.2322503791767422E-03	0.2448034893044764E-03
5	16	0.2426777113780660E-03	0.2448017954351509E-03	20	0.2387139036026090E-03	0.2448002168460955E-03
6	8	0.2437382985741919E-03	0.2448006914824931E-03	24	0.2418586009244202E-03	0.2448002405705739E-03
7	4	0.2442691075095314E-03	0.2448001658003823E-03	28	0.2433802831821293E-03	0.2448002405705739E-03
8	2	0.2445346343652990E-03	0.2448002390032324E-03	32	0.2441151680705952E-03	0.2448002403563278E-03
9	1	0.2446674275837987E-03	0.2448002403591474E-03	36	0.2444697959556787E-03	0.2448002403550793E-03
10	0.5	0.2447338315398706E-03	0.2448002403553692E-03	40	0.2446408673684701E-03	0.2448002403552772E-03
		Analytic	0.244800240355414541E-03			

Fig. 5. Three-point vertex,  $\varepsilon$ -algorithm results for  $G$  with  $b = 2$  (left) and  $b = 1.2$  (right)

As a special case of Corollary 3 with  $\kappa = 0$  and constant  $u_j = u_{j0}(\iota)$  ( $\rho = 1$ ), if furthermore

$$\beta_\iota - S = \sum_{j=1}^{\nu} u_j \varepsilon_j^\iota, \quad (5)$$

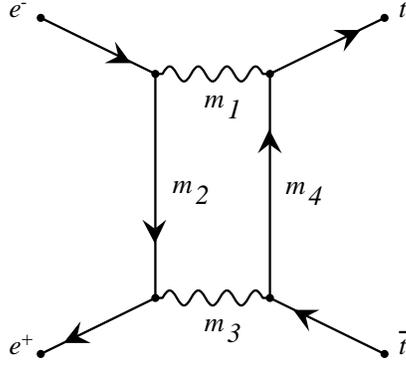
then  $\tau_{\iota, 2\nu} = S$  (pending  $\tau_{\iota, 2\nu}$  exists).

Note the correspondence of (5) with (3) for a geometric sequence  $\varepsilon_j = b^{-j}$ . Indeed if we let  $u_j = a_j \varepsilon_0^j$ , then  $u_j \varepsilon_j^\iota = a_j \varepsilon_0^j b^{-j\iota} = a_j \varphi_j(\varepsilon_\iota)$ .

It should be noted that Corollary 3 allows extrapolation with the  $\varepsilon$ -algorithm for significantly more complicated expansions (3); for instance  $\varphi_j(\varepsilon)$  may be of the form  $\varepsilon^\gamma \log^\delta \varepsilon$  where  $\delta \geq 0$  integer and real  $\gamma > 0$  (as long as a geometric sequence is used for  $\varepsilon$ ).

Figure 5 gives results obtained for the sequences  $\varepsilon_\iota = \varepsilon_0 b^{-\iota}$ ,  $\iota = 0, 1, \dots$ , with  $b = 2$  and  $\varepsilon_0 = 256$  (left), and with  $b = 1.2$  and  $\varepsilon_0 = 1.2^{40}$  (right). The integrals  $I(\varepsilon)$  were approximated to a relative tolerance of

$10^{-12}$ .



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Fig. 6. Feynman diagram for  $e^- e^+ \rightarrow t \bar{t}$

#### 4 Four-point function

Here the integral is  $I_4$  given by (2), over the three-dimensional unit simplex  $\mathcal{S}_3$ . The quadratic  $D_4$  is expressed as

$$D_4 = {}^t \mathbf{x} A \mathbf{x} + 2\mathbf{v} \cdot \mathbf{x} + C,$$

where  $A_{lj} = q_l \cdot q_j$ ,  $q_1 = -p_1$ ,  $q_2 = p_2$ ,  $q_3 = p_2 + p_3$ ,  $C = M_0^2 = m_2^2$  and  $v_l = \frac{1}{2}(-q_l^2 + M_l^2 - M_0^2)$  with  $M_1 = m_1, M_2 = m_3, M_3 = m_4$  (Fujimoto et al.)

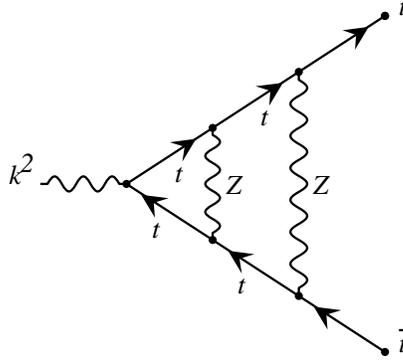
eps	Q(eps)	extrapolated results				analytic	#evals
Real part							
64	-0.6144285e-09						5.9e07
32	-0.5822585e-09	-0.5497390e-09					8.7e07
16	-0.5660866e-09	-0.5502165e-09	-0.5502169e-09				1.2e08
8	-0.5580768e-09	-0.5502169e-09	-0.5502165e-09	-0.5501814e-09			1.6e08
4	-0.5541097e-09	-0.5501958e-09	-0.5501784e-09	-0.5501814e-09	-0.550181e-09		2.2e08
2	-0.5521396e-09	-0.5501862e-09	-0.5501811e-09				2.8e08
1	-0.5511587e-09	-0.5501829e-09					3.5e08
0.5	-0.5506695e-09						4.4e08
Imaginary part							
64	0.1212204e-08						6.3e07
32	0.1209017e-08	0.1217354e-08					9.1e07
16	0.1203859e-08	0.1185690e-08	0.1193325e-08				1.3e08
8	0.1199841e-08	0.1192165e-08	0.1193381e-08	0.1193449e-08			1.7e08
4	0.1197203e-08	0.1193069e-08	0.1193413e-08	0.1193430e-08	0.119343e-08		2.3e08
2	0.1195593e-08	0.1193304e-08	0.1193424e-08				2.9e08
1	0.1194648e-08	0.1193381e-08					3.6e08
0.5	0.1194106e-08						4.5e08

Fig. 7.  $e^-e^+ \rightarrow t\bar{t}$  extrapolation table for  $\cos\theta = -0.5$

Table 7 illustrates the use of the  $\varepsilon$ -algorithm for the integral computation. Here  $m_1 = m_3 = M_Z = 91\text{GeV}$ ,  $m_2 = m_e = 0.511\text{MeV}$ ,  $m_4 = m_t = 150\text{ GeV}$ ,  $\sqrt{s} = 500\text{ GeV}$  and  $\theta = \angle(\mathbf{p}_1, \mathbf{p}_4)$ . In this table the results are given for  $\cos\theta = -0.5$ .

#### Notes:

- The integrals were approximated to an accuracy of  $10^{-7}$  (in each coordinate direction) using an iterated integration with DQAGE.
- The final extrapolation results agree to the 6-digit accuracy of the analytic results reported by Fujimoto et al.
- In this case, DCUHRE was not able to approximate the integrals to the desired accuracy. Significant portions of the domain appeared to be neglected, which may be due to inaccurate error estimates near the singularity.



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Fig. 8. Feynman diagram for vertex correction of top quark with two  $Z^0$  boson exchanges

## 5 Two-loop vertex

A general Feynman loop diagram for  $n$  external particles,  $N$  internal lines and  $L$  independent loops is proportional to

$$\mathcal{I}[\varphi] = \int \prod_{a=1}^L \left( \frac{d^4 l_a}{(2\pi)^{4i}} \right)^L \prod_{\iota=1}^N \frac{1}{k_\iota^2 - m_\iota^2 + i\epsilon} \varphi(k_1, \dots, k_N) = \left( \frac{1}{16\pi^2} \right)^L I[\varphi], \quad (6)$$

where the momenta on the  $\iota$ -th propagator and the according masses are denoted by  $k_\iota$  and  $m_\iota$ , respectively, for  $\iota = 1, \dots, N$ ; the independent loop momenta are  $l_a$ ,  $a = 1, \dots, L$ ; the external momenta are  $p_j$ ,  $j = 1, \dots, n$ ; and the internal momentum  $k_\iota$  is a linear combination of the  $l_a$  and the external momenta.

We treated the scalar integral (with  $\varphi = 1$ ) for which corresponds to (6) with  $L = 2$ ,  $N = 6$  internal lines and  $n = 3$  external lines.

Introducing of the Feynman parameters and integrating over the loop momenta leads to the 5-dimensional integral

$$I[1] = \int_{\mathcal{S}_5} \frac{1}{(D(\mathbf{x}) + i\varepsilon)^2} d\mathbf{x} \quad (7)$$

with

$$\begin{aligned} D = & C(x_1(p_1^2 - m_1^2) + x_2(p_2^2 - m_2^2) - x_3m_3^2 \\ & + x_4(p_1^2 - m_4^2) + x_5(p_2^2 - m_5^2) - x_6m_6^2) \\ & - C_1(x_5^2p_2^2 + x_4^2p_1^2 - x_4x_5(p_3^2 - p_1^2 - p_2^2)) \\ & - C_2(x_2^2p_2^2 + x_1^2p_1^2 - x_1x_2(p_3^2 - p_1^2 - p_2^2)) \\ & - 2x_2x_3x_5p_2^2 - 2x_1x_3x_4p_1^2 + x_3(x_2x_4 + x_1x_5)(p_3^2 - p_1^2 - p_2^2) \end{aligned}$$

and where

$$x_6 = 1 - x_1 - x_2 - x_3 - x_4 - x_5$$

$$C_1 = x_1 + x_2 + x_3$$

$$C_2 = 1 - x_1 - x_2$$

$$C = x_3(1 - x_1 - x_2 - x_3) + (x_1 + x_2)(1 - x_1 - x_2)$$

In view of the fact that  $D$  in (7) vanishes for  $x_1 = x_2 = x_3 = 0$  we perform a *regularizing* transformation,

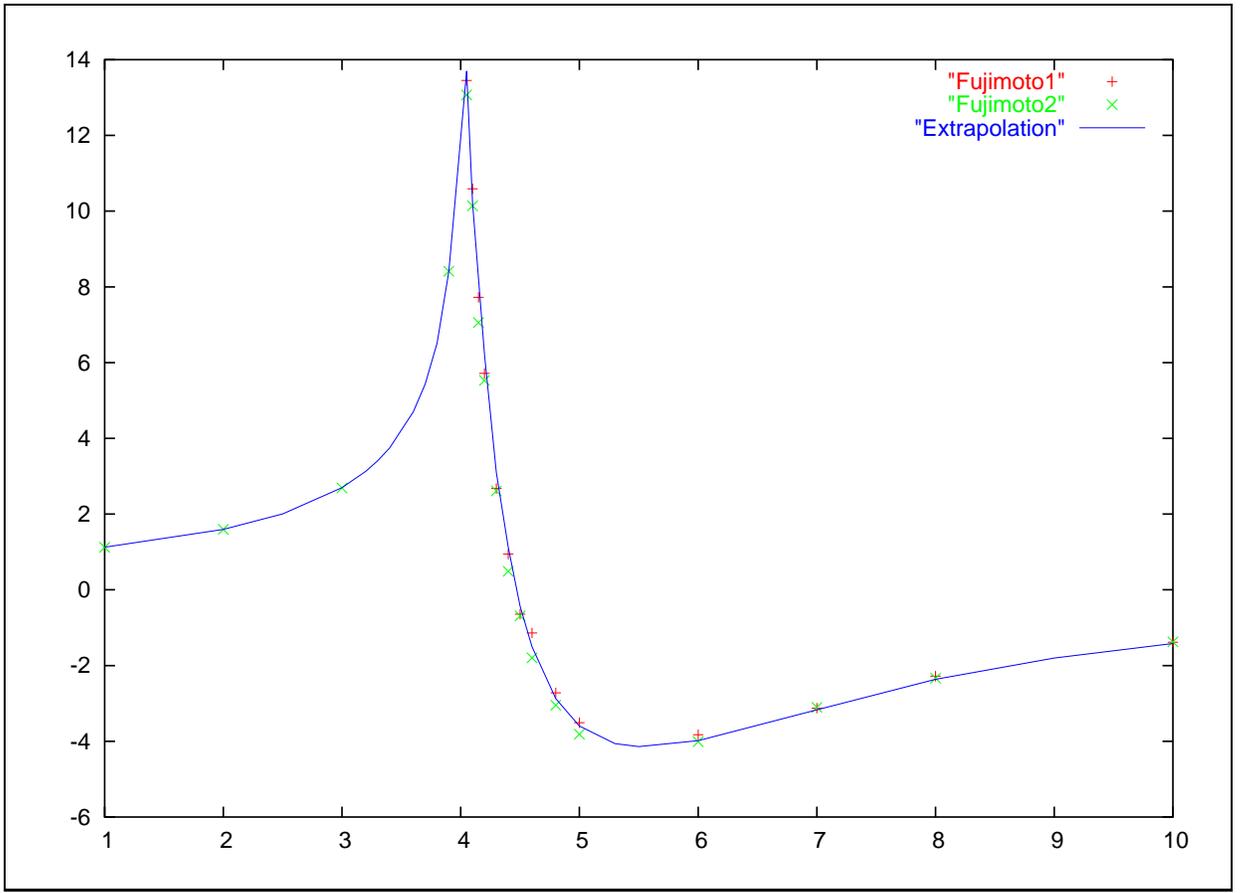


Fig. 9. Approximations to  $I$  (real part) as a function of  $\frac{k^2}{m_{top}^2}$ ,  $m_{top} = 150$  GeV and  $M_Z = 91.17$  GeV

$$\begin{aligned}
 x_1 &= x_1(r_0, \tilde{x}_1) = r_0 \tilde{x}_1 \\
 x_2 &= x_2(r_0, \tilde{x}_1) = r_0(1 - \tilde{x}_1) \\
 x_3 &= x_3(r_0, \tilde{x}_1, \tilde{x}_2) = r_0(1 - \tilde{x}_1 - \tilde{x}_2)
 \end{aligned} \tag{8}$$

Denoting the integrand of (7) by  $f(\mathbf{x})$ , the transformation (8) results in

$$I = \int_0^1 dr_0 r_0^2 \int_0^1 d\tilde{x}_1 \int_{2-\frac{1}{r_0}-\tilde{x}_1}^{1-\tilde{x}_1} \int_0^{1-x_1-x_2-x_3} dx_4 \int_0^{1-x_1-x_2-x_3-x_4} dx_5 f.$$

Using  $\tilde{x}_2 = 2 - \frac{1}{r_0} - \tilde{x}_1 + (\frac{1}{r_0} - 1)t_3$  to map the outer 3-dimensional integral to the 3d unit hypercube we obtain

$$I = \int_0^1 dr_0 r_0(1 - r_0) \int_0^1 d\tilde{x}_1 \int_0^1 dt_3 \int_0^{1-x_1-x_2-x_3} dx_4 \int_0^{1-x_1-x_2-x_3-x_4} dx_5 f. \quad (9)$$

### Notes:

- To approximate the integral (9) we can transform it to the 5-dim. unit hypercube and use DCUHRE.
- Alternatively, we used DCUHRE for the outer 3-dimensional integration and DQAGE for the inner two dimensions, in order to generate the results displayed on the curve labeled “Extrapolation” in Figure 9, which represents the integral approximations to (9) as a function of  $\frac{k^2}{m_{top}^2}$  pertaining to the vertex correction of the top quark with two  $Z^0$  boson exchanges. Also shown (as points) are corresponding results from Fujimoto et al.
- We used the  $\varepsilon$ -algorithm for the extrapolation, using a sequence of  $\varepsilon_j = 2.1^{35-j}$  for  $j = 0$  to about 14 on the right of and away from the  $\frac{k^2}{m_{top}^2}$  threshold, and  $\varepsilon_j = 2.1^{25-j}$  near the threshold.

## 6 Remarks regarding large mass ratios

It is clear that the success (or otherwise) of an extrapolation procedure will depend on whether it is able to intrinsically model the behavior of the entry sequence. For the three-point vertex function:

- The ratio  $\frac{M}{m}$  is a determining factor. The problem is more pronounced for higher values of the energy  $\sqrt{s}$ . In this case, while the  $\varepsilon$ -algorithm appears to become unstable when  $\varepsilon$  is relatively large, the polynomial model delivers an extrapolated sequence only a little “ahead” of the entry sequence but with a behavior similar to the latter.

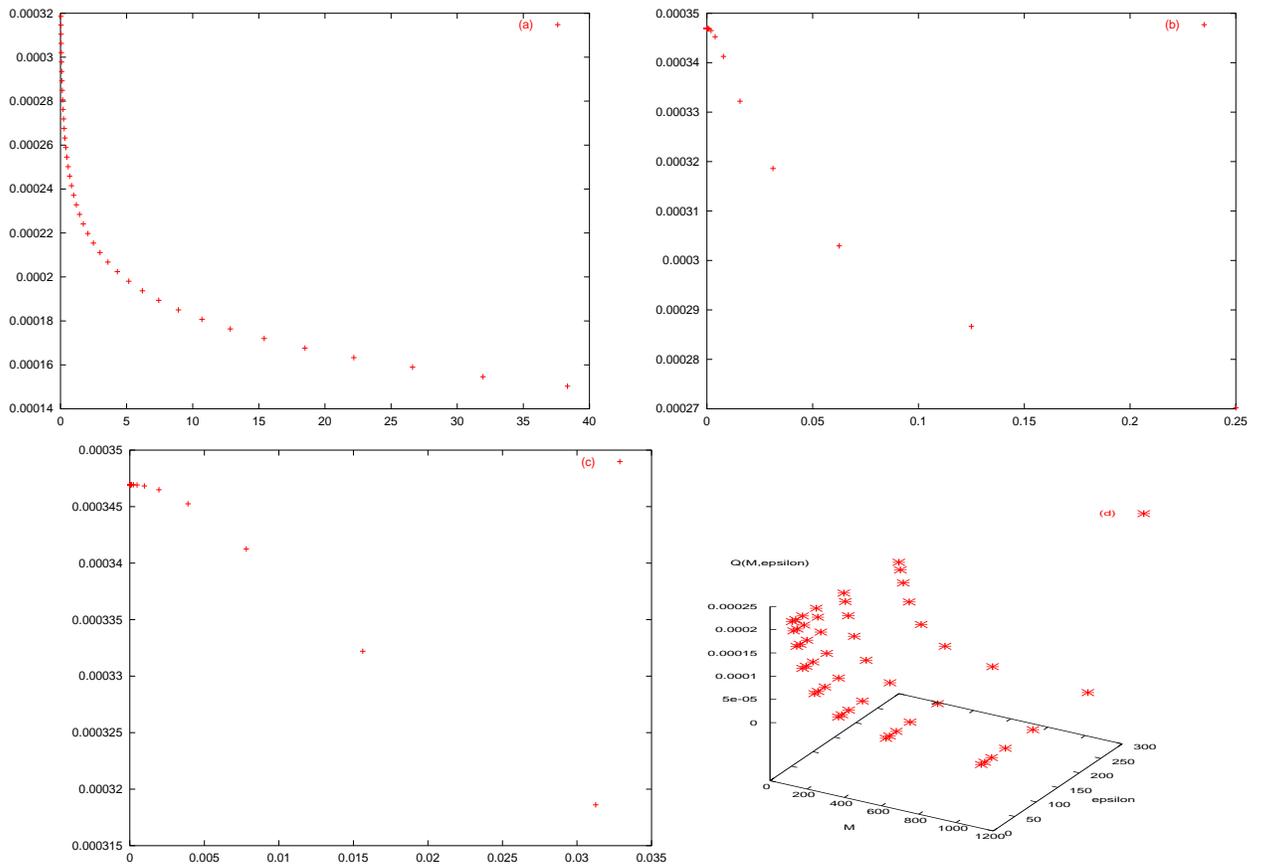


Fig. 10. (a-c)  $Q(\epsilon)$  vs.  $\epsilon$  for  $s = 9000 \text{ GeV}^2$ ,  $m = 150 \text{ GeV}$ ,  $M = 0.1 \text{ GeV}$  and various ranges of  $\epsilon$ ; (d)  $Q(M, \epsilon)$  for  $s = 9000 \text{ GeV}^2$ ,  $m = 40 \text{ GeV}$

- Figure 10 shows  $Q(\epsilon)$  as a function of  $\epsilon$  for  $s = 9000 \text{ GeV}^2$ ,  $m = 150 \text{ GeV}$ ,  $M = 0.1 \text{ GeV}$ . The graph follows a logarithmic behavior over a large range (see (a)). However note how in (b) and (c) the curve gradually becomes as that of Figure 3 over ranges of smaller  $\epsilon$ . The behavior transient for very small  $\epsilon$  needs to be accurately modeled in order to treat small values of  $M$ .
- For  $s = 9000 \text{ GeV}^2$ ,  $m = 150 \text{ GeV}$ , (d) depicts the divergence of the integral as  $M$  decreases.

eps	Q(eps)	Extrapolated	#evals.	eps	Q(eps)	Extrapolated	#evals
0.381680e-03	-0.185652e-03		2.32e06				
0.293600e-03	-0.181426e-03	-0.167341e-03	2.37e06	0.46790e-04	-0.989755e-04	-0.446396e-04	2.88e06
0.225847e-03	-0.174782e-03	-0.131318e-03	2.40e06	0.35992e-04	-0.878292e-04	-0.447056e-04	2.94e06
0.173728e-03	-0.165541e-03	-0.899977e-04	2.49e06	0.27686e-04	-0.785577e-04	-0.443406e-04	3.01e06
0.133637e-03	-0.153854e-03	-0.549082e-04	2.57e06	0.21297e-04	-0.710526e-04	-0.442841e-04	3.10e06
0.102798e-03	-0.140320e-03	-0.363915e-04	2.63e06	0.16382e-04	-0.650854e-04	-0.442955e-04	3.15e06
0.790751e-04	-0.125943e-03	-0.353284e-04	2.74e06	0.12602e-04	-0.603956e-04	-0.442976e-04	3.21e06
0.608270e-04	-0.111854e-03	-0.413304e-04	2.77e06	0.96938e-05	-0.567369e-04	-0.442978e-04	3.36e06

Fig. 11. One-loop vertex illustration of large  $m/M$  ratio:  $\sqrt{s} = 310$  GeV,  $m = 150$  GeV,  $M = 0.01$  GeV. This should be compared with the analytic value  $-0.442975219528810759e - 04$

- With DQAGE we are able to calculate the integrals for the smaller values of  $\varepsilon$  needed to handle fairly considerable mass ratios and energy, for example,  $m = 150$  GeV,  $M = 0.01$  GeV and  $\sqrt{s} = 310$  GeV. Convergence results for this case are given in the table of Figure 11, based on the expansion (4) in integer powers of  $\varepsilon$  and using a geometric sequence with  $b = 1.3$  and  $\varepsilon_0 = 1.3^{-30}$ . For this case, the method used yields results for mass ratios through about five orders of magnitude.

## 7 Conclusions and future work

- We presented a class of methods for the evaluation of loop integrals based on extrapolation. The extrapolation is based on generating a sequence of approximations which converge to the loop integral value as a parameter  $\varepsilon$  introduced in the integrand tends to zero.
- Using the  $\varepsilon$ -algorithm for extrapolation this delivers an automatic method.
- Further work is needed to establish properties of the transient behavior as a function of  $\varepsilon$  as it relates to the convergence properties of the extrapolation process.
- As an alternative method, the generalized Richardson extrapolation process by Sidi et al. should also be investigated.
- Tailoring of multivariate integration codes is warranted to make these calculations more efficient. Furthermore, the large granularity of the integrands as well as the number of integrals involved makes the application an excellent candidate for parallel approaches.

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