Precision tests of Israel-Stewart hydrodynamics

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with
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Introduction
- Boltzmann equation
- Israel-Stewart equations from the kinetic theory
- Israel-Stewart equations in 1+1 dimensions

Riemann problem: Perfect fluid solution

Solutions with shock wave
- Riemann problem
- smoothed Riemann problem
- smoothed Riemann problem: shock front
- smoothed Riemann problem: Heat flow

Summary

References

- Viscous shock waves:
  I. Bourar, E. Molnár, H. Niemi, Z. Xu, A. El, O. Fochler, C. Greiner and D. H. Rischke
  Phys. Rev. Lett. **103**, 032301 (2009) [arXiv:0902.1927 [hep-ph]].
- Testing numerical methods for IS hydrodynamics:
  E. Molnar, H. Niemi and D. H. Rischke, to be published in Eur. Phys. J. C, arXiv:0907.2583 [nucl-th]
- Kinetic theory (BAMPS):
  Z. Xu and C. Greiner, Phys. Rev. C **71** (2005) 064901
  Z. Xu and C. Greiner, Phys. Rev. Lett. **100** (2008) 172301
- IS hydrodynamics:
  W. Israel and J. M. Stewart, Annals Phys. **118**, 341 (1979)

Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = C(x,p)$$

1st and 2nd moments of the Boltzmann equation $\longrightarrow$ conservation laws:

$$\partial_\mu N^\mu = \int \frac{d^3p}{E}\, p^\mu \partial_\mu f = \int \frac{d^3p}{E}\, C(x,p) = 0$$

$$\partial_\mu T^{\mu\nu} = \int \frac{d^3p}{E}\, p^\nu p^\mu \partial_\mu f = \int \frac{d^3p}{E}\, p^\nu C(x,p) = 0$$

3rd moment:

$$\partial_\lambda S^{\mu\nu\lambda} = \int \frac{d^3p}{E}\, p^\mu p^\nu p^\lambda \partial_\lambda f = \int \frac{d^3p}{E}\, p^\mu p^\nu C(x,p)$$

Approximate $f(x,p) = f_{eq}(x,p)\,(1 + \epsilon(x) + \epsilon_\mu p^\mu + \epsilon_{\mu\nu} p^\mu p^\nu)$
$\longrightarrow$ Equations for shear, bulk and heat (Israel & Stewart)

Israel-Stewart equations from the kinetic theory ($\Pi = 0$)

$$
\begin{aligned}
\Delta^{\mu}_{\lambda} Dq^{\lambda} &= -\frac{1}{\tau_q}\left[q^{\mu} + \kappa_q \frac{T^2 n}{e+p}\nabla^{\mu}\left(\frac{\mu}{T}\right)\right] - \frac{1}{2}q^{\mu}\left(\nabla_{\lambda}u^{\lambda} + D\ln\frac{\beta_1}{T}\right) \\
&\quad -\omega^{\mu\lambda}q_{\lambda} + A_1\sigma^{\mu\lambda}q_{\lambda} - \frac{a_1}{\beta_1}\pi^{\lambda\mu}Du_{\lambda} + \frac{\alpha_1}{\beta_1}(\partial_{\lambda}\pi^{\lambda\mu} + u^{\mu}\pi^{\lambda\nu}\partial_{\lambda}u_{\nu}) \\
D\pi^{\langle\mu\nu\rangle} &= -\frac{1}{\tau_{\pi}}\left(\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu}u^{\nu\rangle}\right) - \frac{1}{2}\pi^{\mu\nu}\left(\nabla_{\lambda}u^{\lambda} + D\ln\frac{\beta_2}{T}\right) \\
&\quad +2\pi_{\lambda}{}^{\langle\mu}\omega^{\nu\rangle\lambda} - 2\pi_{\lambda}{}^{\langle\mu}\sigma^{\nu\rangle\lambda} - \frac{\alpha_1}{\beta_2}\nabla^{\langle\mu}q^{\nu\rangle} + \frac{a_1'}{\beta_2}q^{\langle\mu}Du^{\nu\rangle} \\
&\quad +A_2 q^{\langle\mu}\nabla^{\nu\rangle}\frac{\mu}{T}
\end{aligned}
$$

- All terms from the framework of Israel and Stewart included
- Equations without red terms refered as **'reduced' IS equation**
- Not yet complete IS (would need $O(\varepsilon^2)$ in $f(x,p)$)

IS equations in 1+1 dimensions for massless Boltzmann gas

Only 2 independent dissipative quantities q^z and $\pi = -2\pi^{xx}$

$$
\begin{aligned}
Dq^z &= \frac{1}{\tau_q}\left(q^z_{NS} - q^z\right) - \frac{4}{3}q^z\theta - \frac{14}{15}q^z\theta \\
&\quad +(q^z v_z)\gamma^3\left(\partial_t v_z + v_z\partial_z v_z\right) \\
&\quad -\frac{1}{5}\left[\gamma^2\left(v_z\partial_t\pi + \partial_z\pi\right) + \pi\left(\gamma v_z\theta + \gamma^2\partial_t v_z\right)\right] \\
D\pi &= \frac{1}{\tau_\pi}\left(\pi_{NS} - \pi\right) - \frac{4}{3}\pi\theta - \frac{2}{3}\pi\theta \\
&\quad -\frac{10}{9}(q^z\gamma^2)\left(\partial_t v_z + v_z\partial_z v_z\right) \\
&\quad -\frac{2}{9}\left(v_z\partial_t q^z + \partial_z q^z - \frac{q^z v_z}{\gamma}\theta\right) - \frac{1}{18}q^z\left(v_z\frac{\partial_t\lambda}{\lambda} + \frac{\partial_z\lambda}{\lambda}\right)
\end{aligned}
$$

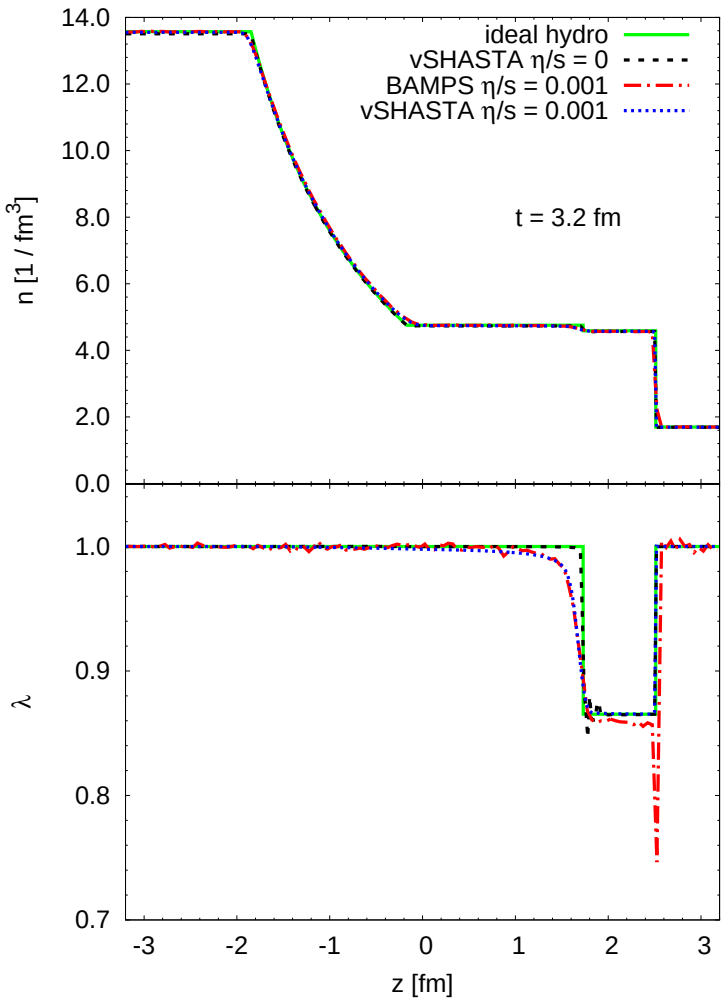
- Equation of State $e = 3p = 3nT$
- $p = \lambda\frac{g}{\pi^2}T^4$, here $g = 16$

Navier-Stokes values and transport coefficients

- Isotropic 2-2 scatterings
- cross-section tuned to give constant η/s

$$
\begin{aligned}
\pi_{NS} &= -\frac{4}{3}\eta\theta \\
q^z_{NS} &= \kappa_q \frac{T^2 n}{e+p}\gamma^2\left(v_z\frac{\partial_t\lambda}{\lambda}+\frac{\partial_z\lambda}{\lambda}\right) \\
\eta &= \frac{4}{5}\frac{T}{\sigma^t} \\
\kappa_q &= \frac{1.3}{\sigma^t} \\
\tau_\pi &= 2\beta_2\eta = 2\frac{3}{4p}\eta \\
\tau_q &= T\beta_1\kappa_q = \frac{5T}{4p}\kappa_q
\end{aligned}
$$

Riemann problem BAMPS and hydro vs analytic solution



- Riemann problem:

$$T(x, t = 0) = \begin{cases} 0.4 \text{ GeV} & z < 0 \\ 0.2 \text{ GeV} & z > 0 \end{cases}$$

- The Boltzmann equation is solved by BAMPS algorithm
- The IS equations are solved using SHASTA
- We can numerically reproduce the exact solution (SHASTA)
- Small discrepancy between BAMPS and exact solution results from small but non-zero residual viscosity in BAMPS calculation (limited by computational time)

Riemann problem: viscous solutions

- Applicability of hydrodynamics determined by Knudsen number

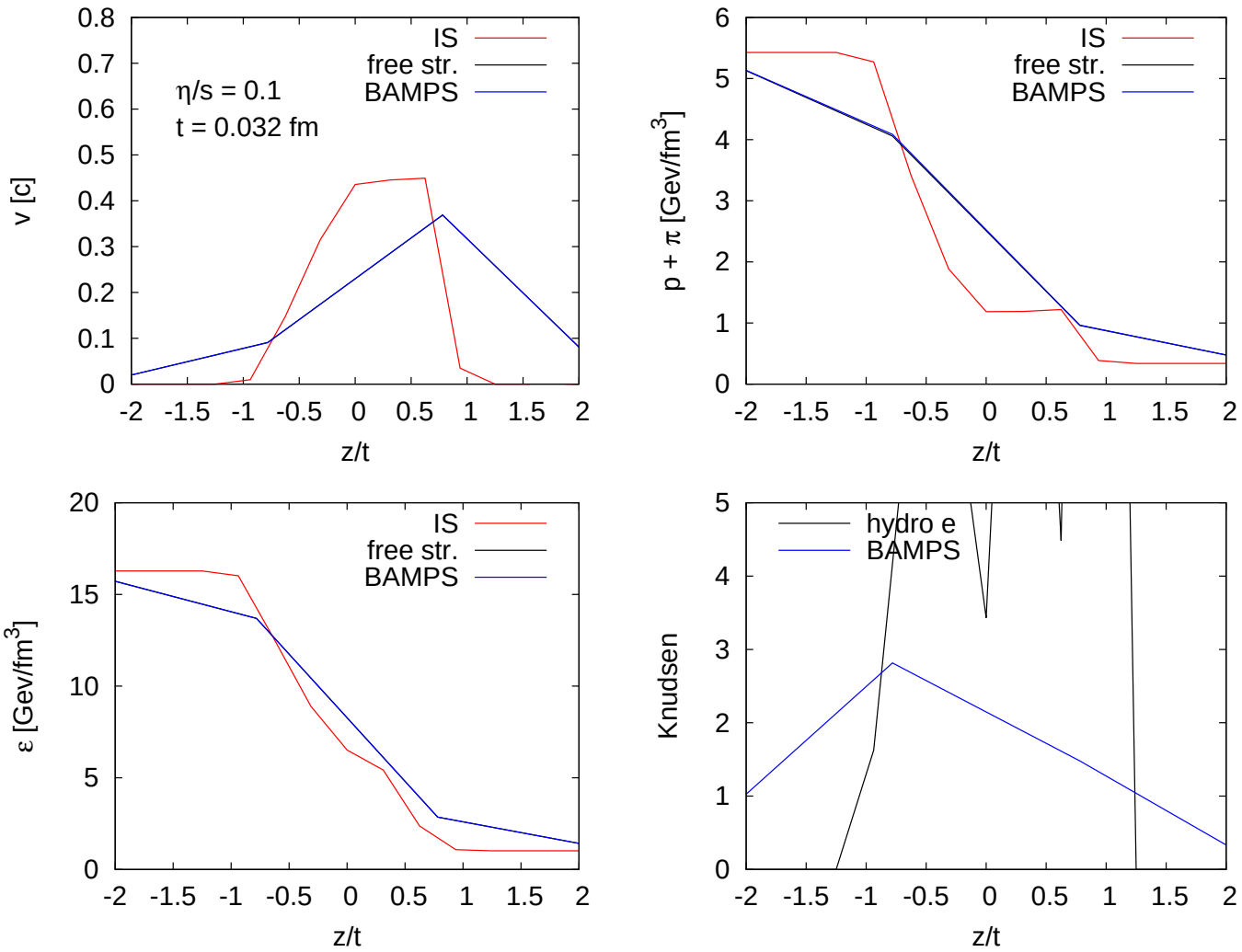
$$Kn = \frac{l_{micr}}{L_{macr}}$$

- For this particular problem shortest macroscopical lenght scale is given by energy density gradient

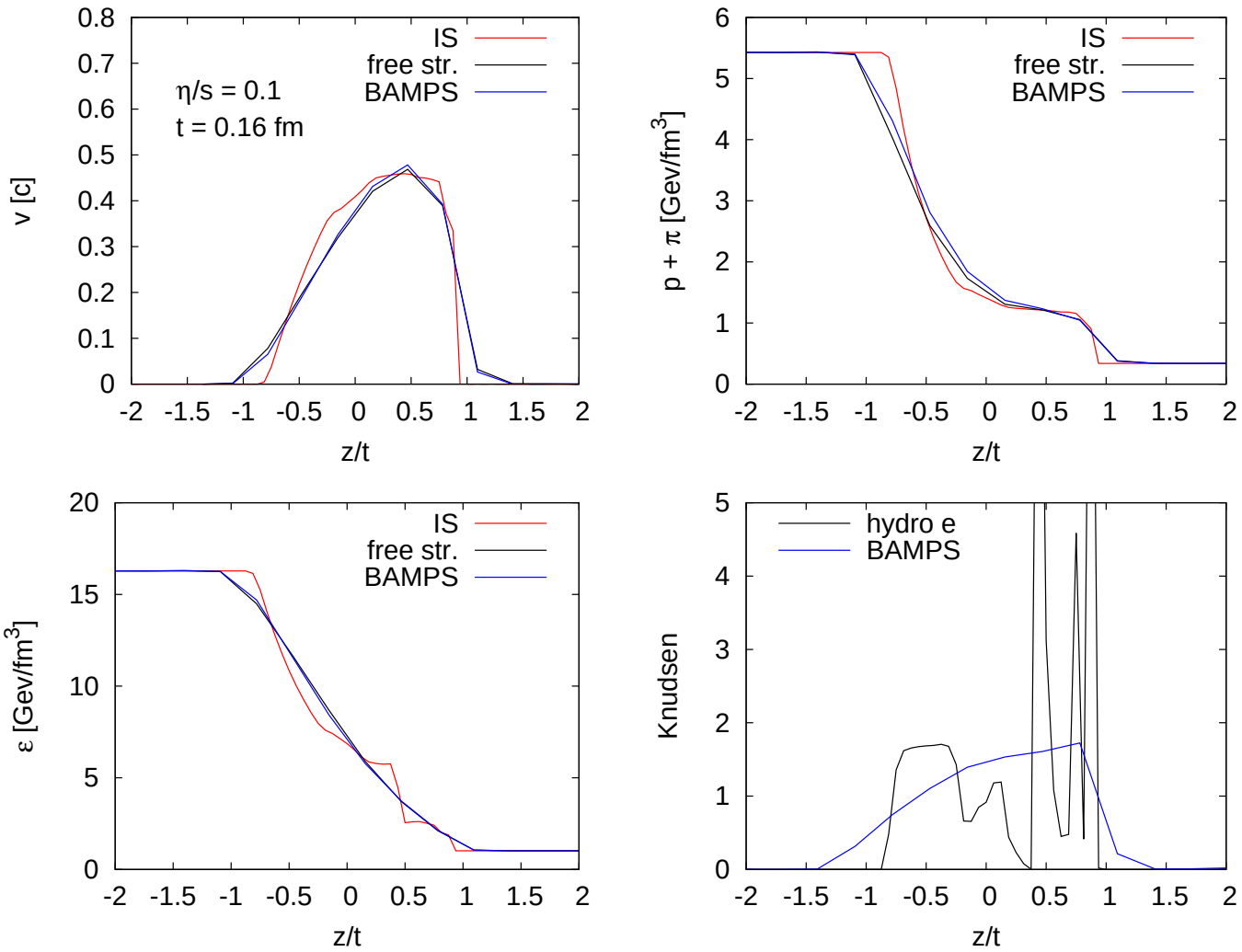
$$L_{macr}^{-1} = \frac{1}{e}\sqrt{\nabla^{\mu} e \nabla_{\mu} e} \tag{1}$$

- l_{micr} = mean free path
- When $Kn \ll 1 \longrightarrow$ hydrodynamics
- When $Kn \gg 1 \longrightarrow$ free streaming
- In the kinetic simulation we can see transition from free streaming to hydrodynamical behavior

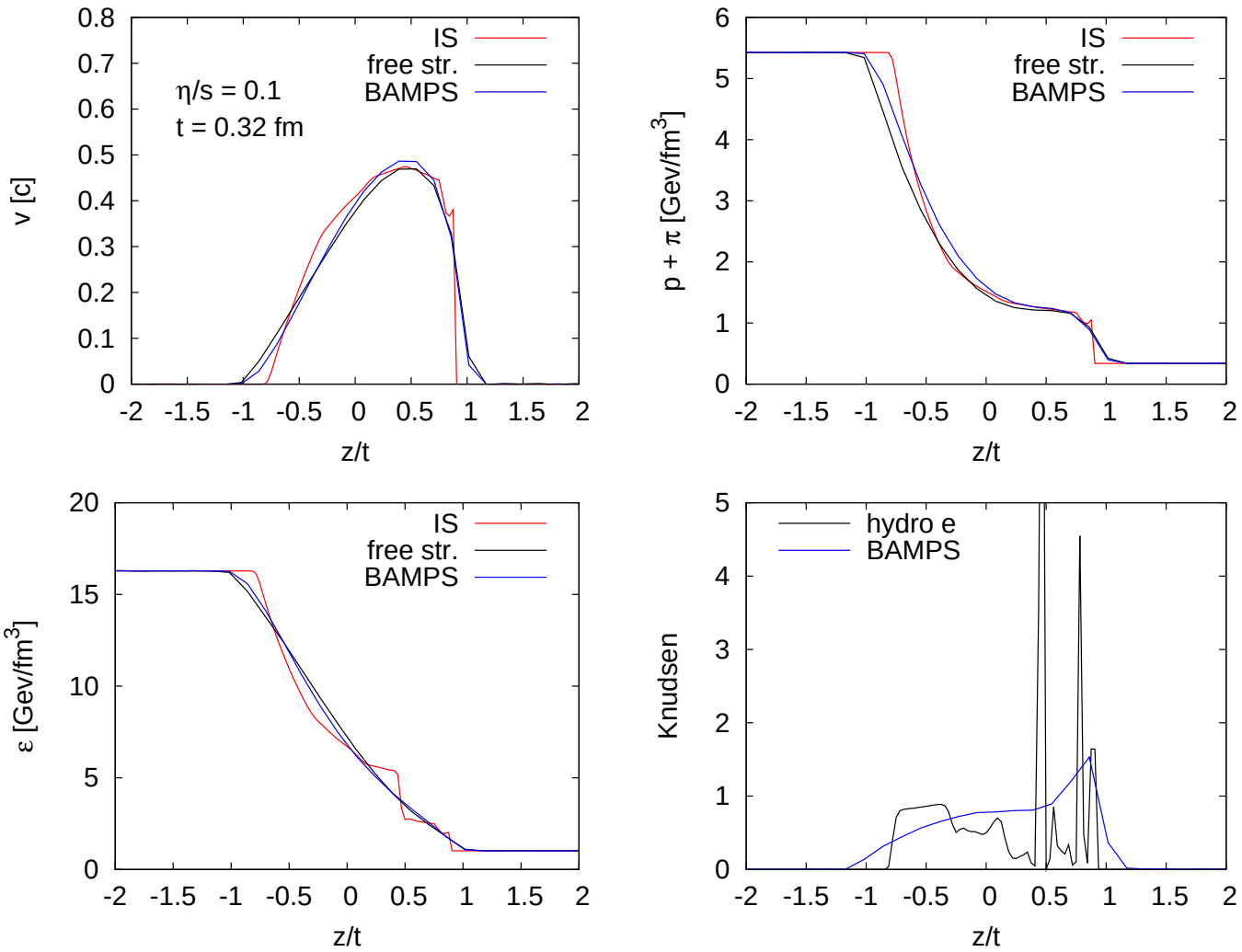
Riemann problem $\eta/s = 0.1$ $t = 0.032$ fm



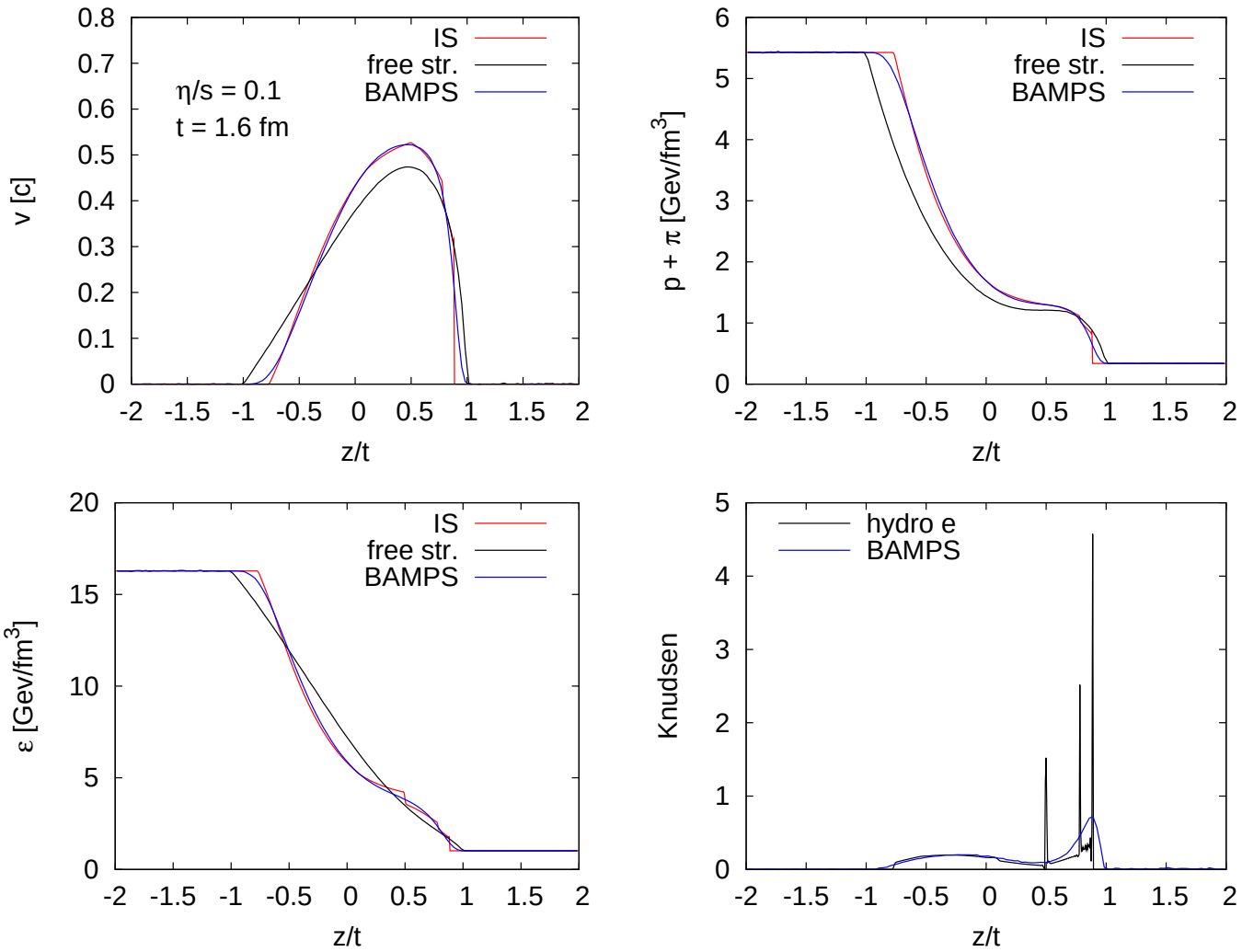
Riemann problem $\eta/s = 0.1$ $t = 0.16$ fm



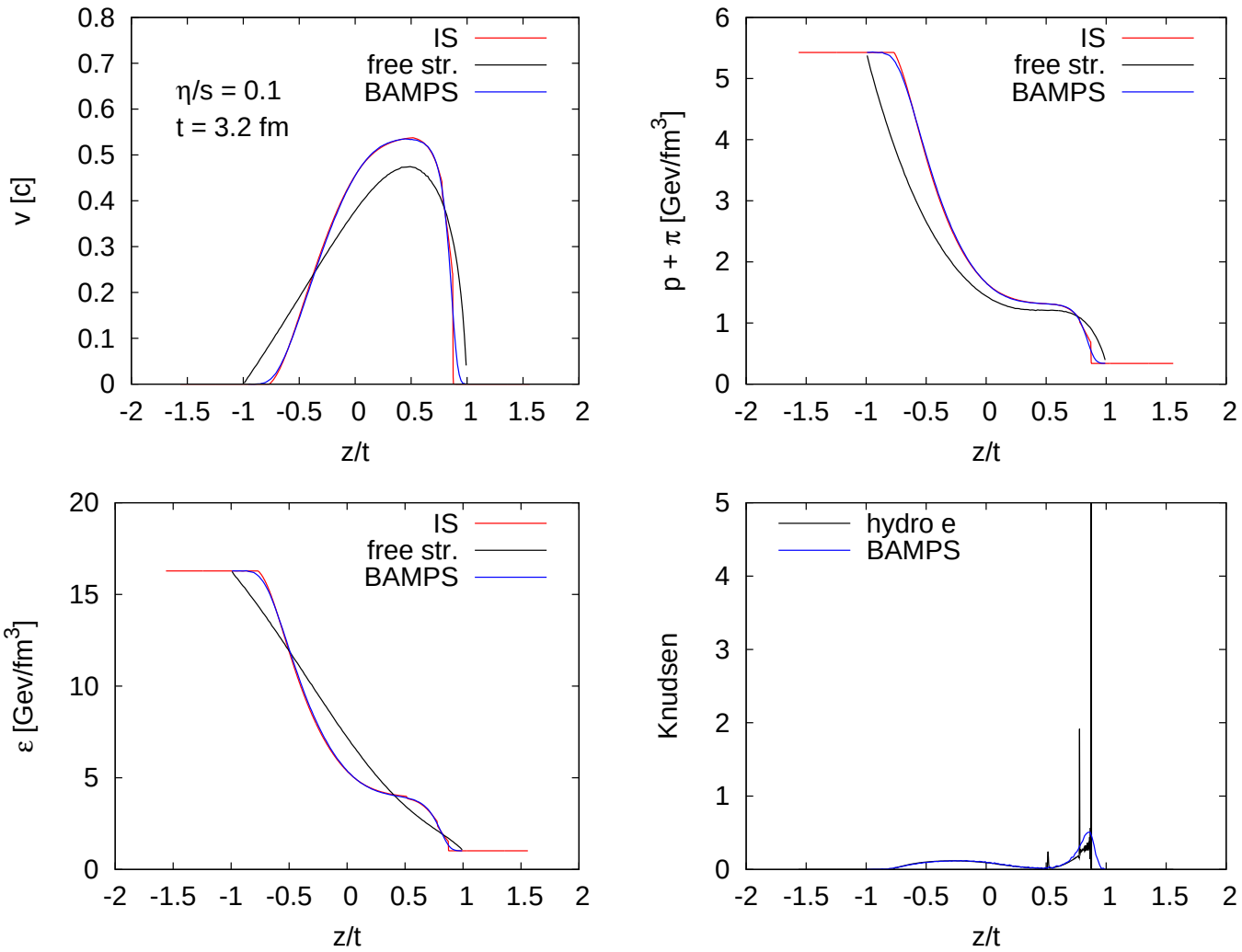
Riemann problem $\eta/s = 0.1$ $t = 0.32$ fm



Riemann problem $\eta/s = 0.1$ $t = 1.6$ fm



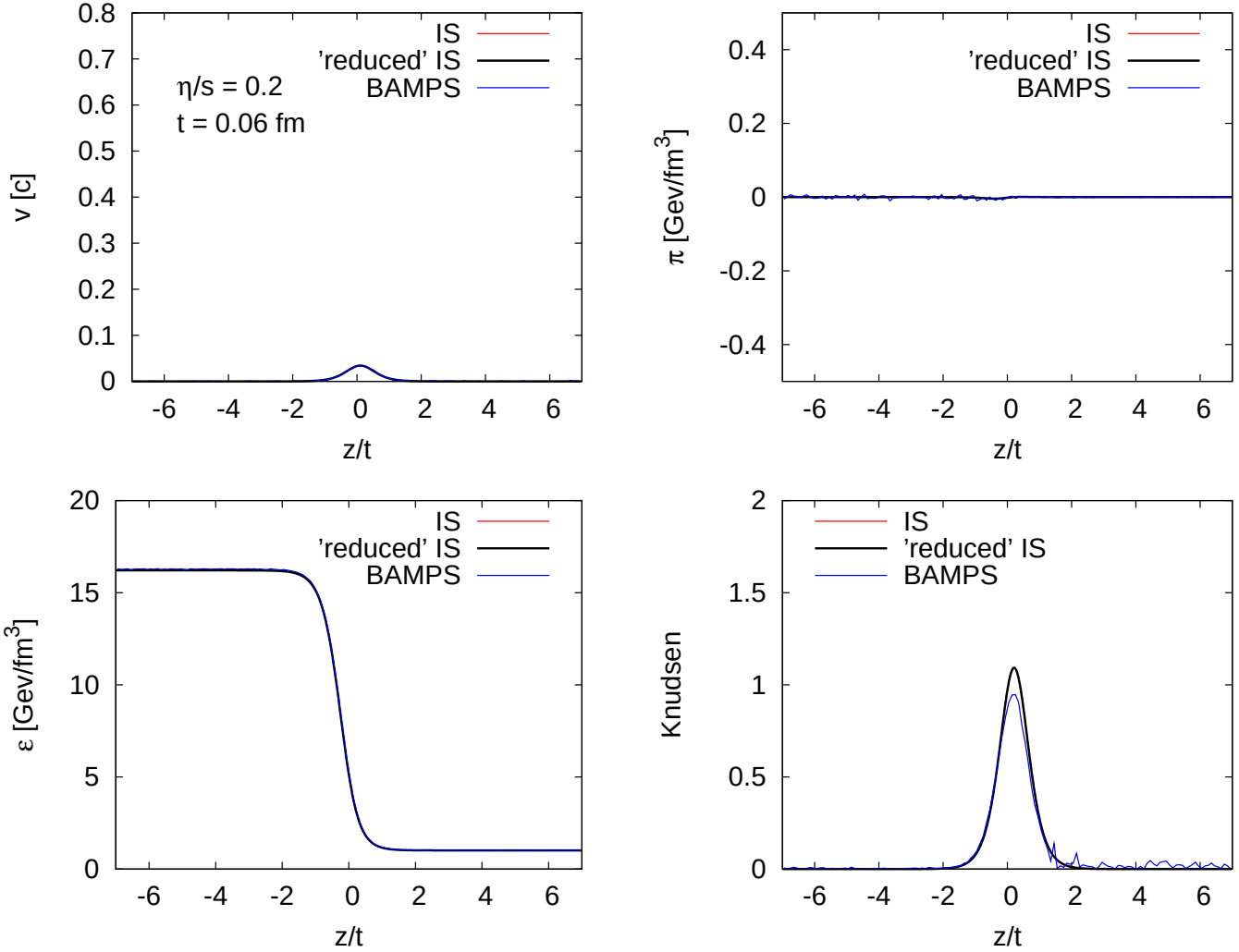
Riemann problem $\eta/s = 0.1$ $t = 3.2$ fm



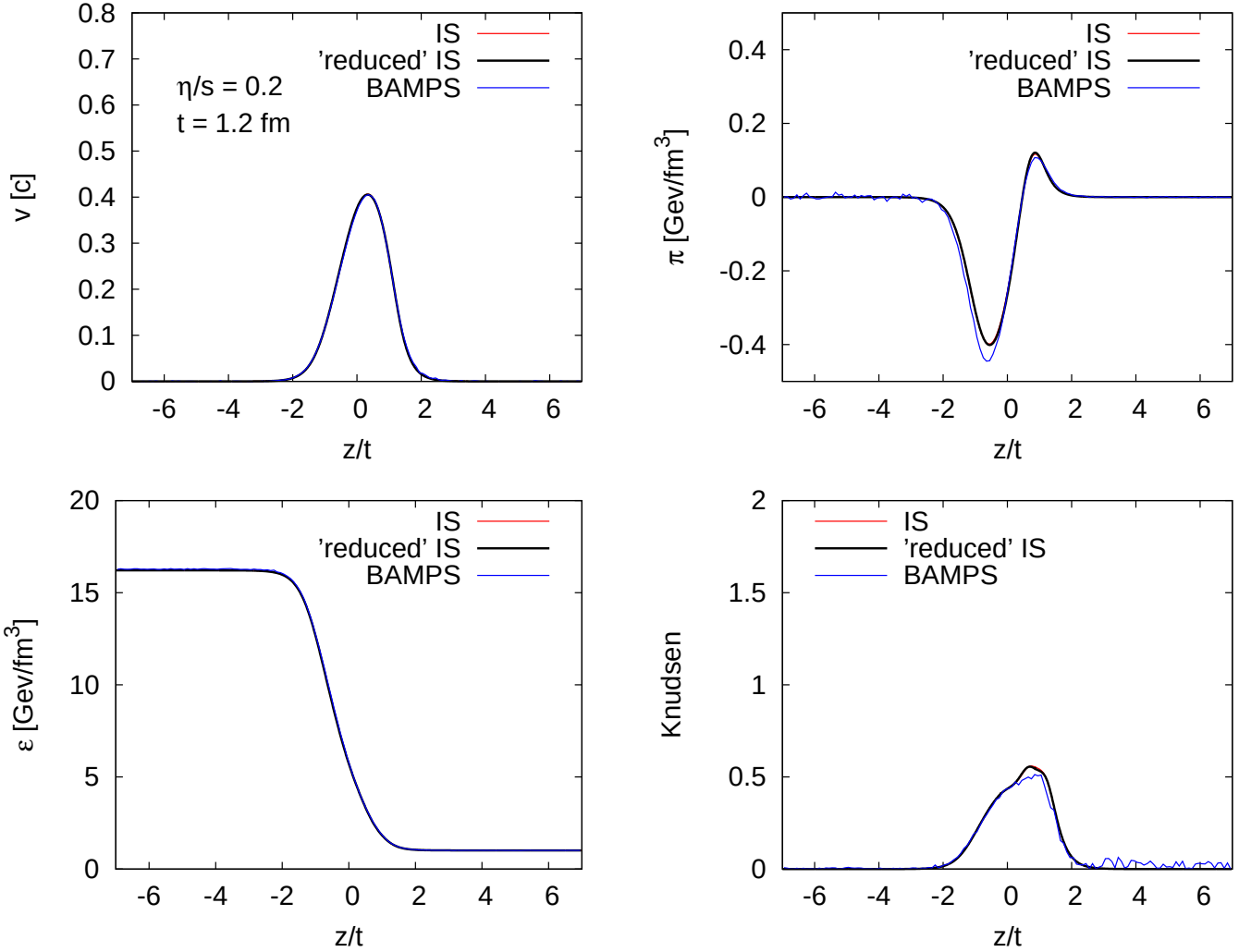
smoothed Riemann problem

- Early time evolution not desribed by IS hydrodynamics (additional structures in the profiles not seen in the kinetic calculation)
- Same kind of structures seen in the hydrodynamical calculation by G. S. Denicol, T. Kodama, T. Koide and Ph. Mota, Phys. Rev. C **78**, 034901 (2008) [arXiv:0805.1719 [hep-ph]].
  (They use complete different numerical scheme) $\longrightarrow$ structures are not numerical artifacts
- In the Riemann problem hard to see which structures come from the initial failure of hydrodynamics and which are results of real failure of hydrodynamics at late times
- $\longrightarrow$ smooth the initial condition by Woods-Saxon profile

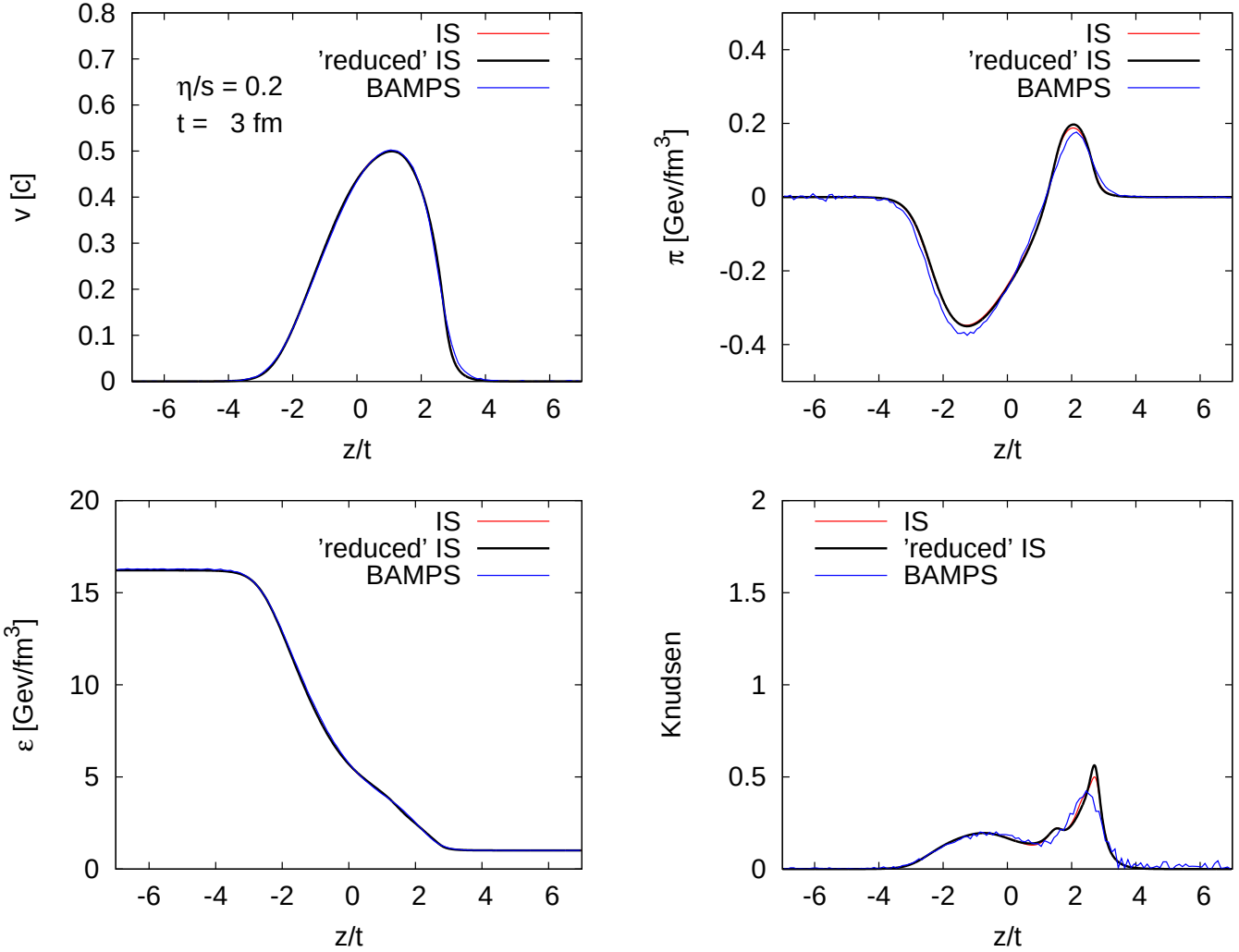
smoothed Riemann problem $\eta/s = 0.2$ $t = 0.06$ fm



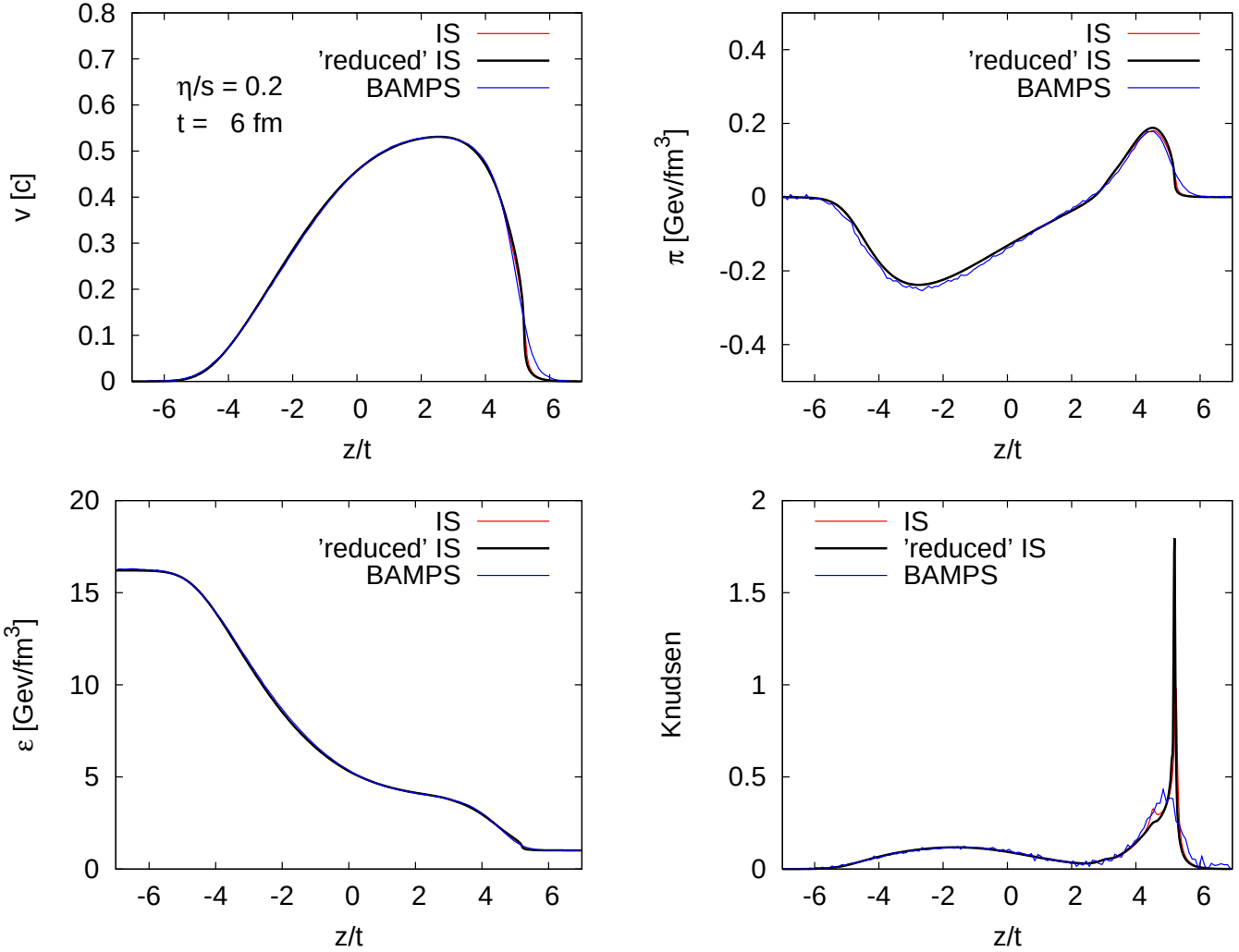
smoothed Riemann problem $\eta/s = 0.2$ $t = 1.2$ fm



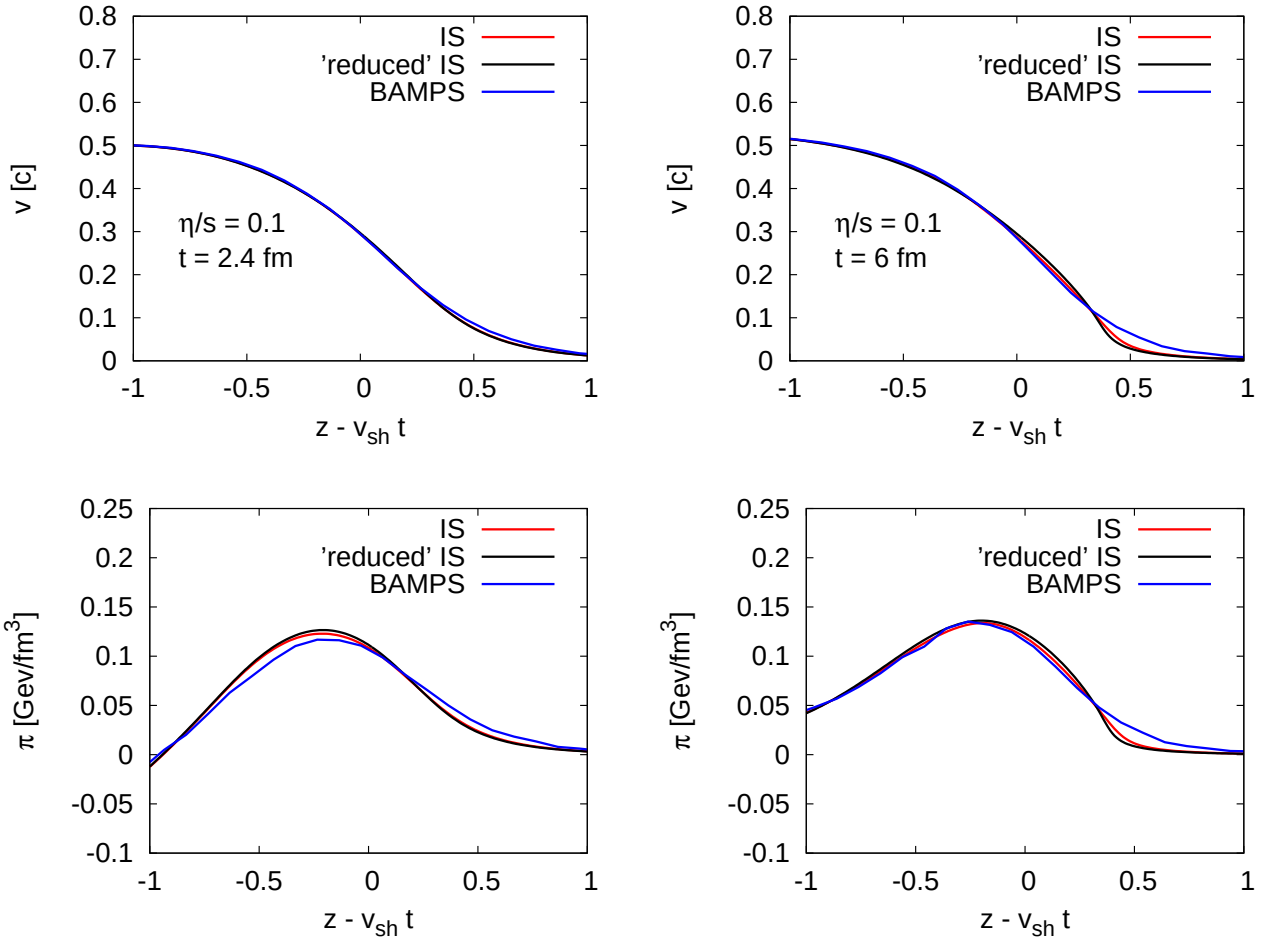
smoothed Riemann problem $\eta/s = 0.2$ $t = 3.0$ fm



smoothed Riemann problem $\eta/s = 0.2$ $t = 6.0$ fm



smoothed Riemann problem: Shockfront

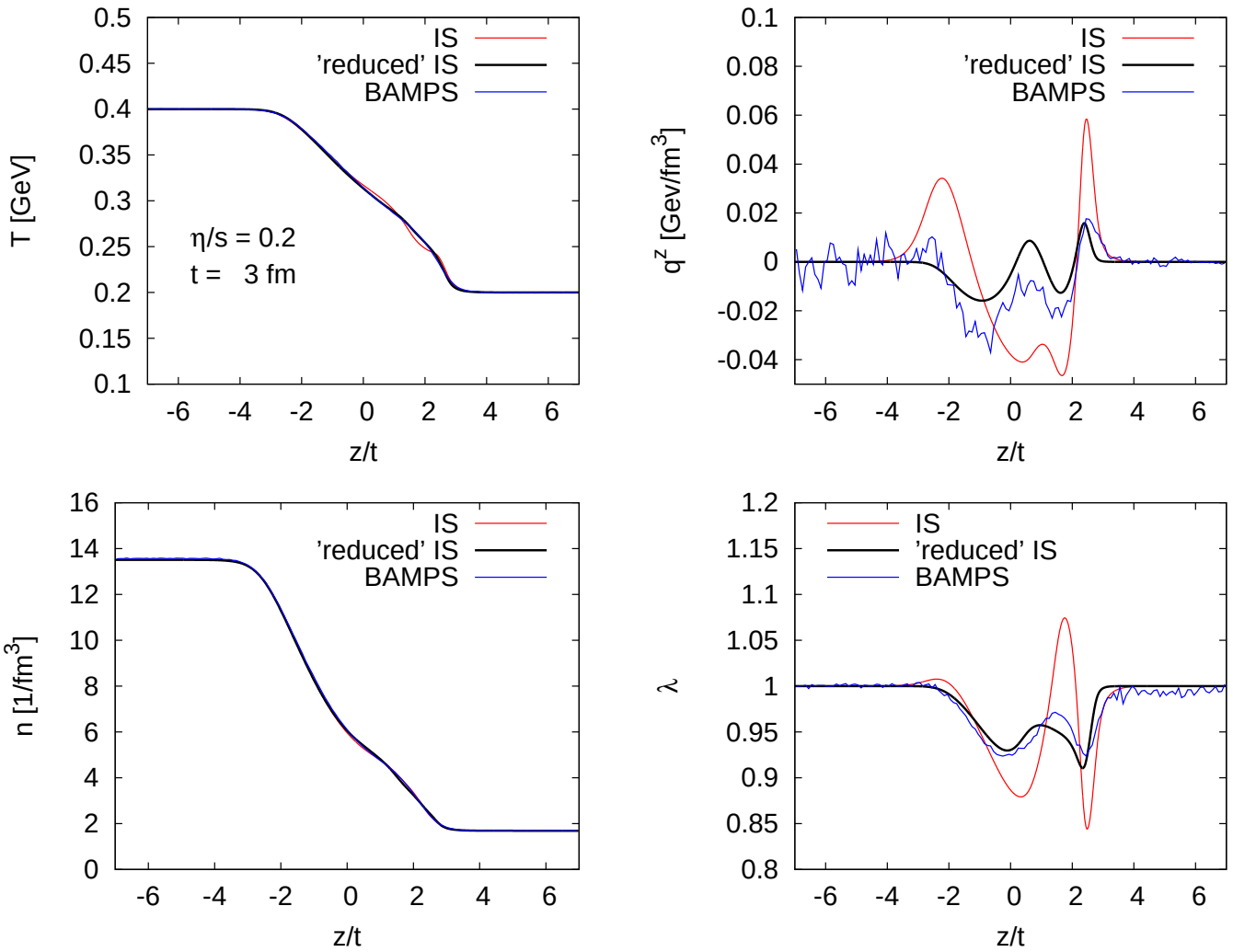


- Early times: $\sim$ perfect agreement
- The difference in the shock front dynamically created during the evolution

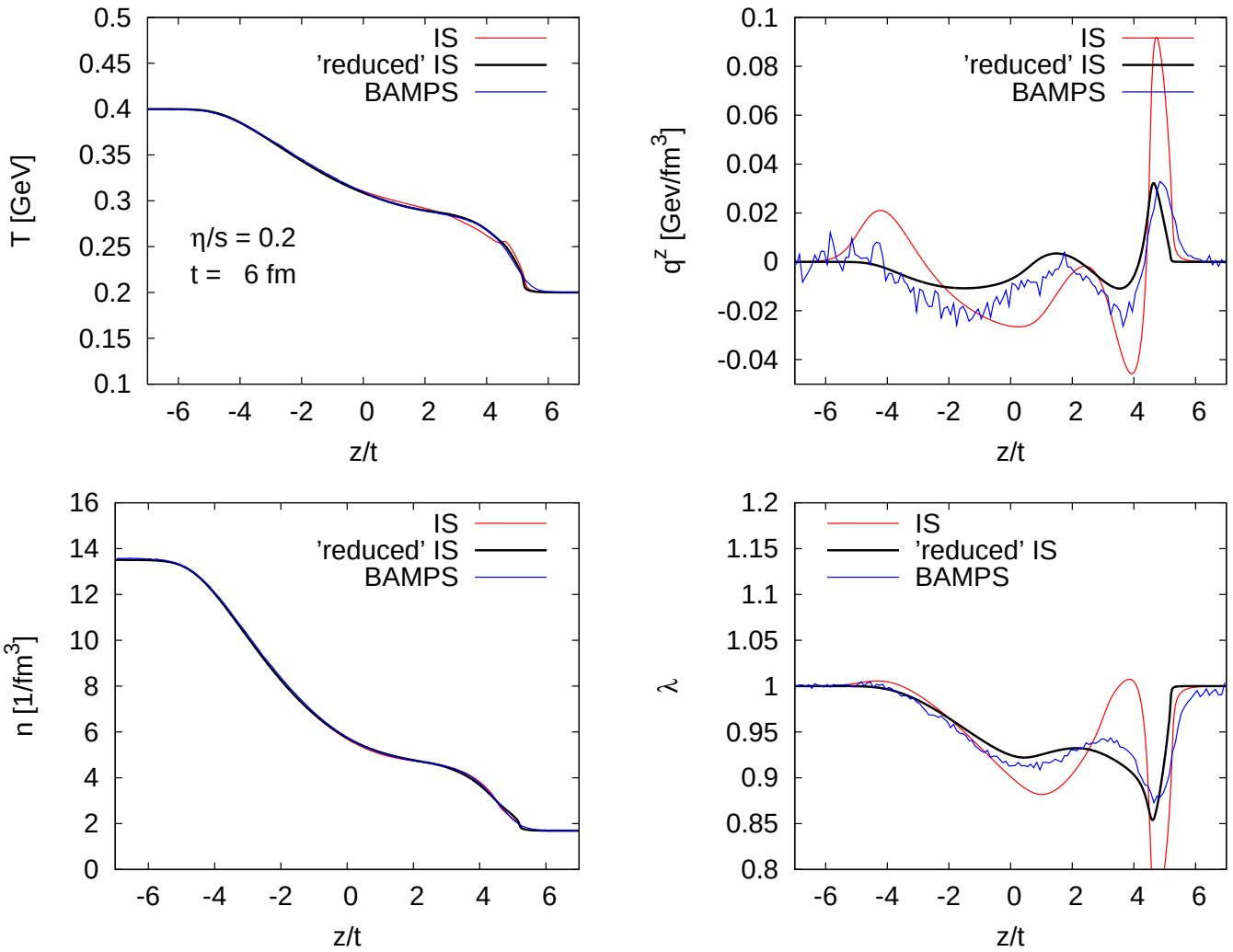
Heat flow

- Shear pressure: good agreement between kinetic theory and IS hydrodynamics
- Heat flow and fugacity?
- In this problem $\pi \gg q \longrightarrow$ coupling of q to π dominates evolution of heat flow $\longrightarrow$ Big difference between IS and 'reduced' IS

smoothed Riemann problem: Heat flow $\eta/s = 0.2$ $t = 3.0$ fm



smoothed Riemann problem: Heat flow $\eta/s = 0.2$ $t = 6.0$ fm



Summary

- When Knudsen number is large $\gg 1$, hydrodynamical describtion breaks
- When Knudsen number is small $\ll 1$, overall agreement between kinetic theory and IS hydrodynamics is very good.
- Exception: Shock front and heat flow.
- TODO: Complete IS hydrodynamics (by D. Rishcke)