# Non-equilibrium steady states in (near) critical quantum systems

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based on works (many in preparation) with
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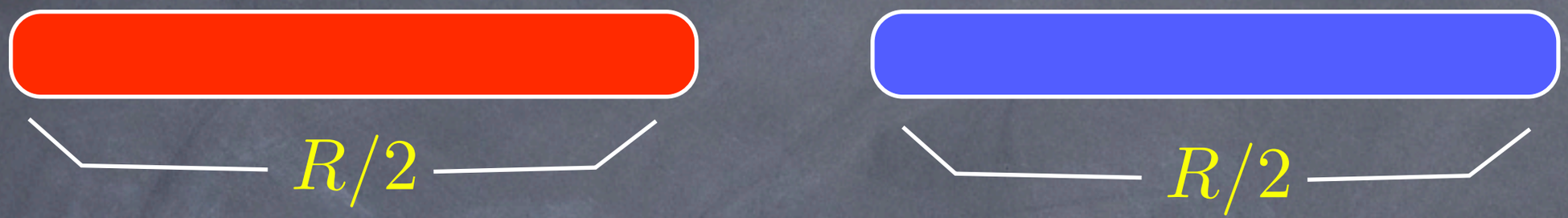
- J. Phys. A: Math. Theor. 45 (2012) 362001, arXiv:1202.0239
- arXiv:1212.1077
- arXiv:1302.3125

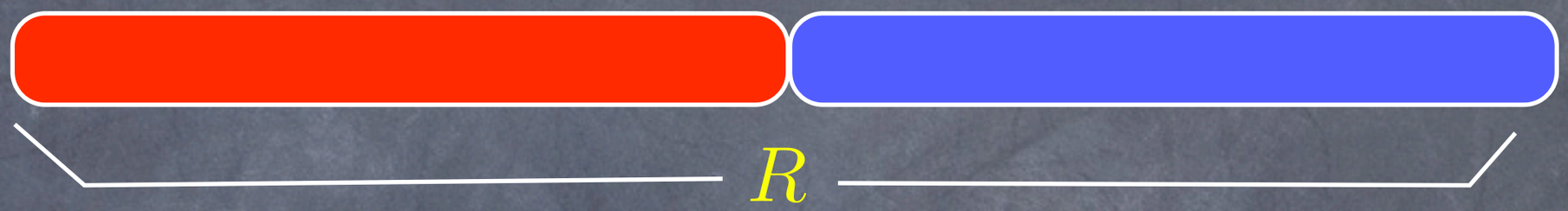
Brookhaven, 25 April 2013

# Physical situation

$t < t_0$ :




$t = t_0$ :




$t > t_0$ :

$$R \gg (t - t_0) v_F \gg \text{observation lengths}$$

# Physical situation

Initially: $\rho_0 = e^{-\beta_l(H^l+\mu_l N^l)-\beta_r(H^r+\mu_r N^r)}$

Evolution: $H = H^l + H^r + H_{\text{contact}}$

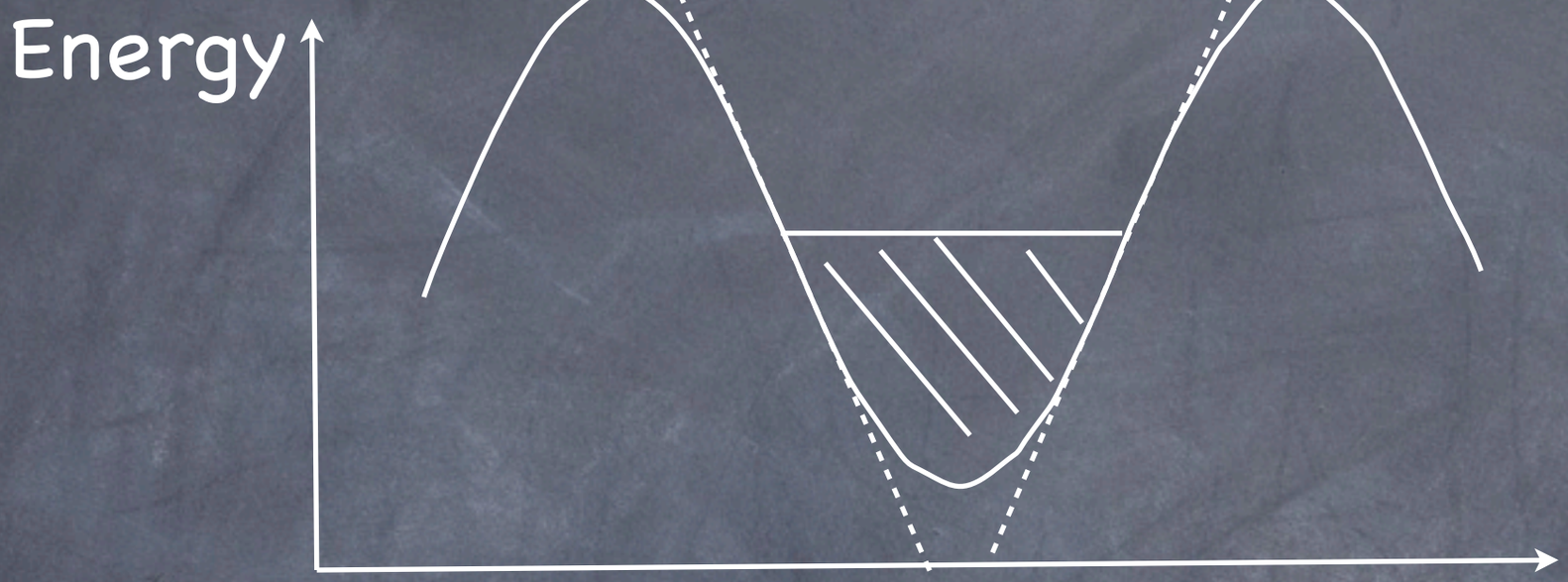
Steady state limit:

$$\langle \cdots \rangle_{\text{ness}} = \lim_{t_0\to-\infty} \lim_{R\to\infty} \frac{\text{Tr}\left(e^{iHt_0}\rho_0 e^{-iHt_0}\cdots\right)}{\text{Tr}\left(\rho_0\right)}$$

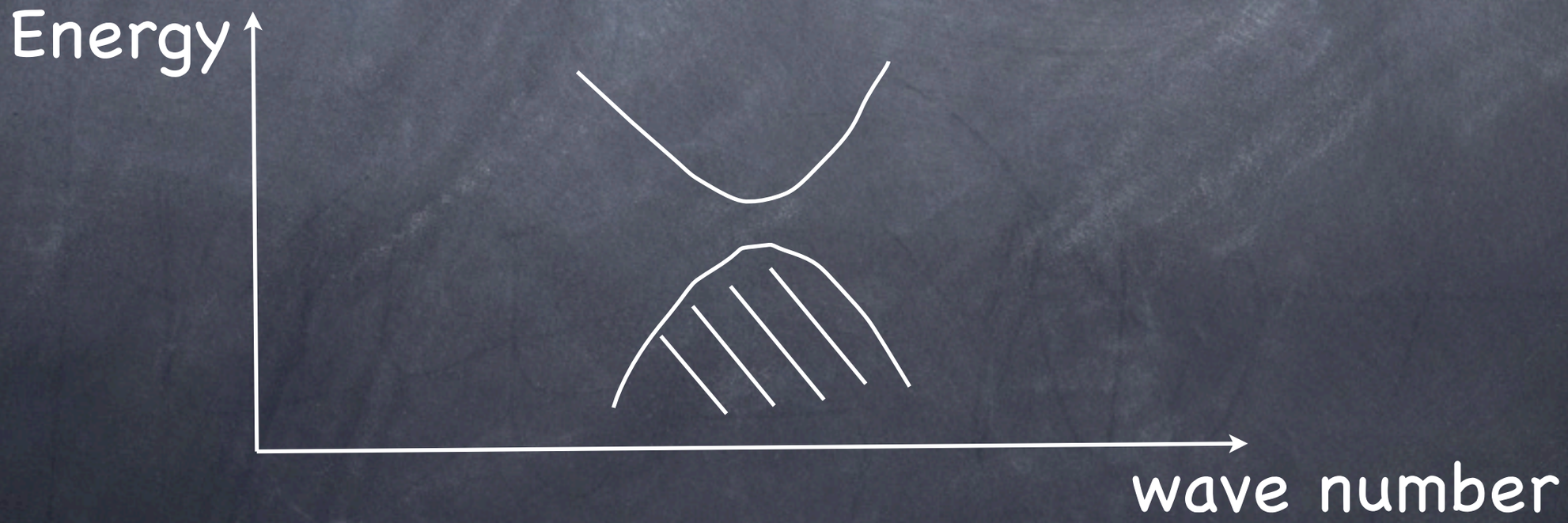
Observables supported on a finite region

# Scaling limit: (relativistic) QFT

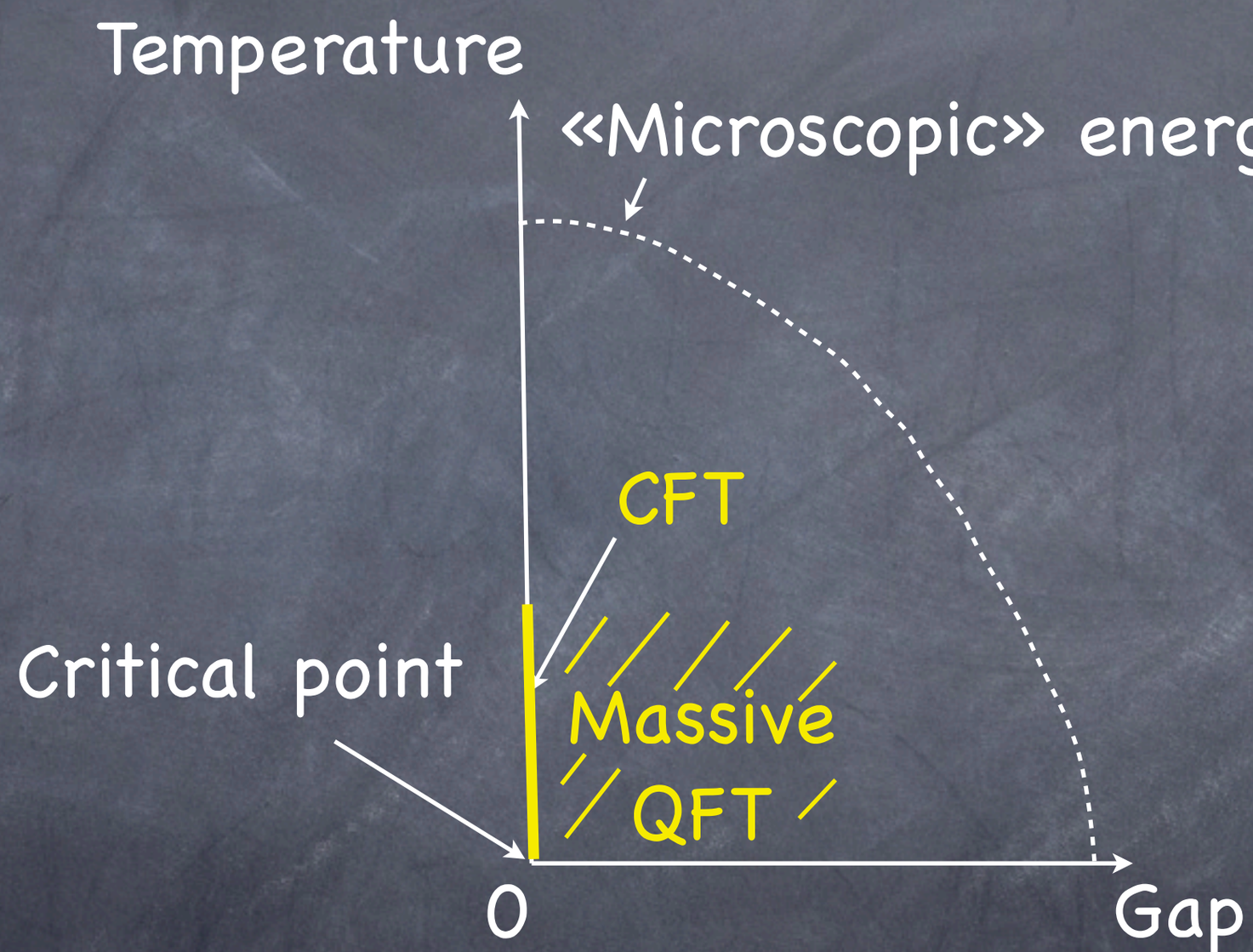
Linear dispersion relation: CFT




Relativistic dispersion relation: QFT



# Scaling limit: (relativistic) QFT



# Formal description of steady state

in massive QFT
(temperature difference only)

[D. Bernard & BD]
[BD]

$$\langle \cdots \rangle_{\rm ness} = {\rm Tr}\left(e^{-W}\cdots\right) \,/\, {\rm Tr}\left(e^{-W}\right)$$

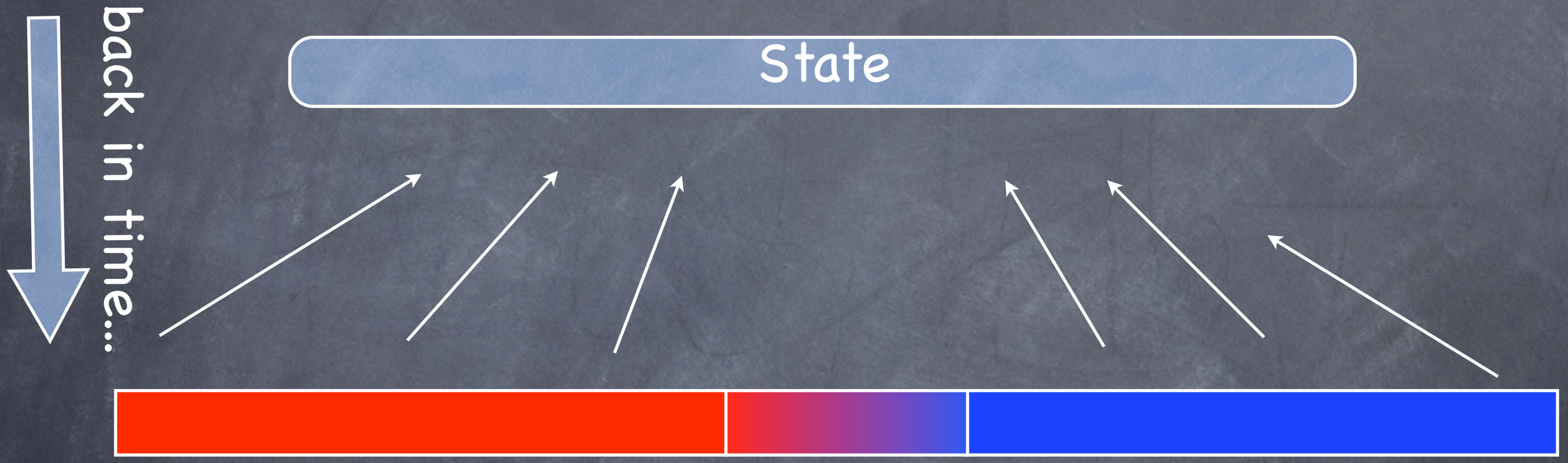
$$W = \beta_l \int_0^\infty d\theta E_\theta n_\theta + \beta_r \int_{-\infty}^0 d\theta E_\theta n_\theta$$

Total energy of right-moving asymptotic particles

Total energy of left-moving asymptotic particles

*In massive QFT: states are described by asymptotically free particles characterized by their rapidities

# Formal description of steady state




agreement with free fermion calculations
[W. H. Aschbacher & C.-A. Pillet, 2003]

# Formal description of steady state

in CFT

(temperature difference, but more general case known)

[D. Bernard & BD]

$$\boxed{\langle \cdots \rangle_{\rm ness} = {\rm Tr}\left(e^{-W}\cdots\right) \,/\, {\rm Tr}\left(e^{-W}\right)}$$

$$W = \beta_l \frac{L_0}{2\pi R} + \beta_r \frac{\bar{L}_0}{2\pi R} \qquad (R\to\infty)$$

Total energy of right-movers  Total energy of left-movers

*In CFT: densities separate into right-movers and left-movers

energy density: $h(x) = h_+(x) + h_-(x)$

momentum density: $p(x) = h_+(x) - h_-(x)$

# Approach to steady state

## in CFT

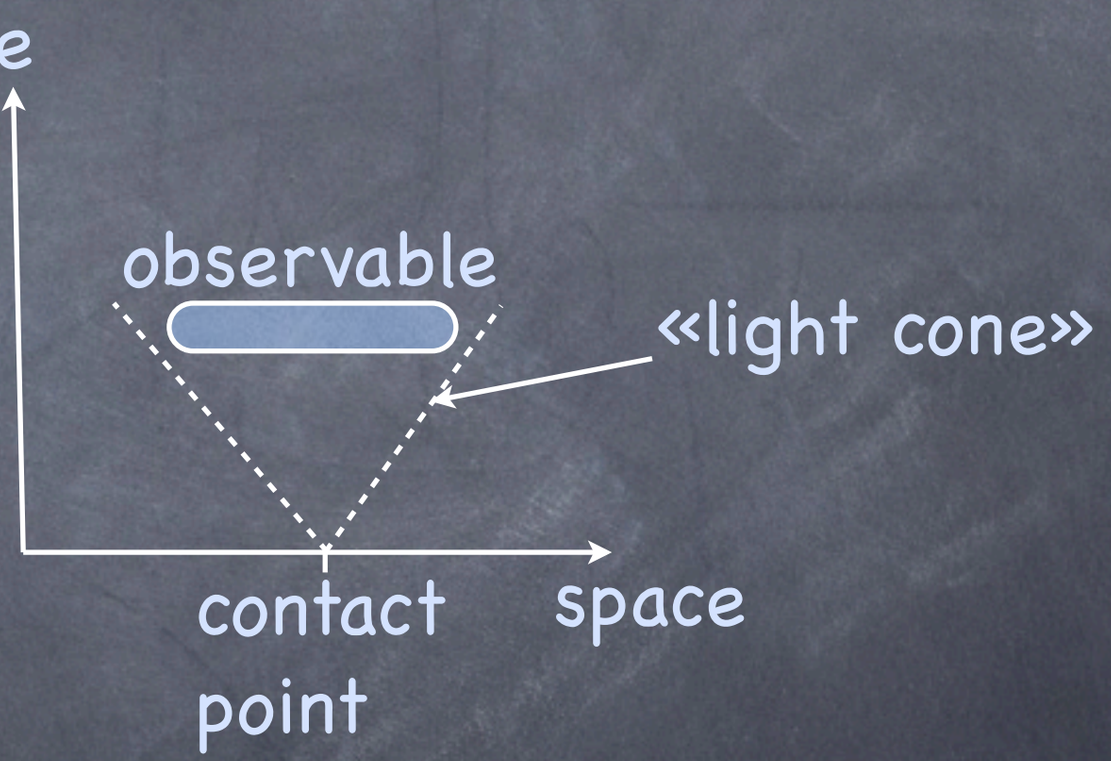
## (observables being symmetry currents and descendants)

[D. Bernard & BD]

Two time scales:

- Universal time scale

$$\tau_{\rm uni} = \frac{\text{observation length}}{v_F}$$




- Microscopic time scale (light cone not sharp)

agreement with numerics [C Karrasch, R. Ilan & J. E. Moore, 2012]

# The energy current in CFT

Using well known character formulas in order to calculate

$$\langle p(0)\rangle_{\rm ness} = \frac{{\rm Tr}\left(e^{-\beta_l \frac{L_0}{2\pi R}} h_+(0)\right)}{{\rm Tr}\left(e^{-\beta_l \frac{L_0}{2\pi R}}\right)} - \frac{{\rm Tr}\left(e^{-\beta_r \frac{L_0}{2\pi R}} h_-(0)\right)}{{\rm Tr}\left(e^{-\beta_r \frac{\bar{L}_0}{2\pi R}}\right)}$$

in the limit $R \to \infty$

# The energy current in CFT

[D. Bernard & BD]

$$J_{\Delta\mu=0} = \frac{\pi c}{12}(\beta_l^{-2} - \beta_r^{-2}) = \frac{\pi c k_B^2}{12\hbar}(T_l^2 - T_r^2)$$

central charge of Virasoro algebra

$$h_+(x) \propto -\frac{c}{24} + \sum_{n\in Z} L_n e^{-\frac{2\pi i n x}{R}}$$

Virasoro

$$J_{\Delta\beta=0} = \frac{1}{8\pi\hbar}(\mu_l^2 - \mu_r^2)$$

# The energy current in CFT

## Agreement with other works

- confirmed by numerics

[C Karrasch, R. Ilan & J. E. Moore, 2012]

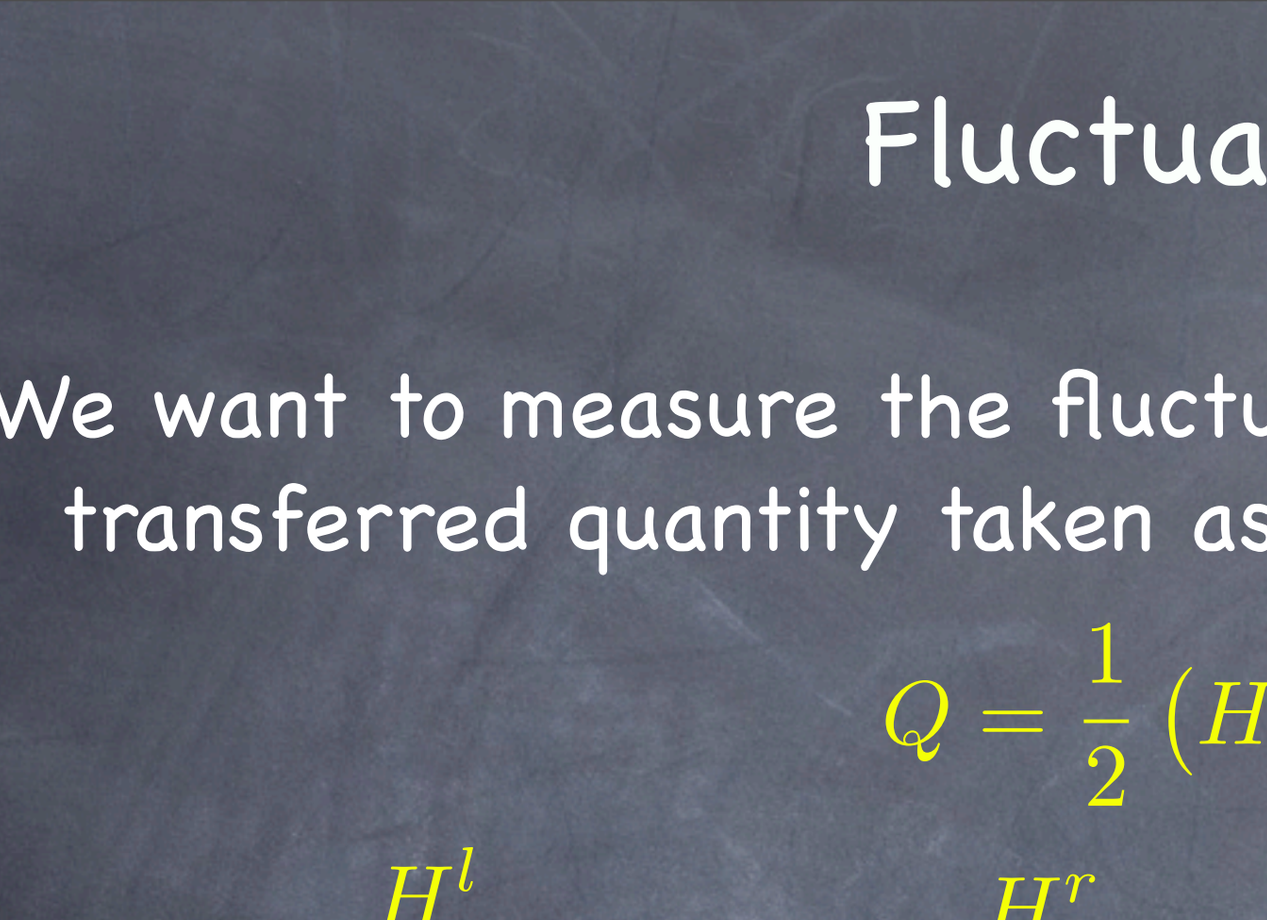
- agreement with Luttinger liquid results

[M. Mintchev & P. Sorba, 2012]
[D. B. Gutman, Yu. Gefen & A. D. Mirlin, 2010]

# Fluctuations

We want to measure the fluctuations of transfer, with the transferred quantity taken as, e.g. for energy transfer:

$$Q = \frac{1}{2}\left(H^r - H^l\right)$$




$$Q = q_0$$

$$Q = q_0 + q$$

# Fluctuations

$$P(q,t) = \sum_{q_0} \mathrm{Tr}\left( P_{q_0+q} e^{-iHt} P_{q_0} \rho P_{q_0} e^{iHt} P_{q_0+q} \right)$$

Starting point:

either in steady state $\rho_{\mathrm{ness}}$

or at the connection time $\rho_0$

# Fluctuations

$$P(\lambda, t) = \sum_q e^{i\lambda q} P(q, t)$$

$$\log P(\lambda, t) \sim t F(\lambda) + O(1)$$

Cumulant generating function

$$= i\lambda J + \dots$$

# Fluctuations in CFT

[D. Bernard & BD]

Independently of starting point for charge transfer
Starting point at connection time for energy transfer
(or: indirect measurement (cf Levitov & Lesovik, 1993))

$$P(\lambda,t) = \frac{\mathrm{Tr}\left(\rho_{\mathrm{ness}}\, e^{i\lambda Q(t)} e^{-i\lambda Q}\right)}{\mathrm{Tr}\,\rho_{\mathrm{ness}}}$$

# Fluctuations in CFT

[D. Bernard & BD]
[D. Bernard, BD, M. Hoogeveen]

Measuring fluctuations of both energy and U(1) charge:

$\lambda$ $\qquad$ $\nu$

$$F(\lambda,\nu) = f(\lambda,\nu;\beta_l,\mu_l) + f(-\lambda,-\nu,\beta_r,\mu_r)$$

$$f(\lambda,\nu;\beta,\mu) = \frac{c\pi}{12\hbar}\left(\frac{1}{\beta-i\lambda}-\frac{1}{\beta}\right)+\frac{(\beta\mu+i\nu)^2}{8\pi\hbar(\beta-i\lambda)}-\frac{\beta\mu^2}{8\pi\hbar}$$

Gaussian for charge transfer only
Poissonian for energy transfer only

# Fluctuation relation

$$F(\lambda,\nu) = F(i(\beta_r - \beta_l) - \lambda, i(\beta_l\mu_l - \beta_r\mu_r) - \nu)$$

Equivalent to (e.g. for energy transfer):

$$P(q, t \to \infty) = e^{(\beta_l - \beta_r)q} P(-q, t \to \infty)$$

Similar relation was argued for in: Jarzynski, Wojcik (PRL 2004)

See the nice review by: Esposito, Harbola, Mukamel (RMP 2009)

Basic ideas: Gallavoti, ...

# Energy current in massive integrable QFT

Using Thermodybamic Bethe ansatz (Al. B. Zamolodchikov, 1990) in order to evaluate

$$J = \mathrm{Tr}\left(e^{-W} p(0)\right) \ / \ \mathrm{Tr}\left(e^{-W}\right)$$

*In massive integrable QFT: scattering matrix factorizes into two-particle processes, and scattering is elastic, set of rapidities is preserved.

# Energy current in massive integrable QFT

[O. Castro Alvaredo, Y. Chen & BD]

[F. Essler and BD]

We find: $$J = \frac{d}{da} f_a \Big|_{a=0}$$

$$f_a = -\int \frac{d\theta}{2\pi} m \cosh\theta \, \log\left(1 + e^{-\epsilon_a(\theta)}\right)$$

$$\epsilon_a(\theta) = W(\theta) + a\, m \sinh\theta - \int \frac{d\gamma}{2\pi} \varphi(\theta-\gamma) \, \log\left(1 + e^{-\epsilon_a(\gamma)}\right)$$

One-particle eigenvalues of $W$

$$\varphi(\theta) = -i \frac{d}{d\theta} S(\theta)$$

Two-particle scattering matrix

# Fluctuations and PT symmetry

[D. Bernard & BD]

If there is dynamical PT symmetry and «asymptotic» PT invariance:

$$-i\frac{d}{d\lambda}F(\lambda) = J(\beta_l - i\lambda, \beta_r + i\lambda)$$

This holds in CFT, and in massive integrable QFT with energy transfer

Something similar for charge current in CFT, and currents associated to other conserved quantities in integrable QFT

# Correlation functions in steady state (Ising model)

[Y. Chen & BD]

We use a «spectral decomposition» in an appropriate space (Liouville space) in order to obtain a convergent expansion, valid at large distances:

$$\langle\sigma(x)\sigma(0)\rangle_{\rm ness} = \int d\theta \cdots \ |\text{form factors}|^2 \ e^{imx\sum \sinh\theta}$$

Two-particle form factors:

$$f(\theta_1,\theta_2) \propto h(\theta_1)h(\theta_2)\tanh\left(\frac{\theta_2-\theta_1}{2}\right)$$

$$h(\theta) \propto \exp\left[\int \frac{d\gamma}{2\pi i}\frac{1}{\sinh(\theta-\gamma)}\log\tanh\frac{W(\gamma)}{2}\right]$$

-> Has a branch cut in rapidity space!

We find an oscillatory behaviour at large distances, e.g. in the disordered phase:

$$\frac{e^{-x\mathcal{E}_{\rm ness}}}{mx}\cos(\nu\log(mx)+B)$$

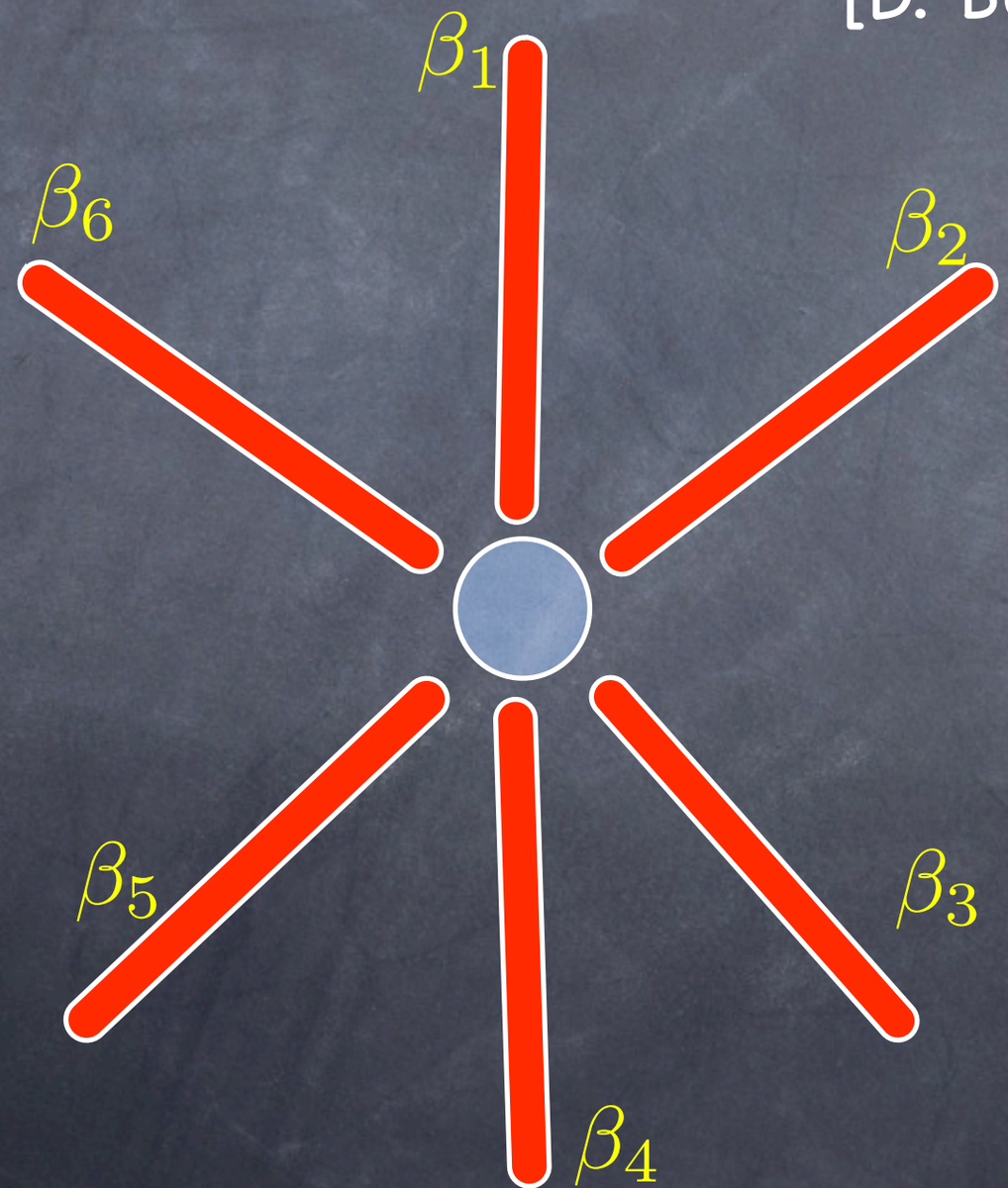
with a frequency determined by the temperatures:

$$\nu=\frac{1}{\pi}\log\left(\coth\frac{m}{2T_r}\ \tanh\frac{m}{2T_l}\right)$$

# Star-graphs in CFT...

Generalization of energy current and fluctuations
to certain star-graphs in CFT

[D. Bernard, BD & M. Hoogeveen]



# Conclusion and perspectives

- Generalization to presence of impurities
- Non-equilibrium correlation functions in integrable models
- Higher dimensions
- Full stochastic interpretation of fluctuation formulas