The Integrable Bootstrap Program at Large N and its Applications in Gauge Theory

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The Principal Chiral Sigma Model (PCSM)

$$\text{Action}: S = \frac{N}{2g^2}\int d^2x\, \mathrm{Tr}\partial_\mu U^\dagger(x)\partial^\mu U(x),$$

$$U(x) \in SU(N):$$

$$SU(N)\times SU(N)\,\text{symmetry}: \quad U(x) \to V_L U(x) V_R,\ V_{L,R} \in SU(N).$$

Associated Noether currents:

$$j^L_\mu(x)^c_a = \frac{-iN}{2g^2}\partial_\mu U_{ab}(x)\, U^{\dagger\, bc}(x),$$

$$j^R_\mu(x)^d_b = \frac{-iN}{2g^2}U^{\dagger\, da}(x)\partial_\mu U_{ab}(x)$$

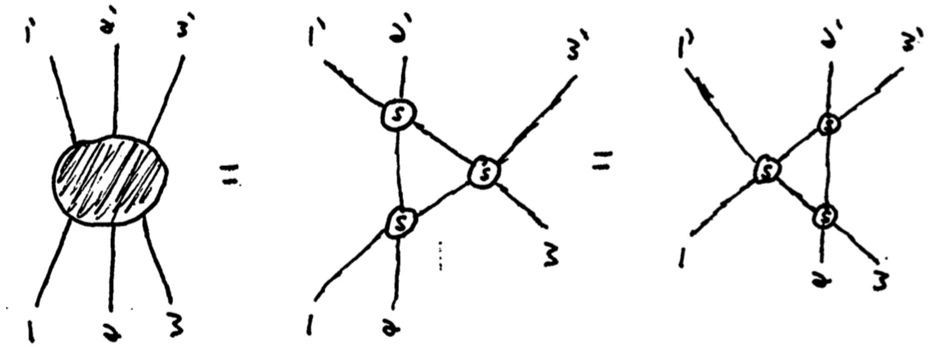
Theory of asymptotically free massive particles, with left and right color.

We work in the 'tHooft (planar) limit.

Integrable Quantum Field Theory

Integrability: Equal number of conservation laws and degrees of freedom (infinite in QFT)

In Quantum field Theory there is no particle production. Set of momenta is conserved $\{p\}_{\text{in}} = \{p\}_{\text{out}}$. Scattering is factorizable.



Yang-Baxter equation

The S-Matrix

Particles and antiparticles have two color charges (color dipoles). Two-particle S-matrix determined by Yang-Baxter equation, unitarity and crossing symmetry.

$$ {}_{\rm out}\langle P, \theta_1', c_1, d_1; P, \theta_2', c_2, d_2 | P, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2 \rangle_{\rm in} $$

$$ = S(\theta, N)\left(\delta^{c_1}_{a_1}\delta^{c_2}_{a_2} - \frac{2\pi i}{N\theta}\delta^{c_2}_{a_1}\delta^{c_1}_{a_2}\right)\times\left(\delta^{d_1}_{b_1}\delta^{d_2}_{b_2} - \frac{2\pi i}{N\theta}\delta^{d_2}_{b_1}\delta^{d_2}_{b_2}\right)\langle\theta_1'|\theta_1\rangle\langle\theta_2'|\theta_2\rangle $$

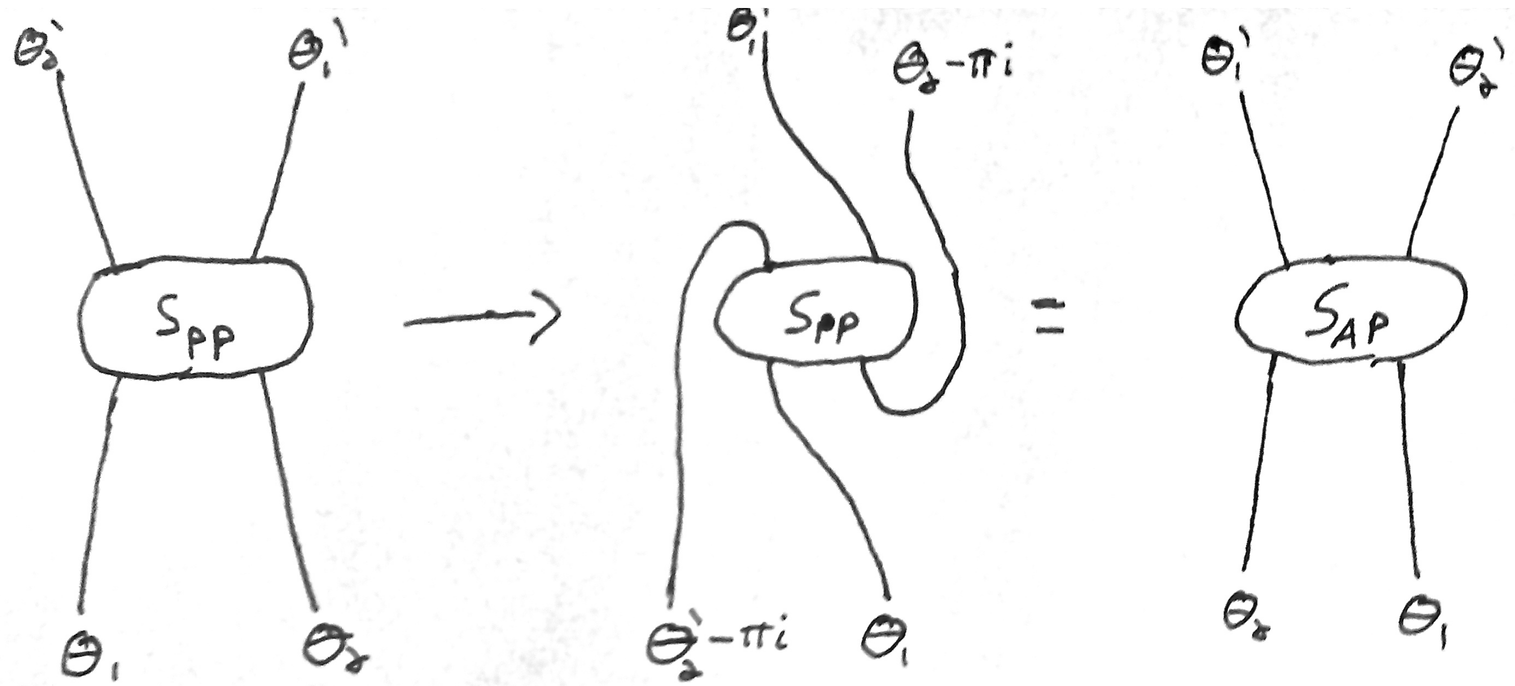
$$ \theta = \text{rapidity} : E = m\cosh\theta,\ \ p = m\sinh\theta,\ \ E^2 = p^2 + m^2 $$

$$ \text{rapidity difference}\, \theta = \theta_1 - \theta_2 $$

$$ \text{At large}\, N: \ \ S(\theta, N) = 1 + \mathcal{O}\left(\frac{1}{N^2}\right). $$

Particle-antiparticle related by crossing $\theta \to \hat{\theta} = \pi i - \theta$.

Particle-antiparticle scattering



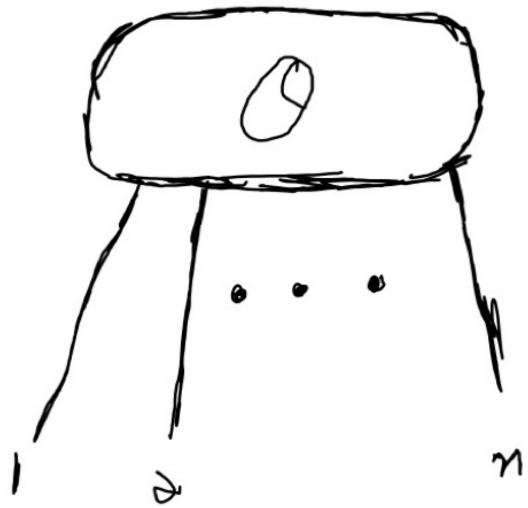
$$S_{AP}(\theta) = S_{PP}(\pi i - \theta)$$

General Form Factors

% Short-hand notation: $|A_1\rangle = |A, \theta_1, b_1, a_1\rangle, \quad |P_1\rangle = |P, \theta_1, a_1, b_1\rangle$

$$\text{Form factor of operator } \mathcal{O}(x) \; :$$

$$\langle 0|\mathcal{O}(x)|A_1, A_2, \ldots, A_l, P_{l+1}, \ldots, P_n\rangle = e^{-ix\cdot\sum p}\mathcal{F}(\{\theta\})_{\{a\}\{b\}} =$$



We eventually want to calculate correlation functions

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle = \sum_{\Psi}\langle 0|\mathcal{O}(x)|\Psi\rangle\langle\Psi|\mathcal{O}(0)|0\rangle$$

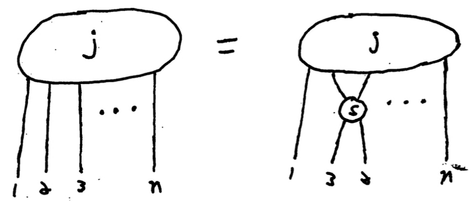
The current operator ansatz

$$\langle 0|j^L_\mu(x)_{a_0 a_{2M+1}}|A_1;\ldots;A_M;P_{M+1};\ldots;P_{2M}\rangle$$
$$= [p_1+\cdots+p_M-(p_{M+1}+\cdots+p_{2M})]_\mu \frac{e^{-ix\cdot\sum p}}{N^{M-1}} \sum_{\sigma,\tau\in S_M} F_{\sigma\tau}(\theta_1,\ldots,\theta_{2M})$$
$$\times \Big[\textstyle\prod_{j=0}^M \delta_{a_j a_{\sigma(j)+M}} \prod_{k=1}^M \delta_{b_k b_{\tau(k)+M}}$$
$$-\tfrac{1}{N}\delta_{a_0 a_{2M+1}}\delta_{a_{l_\sigma} a_{\sigma(0)+M}} \textstyle\prod_{j=1,\,j\neq l_\sigma} \delta_{a_j a_{\sigma(j)+M}} \prod_{k=1}^M \delta_{b_k b_{\tau(k)+M}}\Big],$$

$\sigma \in S_M$, takes $\{1,2,\ldots,M\}$ to $\{\sigma(1),\sigma(2),\ldots,\sigma(M)\}$

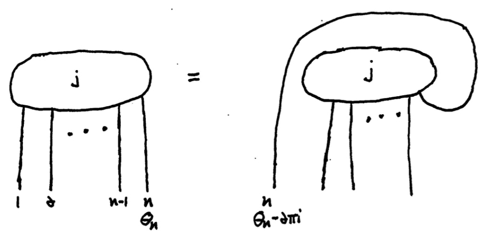
Smirnov's form factor axioms

Scattering Axiom (Watson′s theorem)



$$\langle 0|j|P_2, A_1\rangle = S^{12}_{AP}\langle 0|j|A_1, P_2\rangle$$

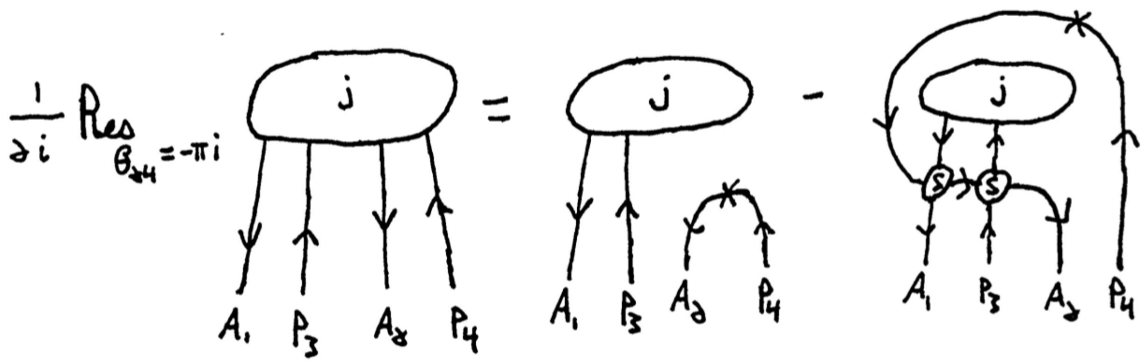
Periodicity axiom



$$\langle 0|j|A_1(\theta_1), P_2(\theta_2)\rangle = \langle 0|j|P_2(\theta_2 - 2\pi i), A_1(\theta_1)\rangle$$

Smirnov's form factor axioms

Annihilation pole axiom



The antiparticle A_2 and the particle P_4 can annihilate. The four particle form factor needs to have an annihilation pole at $\theta_{24} = -\pi i$.

Underlying Abelian Structure at Large N

The excitations in the incoming state of the form factor only interact with each other if they have color indices contracted together.

We can order incoming particle such that they only interact with their two nearest neighbors. Particles now have the simple commutation relation

$$\mathfrak{A}^\dagger(\theta_j)\mathfrak{A}^\dagger(\theta_k) = \frac{\theta_k - \theta_j + \pi i}{\theta_k - \theta_j - \pi i}\mathfrak{A}^\dagger(\theta_k)\mathfrak{A}^\dagger(\theta_j), \text{ if } k = j + 1$$

Behaves like colorless Abelian particles at large N.

This is not related to integrability, but to the large N limit.

Is a nonintegrable large N bootstrap possible?

Solution from Smirnov's axioms

$$F_{\sigma\tau}(\theta) = \frac{g_{\sigma\tau}}{\prod_{j=1,\, j\neq l_\sigma}^{M}(\theta_j - \theta_{\sigma(j)+M} + \pi i) \prod_{k=1}^{M}(\theta_k - \theta_{\tau(k)+M} + \pi i)},$$

From the annihilation pole axiom:

$$g_{\sigma\tau} = \begin{cases} 2\pi i (4\pi)^{M-1}, & \text{for}\, \sigma(j) \neq \tau(j), \ \text{for all}\, j \\ 0 & , \quad \text{else} \end{cases}$$

unphysical double poles go away!

The two-point function

We can calculate exactly the two-current correlator,

$$W_{\mu\nu}(x)_{a_0c_0e_0f_0} = \frac{1}{N}\langle 0|j^L_\mu(x)_{a_0c_0}\, j^L_\nu(0)_{e_0f_0}|0\rangle$$

$$= \frac{1}{N}\sum_{\Psi}\langle 0|j^L_\mu(x)_{a_0c_0}|\Psi\rangle\;\langle\Psi|j^L_\nu(0)_{e_0f_0}|0\rangle$$

$\langle 0|j^L_\mu(x)_{a_0c_0}|\Psi\rangle$ are the form factors we know

$$W_{\mu\nu}(x)_{a_0c_0e_0f_0} = \sum_{M=1}^{\infty}\int\left(\prod_{j=1}^{2M}\frac{d\theta_j}{4\pi}\right)e^{-ix\sum p}4\pi^2(4\pi)^{2M-2}$$

$$\times[p_1+p_3+\cdots+p_{2M-1}-(p_2+\cdots+p_{2M})]_\mu[p_1+\cdots+p_{2M-1}-(p_2+\cdots+p_{2M})]_\nu$$

$$\times(\delta_{a_0e_0}\delta_{c_0f_0}-\frac{1}{N}\delta_{a_0c_0}\delta_{e_0f_0})\prod_{j=1}^{2M-1}\left[\frac{1}{(\theta_j-\theta_{j+1})^2+\pi^2}\right]$$

The energy-momentum two-point function

$$W^T_{\mu\nu\alpha\beta}(x) = \frac{1}{N^2}\langle 0|T_{\mu\nu}(x)T_{\alpha\beta}(0)|0\rangle$$

$$= \sum_{M=1}^{\infty} \frac{\pi}{8} \int \left(\prod_{j=1}^{2M} d\theta_j \right) e^{-ix\sum p}$$

$$\times[p_1{+}p_3{+}\cdots{+}p_{2M-1}{-}(p_2{+}\cdots{+}p_{2M})]_\mu[p_1{+}\cdots{+}p_{2M-1}{-}(p_2{+}\cdots{+}p_{2M})]_\nu$$

$$\times[p_1{+}p_3{+}\cdots{+}p_{2M-1}{-}(p_2{+}\cdots{+}p_{2M})]_\alpha[p_1{+}\cdots{+}p_{2M-1}{-}(p_2{+}\cdots{+}p_{2M})]_\beta$$

$$\times\frac{1}{[(\theta_1-\theta_{2M})^2+\pi^2]} \prod_{j=1}^{2M-1} \frac{1}{[(\theta_j-\theta_{j+1})^2+\pi^2]}$$

What do we know about finite N?

The S-matrix is known:

$$S_{PP}(\theta, N) = \frac{\sinh(\frac{\theta}{2} - \frac{\pi i}{N})}{\sinh(\frac{\theta}{2} + \frac{\pi i}{N})} \left[\frac{\Gamma(i\theta/2\pi + 1)\Gamma(-i\theta/2\pi - \frac{1}{N})}{\Gamma(i\theta/2\pi + 1 - \frac{1}{N})\Gamma(-i\theta/2\pi)} \right]^2 \times S_{PP}(\theta, N \to \infty)$$

There are r-particle bound states with mass

$$m_r = m \frac{\sin\left(\frac{\pi r}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}, \quad r = 1, \ldots, N-1$$

The presence of bound states makes it *impossible* to calculate the form factors. The possibility of incoming particles fusing must be accounted. (Bound-state pole axiom)

N=2

Form factors of this model have been known for a long time, solved by virtue of

$$SU(2) \times SU(2) \simeq O(4),$$

or explicitly:

$$U(x) = n^0(x)\mathbf{1} + \vec{n}(x) \cdot \vec{\sigma}.$$

The $SU(2)$ theory can be mapped into a vector model (instead of a matrix model). The first form factors for the $O(N)$ sigma model were found long ago by Karowski and Weisz (1978).

Our less ambitious result for finite N

For arbitrary N $(2 < N < \infty)$, only the two-particle form factors can be found. This is possible essentially because there is only one particle and one antiparticle, with no possibility of bound states.

$$\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2\rangle$$

$$= (p_1 - p_2)_\mu \left(\delta_{a_0a_2}\delta_{c_0a_1} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_2}\delta_{b_1b_2}\right)$$

$$\times \frac{2\pi i}{(\theta+\pi i)} \exp \int_0^\infty \frac{dx}{x}\left[\frac{-2\sinh\left(\frac{2x}{N}\right)}{\sinh x} + \frac{4e^{-x}\left(e^{2x/N}-1\right)}{1-e^{-2x}}\right]\frac{\sin^2[x(\pi i-\theta)/2\pi]}{\sinh x}$$

Anisotropic QCD

Longitudinal Rescaling: $x^{0,1} \to \lambda x^{0,1},\ x^{2,3} \to x^{2,3}$

$$A_{0,1} \to \lambda^{-1} A_{0,1},\ A_{2,3} \to A_{2,3}$$

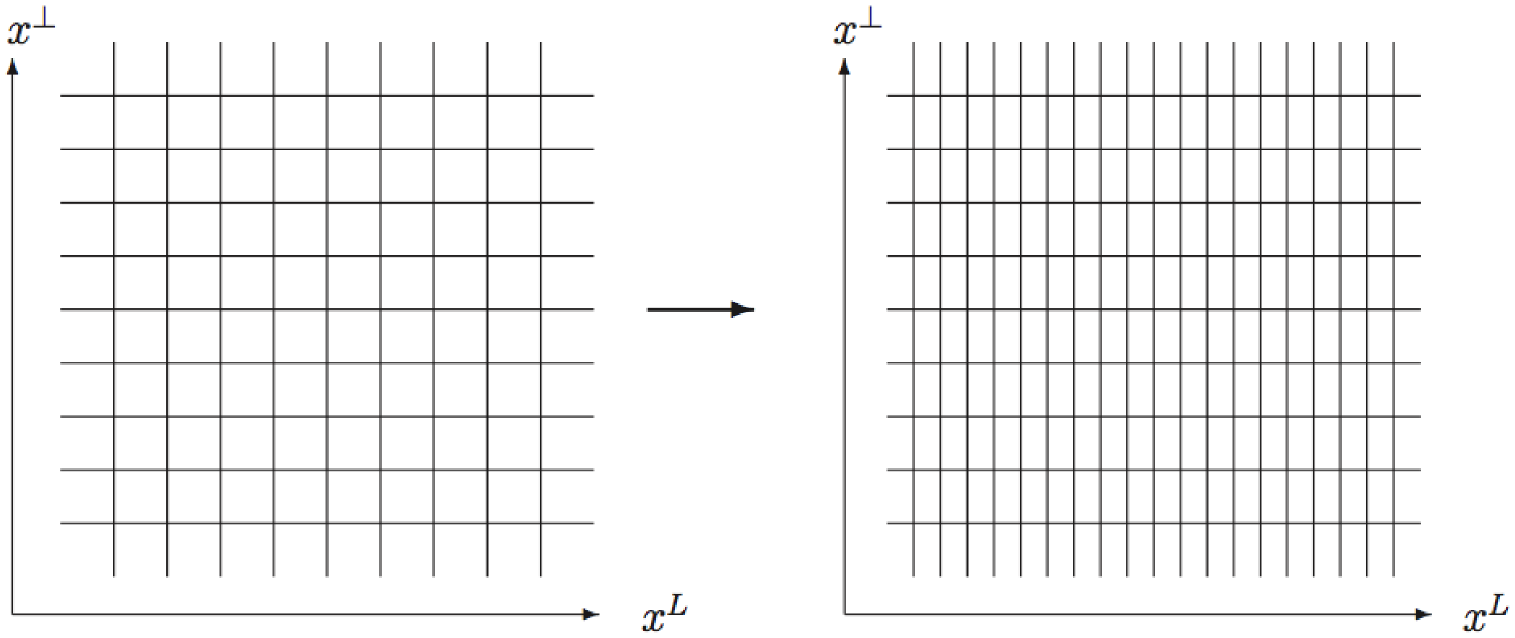
$$H = H_0 + \lambda^2 H_1 + \lambda^2 H_2$$

$$= \left[\int d^3x \left(\frac{g^2}{2} E_\perp^2 + \frac{1}{2g^2} B_\perp^2\right)\right] + \lambda^2 \left[\int d^3x \frac{g^2}{2} E_1^2\right] + \lambda^2 \left[\int d^3x \frac{1}{2g^2} B_1^2\right]$$

Examine the $\lambda \to 0$ limit

no H_2 in 2+1 dimensions

Longitudinal rescaling on the lattice



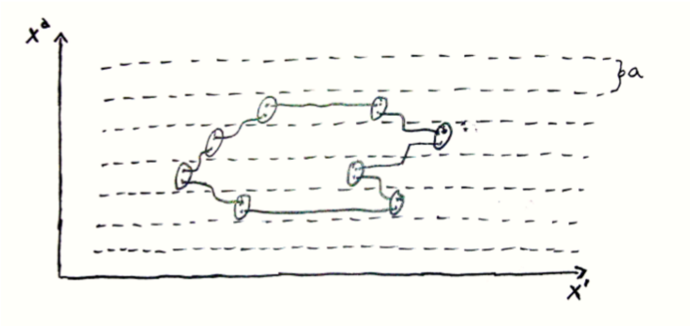
Anisotropic Lattice, 2+1 dimensions

Gauge choice: $A_0 = A_1 = 0$, make x^2 direction discrete.

$$H_0 = \sum_{x^2} H_{PCSM}(x^2), \text{ with } SU(N) \text{ field } U(x) = e^{iaA_2(x)}$$

$$H_1 = -\sum_{x^2} \int dx^1 \int dy^1 \frac{\lambda^2}{4g_0^2 a^2} |x^1 - y^1|$$

$$\times [j_0^L(x^1, x^2) - j_0^R(x^1, x^2 - a)] \times [j_0^L(y^1, x^2) - j_0^R(y^1, x^2 - a)]$$



We compute corrections from $\langle \Psi' | H_1 | \Psi \rangle$ with our form factors

Wait, does that say $A_0 = 0$ and $A_1 = 0$?

E_1 is not 0

This is OK as long as long as you deal with the weird NONLOCAL Gauss's law that is left.

The electric field in the 1 direction is not zero, but determined from Gauss's law:

$$D_\mu E^\mu(x)\Psi = 0 \to E_1(x) = -\int^{x^1} dy^1 D_2(y^1, x^2) E_2(y^1, x^2)$$

with the remaining condition

$$\int dx^1 D_2 E_2(x^1, x^2)\Psi = 0$$

which in the anisotropic lattice becomes

$$\int dx^1 [j_0^L(x^1, x^2) - j_0^R(x^1, x^2 - a)]\Psi = 0.$$

Form factor perturbation theory

We can define a "transfer matrix" to evolve the system in the x^2 direction:

$$T_{x^2,x^2+a} = e^{-\frac{1}{2}H_0(x^2)-\frac{1}{2}H_0(x^2+a)-H_1(x^2,x^2+a)}$$

Truncated spectrum approach: Organize states of H_0 by energy $|1\rangle$, $|2\rangle$, $|3\rangle, \ldots,$ $|n\rangle$.

E_n is the truncation energy.

The (now finite) matrix $T_{jk} = \langle j|T_{x^2,x^2+a}|k\rangle$ can be diagonalized numerically.

Real space renormalization group: we can study the dependence of physical quantities (mass gap, string tensions) on the truncation energy E_n.


Fin