Functional renormalization group for ultracold fermions

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Introduction
Examples many-body fermionic systems

Many-body fermionic systems with nontrivial phases:
- Many-electron system: metal, insulators, magnetism, ....
- Nucleons: nuclear, nucleon superfluid inside neutron stars, ....
- Quarks in the high-density QCD
Effective field approach to strongly-correlated fermions

Microscopic model (Hubbard model, lattice spin model, lattice gauge theory)
↓
Effective field theory
↓
Experiments & Phenomenology
Effective field approach to strongly-correlated fermions

Microscopic model (Hubbard model, lattice spin model, lattice gauge theory) ↓

Effective field theory ↓

Experiments & Phenomenology

Requirements for EFTs:
1. Be simpler than original microscopic models
2. Emerge from renormalizable theories, or lattice models.
Effective field approach to strongly-correlated fermions

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\[ \downarrow \]
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Effective field approach to strongly-correlated fermions

Simple forms of effective action:

\[ \mathcal{L} = \bar{\psi} G^{-1}(\partial \tau, \nabla) \psi + g(\bar{\psi} \psi)^2 \]

or

\[ \mathcal{L} = \bar{\psi} G^{-1}(\partial \tau, \nabla) \psi + \phi G^{-1}_\phi(\partial \tau, \nabla) \phi + g_{\phi \psi} \phi \bar{\psi} \psi \]

At low energies, interactions become strong due to dynamical effects.
\[ \Rightarrow \] Nonperturbative methods of QFT

Important!

Nonperturbative techniques of field theories must be developed in order to describe IR physics using EFT.
Cold atomic physics

Ultracold fermions provides examples of strongly-correlated fermions. High controllability can tune effective couplings with real experiments!

\begin{align*}
\delta(B) &= \text{bound state} \\
S &= 0 \\
\text{closed channel} \\
\text{open channel} & \quad S = 1
\end{align*}

\begin{align*}
E_b &= \frac{1}{ma^2} \\
\text{BCS-BEC crossover} \\
\text{BCS phase} & \quad a < 0 \\
\text{No bound state} \\
\text{BEC phase} & \quad a > 0 \\
\text{Bound state}
\end{align*}

(Typically, $T \sim 100\text{nK}$, and $n \sim 10^{11-14} \text{ cm}^{-3}$)
Introduction

**BCS-BEC crossover**

EFT: Two-component fermions with an attractive contact interaction.

\[
S = \int d^4x \left[ \overline{\psi}(x) \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi(x) + g \overline{\psi}_1(x) \overline{\psi}_2(x) \psi_2(x) \psi_1(x) \right]
\]

**Question**

*Is it possible to treat EFT systematically to describe the BCS-BEC crossover?*

Purpose of this talk

- Develop the **functional renormalization group** (FRG) method for many-body fermions.

- Study the BCS-BEC crossover using the developed formalism of FRG.
  - BCS side: Connection of FRG & BCS theory + GMB correction is made clear. Systematic improvement is considered to go beyond it!
  - BEC side: Describe the Bose gas of dimers /wo auxiliary field methods. This requires a new non-perturbative formalism of FRG.
  - Describe the whole region of the BCS-BEC crossover in this formalism.
Functional renormalization group
General framework of FRG

Generating functional of connected Green functions:

$$\exp(W[J]) = \int \mathcal{D}\Phi \exp (-S[\Phi] + J \cdot \Phi).$$

infinite dimensional integration!

Possible remedy: Construct nonperturbative relations of Green functions!

⇒ Functional techniques

- Dyson-Schwinger equations
- 2PI formalism
- Functional renormalization group (FRG)
**Flow equation of FRG**

\[ \delta S_k[\Phi]: \text{Some function of } \Phi \text{ with a parameter } k. \text{ (IR regulator)} \]

- \( k \)-dependent Schwinger functional
  
  \[ \exp(W_k[J]) = \int \mathcal{D}\Phi \exp \left[ - (S[\Phi] + \delta S_k[\Phi]) + J \cdot \Phi \right] \]

Flow equation

\[ -\partial_k W_k[J] = \langle \partial_k \delta S_k[\Phi] \rangle J \]

\[ = \exp(-W_k[J]) \partial_k (\delta S_k) \left[ \delta/\delta J \right] \exp(W_k[J]) \]

**Consequence**

*We get a (functional) differential equation instead of a (functional) integration!*
Conventional approach: Wetterich equation

At high energies, perturbation theory often works well. ⇒ Original fields control physical degrees of freedom.

IR regulator for bare propagators (\(\sim\) mass term): \(\delta S_k[\Phi] = \frac{1}{2} \Phi^\alpha R_k^{\alpha\beta} \Phi^\beta\).

Flow equation of 1PI effective action \(\Gamma_k[\Phi]\) (Wetterich 1993)

\[
\partial_k \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \frac{\partial_k R_k}{\delta^2 \Gamma_k[\Phi]/\delta \Phi \delta \Phi + R_k} = \partial_k R_k
\]
In the infrared region, collective bosonic excitations emerge quite in common. (e.g.) Another low-energy excitation emerges in the $\Phi \Phi$ channel

**Vertex IR regulator:**

$$\delta S_k = \frac{1}{4!} g^{\alpha \beta \gamma \delta}_k \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta.$$ 

**Flow equation with the vertex IR regulator** (YT, PTEP 2014, 023A04)

$$\partial_k \Gamma_k[\Phi] = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array}$$
Choice of IR regulators $\delta S_k$ is arbitrary.

**Optimization:**
Choose the “best” IR regulator, which validates systematic truncation of an approximation scheme.

**Optimization criterion** (Litim 2000, Pawlowski 2007):
- IR regulators $\delta S_k$ make the system gapped by a typical energy $k^2/2m$ of the parameter $k$.
- High-energy excitations ($\gtrsim k^2/2m$) should decouple from the flow of FRG at the scale $k$.
- Choose $\delta S_k$ stabilizing calculations and making it easier.
Application of fermionic FRG to the BCS-BEC crossover
BCS-BEC crossover

Model:

\[ S = \int d^4x \left[ \overline{\psi}(x) \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi(x) + g\overline{\psi}_1(x)\overline{\psi}_2(x)\psi_2(x)\psi_1(x) \right] \]

\[ n = \frac{k_F^3}{3\pi^2}, \quad \varepsilon_F = \frac{k_F^2}{2m} \]

Purpose of this talk

Nonperturbative FRG can describe the BCS-BEC crossover /wo auxiliary fields!
General strategy

We will calculate $T_c/\varepsilon_F$ and $\mu/\varepsilon_F$.

$\Rightarrow$ Critical temperature and the number density must be calculated.

We expand the 1PI effective action in the symmetric phase:

$$\Gamma_k[\bar{\psi}, \psi] = \beta F_k(\beta, \mu) + \int_p \bar{\psi}_p[\tilde{G}^{-1}(p) - \Sigma_k(p)]\psi_p$$

$$+ \int_{p, q, q'} \Gamma_k^{(4)}(p)\bar{\psi}^{\uparrow, \frac{p}{2} + q}\bar{\psi}^{\downarrow, \frac{p}{2} - q}\psi^{\downarrow, \frac{p}{2} - q'}\psi^{\uparrow, \frac{p}{2} + q'}.$$

Critical temperature and the number density are determined by

$$\frac{1}{\Gamma_0^{(4)}(p = 0)} = 0, \quad n = \int_p \frac{-2}{\tilde{G}^{-1}(p) - \Sigma_0(p)}.$$
Case 1  Negative scattering length \((k_F a_s)^{-1} \ll -1\).

\(\Rightarrow\) Fermi surface exists, and low-energy excitations are fermionic quasi-particles.

Shanker's RG for Fermi liquid (Shanker 1994)
Functional implementation of Shanker’s RG

RG must keep low-energy fermionic excitations under control.

\[ \delta S_k = \int_p \bar{\psi}_p R_k^f(p) \psi_p \] with

\[ R_k^f(p) = \text{sgn}(\xi(p)) \left( \frac{k^2}{2m} - |\xi(p)| \right) \theta \left( \frac{k^2}{2m} - |\xi(p)| \right) \]

Flow equation of the self-energy \( \Sigma_k \) and the four-point 1PI vertex \( \Gamma^{(4)}_k \):

\[ \partial_k \begin{array}{c} \Box \end{array} = \begin{array}{c} \Box \end{array} \]

\[ \partial_k \begin{array}{c} \Box \end{array} = \begin{array}{c} \Box \end{array} \] + \begin{array}{c} \Box \end{array} \]
Flow of fermionic FRG: effective four-fermion interaction

- Particle-particle loop $\Rightarrow$ RPA & BCS theory
- Particle-hole loop gives screening of the effective coupling at $k \sim k_F$

\[ T_{c}^{\text{BCS}} = \varepsilon_F \frac{8e^{-2}e^{-\pi/2k_F|a_s|}}{\pi} \Leftrightarrow T_{c}^{\text{BCS}}/2.2. \] (Gorkov, Melik-Barkhudarov, 1961)
Flow of fermionic FRG: self-energy

Local approximation on self-energy: $\Sigma_k(p) \simeq \sigma_k$.

- High energy: $\sigma_k \simeq (\text{effective coupling}) \times (\text{number density}) \sim 1/k$
- Low energy: $\partial_k \sigma_k \sim 0$ due to the particle-hole symmetry.
Transition temperature and chemical potential in the BCS side


Consequence

- Critical temperature $T_c/\epsilon_F$ is significantly reduced by a factor 2.2 in $(k_Fa_s)^{-1} \lesssim -1$, and the self-energy effect on it is small in this region.
- $\mu(T_c)/\epsilon_F$ is largely changed from 1 even when $(k_Fa_s)^{-1} \lesssim -1$. 

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**BEC side**

**Case 2** Positive scattering length : $(k_F a_s)^{-1} \gg 1$

$\Rightarrow$ Low-energy excitations are one-particle excitations of **composite dimers**.

\[ E_b = \frac{1}{2m_r} a_s^2 \]
Case 2 Positive scattering length : \( (k_F a_s)^{-1} \gg 1 \)

\[ \Rightarrow \text{Low-energy excitations are one-particle excitations of composite dimers.} \]

Several approaches for describing BEC of composite bosons. (Pros/Cons)

- **Auxiliary field method**
  
  (Easy treatment within MFA/ Fierz ambiguity in their introduction)

- **Fermionic FRG** (\( \Leftarrow \text{We develop this method!} \))
  (Unbiased and unambiguous/ Nonperturbative treatment is necessary)
### Vertex IR regulator & Flow equation

Optimization can be satisfied with the vertex IR regulator:

\[
\delta S_k = \int_p \frac{g^2 R_k^{(b)}(p)}{1 - g R_k^{(b)}(p)} \int_{q,q'} \bar{\psi}^\uparrow, \frac{p}{2} + q \bar{\psi}^\downarrow, \frac{p}{2} - q \psi^\downarrow, \frac{p}{2} - q' \psi^\uparrow, \frac{p}{2} + q'
\]

Flow equation up to fourth order (YT, PTEP 2014 023A04, YT, arXiv:1402.0283):

\[
\partial_k \rightarrow \square \rightarrow \partial_k \rightarrow \square \rightarrow \partial_k \rightarrow \square \rightarrow \square = \square \rightarrow \square + \square \rightarrow \square + \square \rightarrow \square
\]

Effective boson propagator in the four-point function:

\[
\frac{1}{\Gamma_k^{(4)}(p)} = -\frac{m^2 a_s}{8\pi} \left( ip^0 + \frac{p^2}{4m} \right) - R_k^{(b)}(p)
\]
Flow of fermionic FRG: self-energy

Flow equation of the self-energy:

\[ \partial_k \Sigma_k(p) = \int_l \frac{\partial_k \Gamma_k^{(4)}(p + l)}{il^0 + l^2/2m + 1/2ma_s^2 - \Sigma_k(l)}. \]

If \( |\Sigma_k(p)| \ll 1/2ma_s^2 \),

\[ \Sigma_k(p) \approx \int_l \frac{\Gamma_k^{(4)}(p + l)}{il^0 + l^2/2m + 1/2ma_s^2} \]

\[ \approx \int \frac{d^3q}{(2\pi)^3} \frac{(8\pi/m^2a_s)n_B(q^2/4m + m^2a_s R_k^{(b)}(q))}{ip^0 + q^2/4m + m^2a_s R_k^{(b)}(q) - (q+p)^2/2m - 1/2ma_s^2}. \]

Estimate of \( |\Sigma_k(p)| \):

\[ |\Sigma_k(p)| \ll \frac{1}{2ma_s^2} \times (\sqrt{2mT}a_s)^3 \times n_B(k^2/4m). \]

\( \Rightarrow \) Our approximation is valid up to \( (k^2/2m)/T \sim (k_Fa_s)^3 \ll 1. \)
Critical temperature in the BEC side

Number density:

\[
  n = \int_{p} \frac{-2}{ip^{0} + p^{2}/2m + 1/2ma_{s}^{2} - \Sigma_{0}(p)} \\
  \approx \frac{(2mT_{c})^{3/2}}{\pi^{2}} \sqrt{\frac{\pi}{2}} \zeta(3/2).
\]

Critical temperature and chemical potential:

\[
  T_{c}/\epsilon_{F} = 0.218, \quad \mu/\epsilon_{F} = -1/(k_{F}a_{s})^{2}.
\]

⇒ Transition temperature of BEC.

Consequence

*FRG with vertex regulator provides a nonperturbative description of many-body composite particles.*
We discuss the whole region of the BCS-BEC crossover with fermionic FRG. ⇒ Combine two different formalisms appropriate for BCS and BEC sides.

Minimal set of the flow equation for $\Sigma_k$ and $\Gamma^{(4)}_k$:(YT, arXiv:1402.0283)

\[ \partial_k = \quad = \quad + \]

\[ \partial_k = \quad = \quad + \]
Flow of fermionic FRG with multiple regulators

Flow of four-point vertex:
Important property: fermions decouple from RG flow at the low energy region.

- In BCS side, fermions decouples due to Matsubara freq. \( (k^2/2m \lesssim \pi T) \).
- In BEC side, fermions decouples due to binding E. \( (k^2/2m \lesssim 1/2ma_s^2) \).

Approximation on the flow of the four-point vertex at low energy:

\[ \partial_k \sim \text{Flow of self-energy:} \]

At a low-energy region, the above approx. gives
Qualitative behaviors of the BCS-BEC crossover from f-FRG

Approximations on the flow equation have physical interpretations.

Four-point vertex: Particle-particle RPA. The Thouless criterion $1/\Gamma^{(4)}(p = 0) = 0$ gives

$$\frac{1}{a_s} = -\frac{2}{\pi} \int_0^\infty \sqrt{2m\varepsilon} d\varepsilon \left[ \tanh \frac{\beta}{2} (\varepsilon - \mu) \frac{\varepsilon - \mu}{2(\varepsilon - \mu)} - \frac{1}{2\varepsilon} \right]$$

$\Rightarrow$ BCS gap equation at $T = T_c$.

Number density: $n = -2 \int 1/(G^{-1} - \Sigma)$.

$$n = -2 \int_p^{(T)} G(p) - \frac{\partial}{\partial \mu} \int_p^{(T)} \ln \left[ 1 + \frac{4\pi a_s}{m} \left( \Pi(p) - \frac{m\Lambda}{2\pi^2} \right) \right].$$

$\Rightarrow$ Pairing fluctuations are taken into account. (Nozieres, Schmitt-Rink, 1985)

Consequence

*We established the fermionic FRG which describes the BCS-BEC crossover.*
Summary & Outlook
Summary

- EFT is a powerful approach to strongly-correlated fermions.
  ⇒ More powerful analytical method is still required for intuitive, unbiased and systematic understandings.

- Fermionic FRG is a promising formalism.
  ⇒ Separation of energy scales can be realized by optimization.
  ⇒ Very flexible form for various approximation schemes.

- Fermionic FRG is applied to the BCS-BEC crossover.
  ⇒ BCS side: GMB correction + the shift of Fermi energy from \( \mu \).
  ⇒ BEC side: BEC without explicit bosonic fields.
  ⇒ whole region: Crossover physics is successfully described at the quantitative level with a minimal setup on f-FRG.
Outlook

- Perform numerical computations for the whole region of the BCS-BEC crossover.
  ⇒ This explicitly confirms that our formalism can be systematically improvable to describe the crossover physics.

- Application of fermionic FRG to other low-density strongly-correlated fermions.
  e.g., Neutron superfluid, dipolar fermions in ultracold atoms, ...