Exploring nucleus structure functions at small x

through factorization scheme independent evolution

Martin Hentschinski

Physics Department
Brookhaven National Laboratory
Upton, NY 11973, USA

In collaboration with Marco Stratmann (arXiv:1311.2825)

I. Non-linear versus linear evolution in DIS at small x

- How to provide evidence for saturation in DIS?
- What prevents currently definite answers

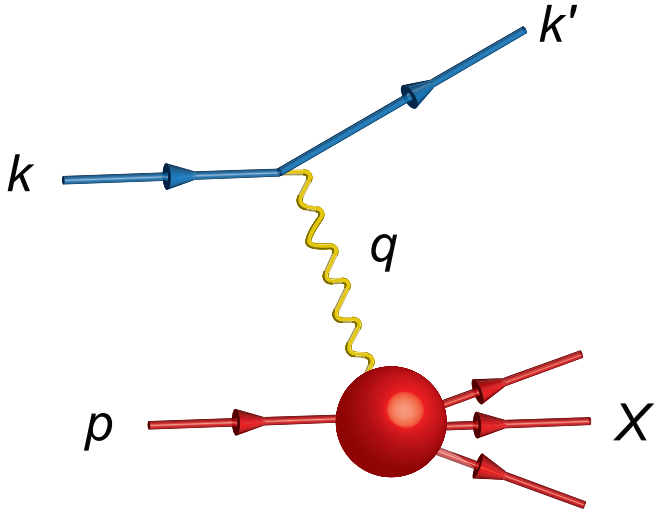
II. RG evolution without scheme dependence

- increasing sensitivity to non-linearities
- introduction to the formalism
- numerical studies up to NNLO

III. Summary & Outlook

Deep Inelastic Scattering - σ_{tot} for γ^*+nucleon/-us$\rightarrow X$

$$e^- + p[A] \rightarrow e^- + X = \gamma^* + p \rightarrow X \text{ (up to QED corrections)}$$



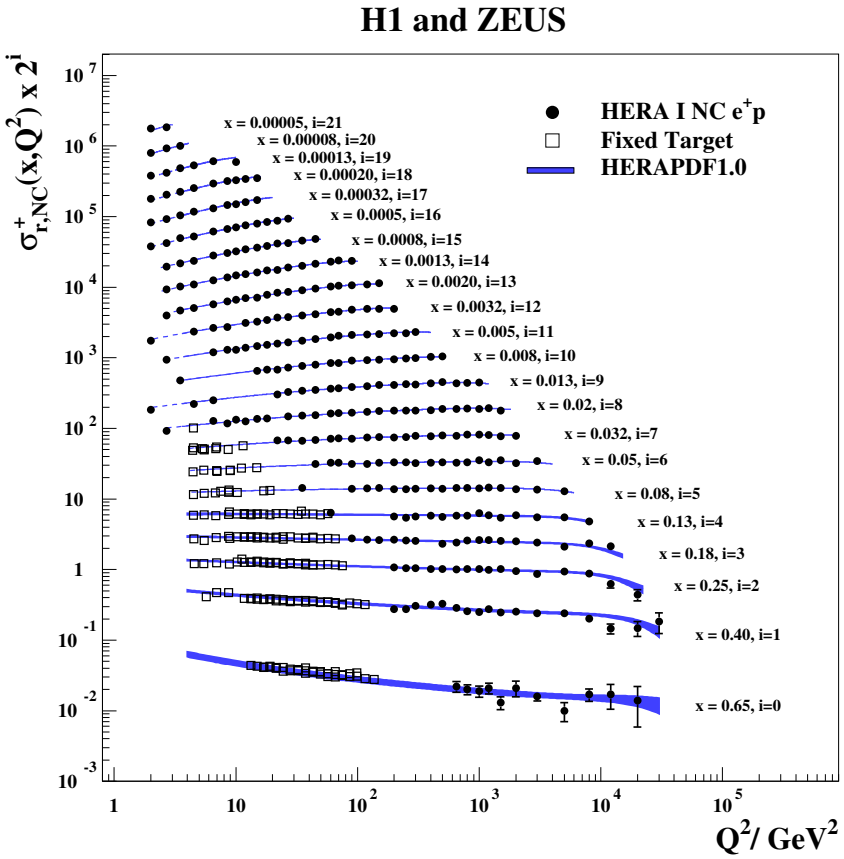
- $y = \dfrac{q \cdot p}{k \cdot p}$ "inelasticity"
- $Q^2 = -q^2 = -(k - k')^2$ "resolution"
- $x_{Bj} = \dfrac{Q^2}{2p \cdot q}$ Parton model: fraction of nucleon momentum carried by struck quark

unpolarized + neutral charge current

$\longrightarrow$ hadronic tensor $\equiv$ proton structure functions F_2 & F_L

$$\frac{d^2\sigma_{\gamma^* p \rightarrow X}}{dx_{Bj.}dQ^2} = \frac{2\pi\alpha^2}{x_{Bj.}Q^4}\left\{\left[1 + (1-y)^2\right]F_2(x_{Bj.}, Q^2) - y^2 F_L(x_{Bj.}, Q^2)\right\}$$

HERA experiment: reduced cross-section → proton structure function F_2



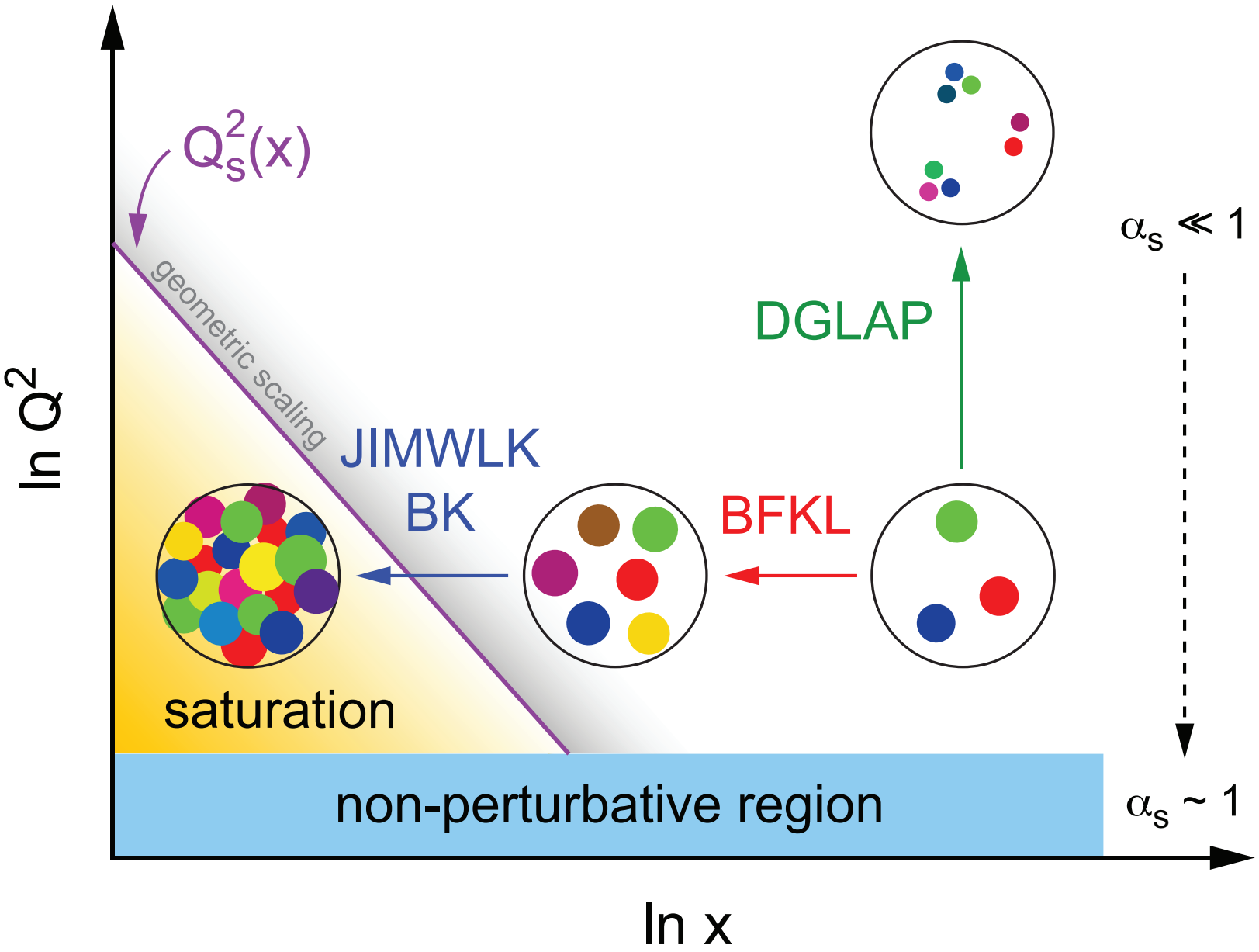
[H1 and ZEUS collaboration, JHEP 1001:109 (2010)]

description with collinear factorization: a success!

$$F_2(x, Q^2) = \text{coeff. funct.} \otimes \text{pdfs} + \mathcal{O}\left(\frac{1}{Q^2}\right)$$

- Q^2 dependence: DGLAP evolution
- essential building block: parton distribution functions (pdfs)
- pdf: probability that struck parton carries fraction x of hadron momentum at scale μ_F^2

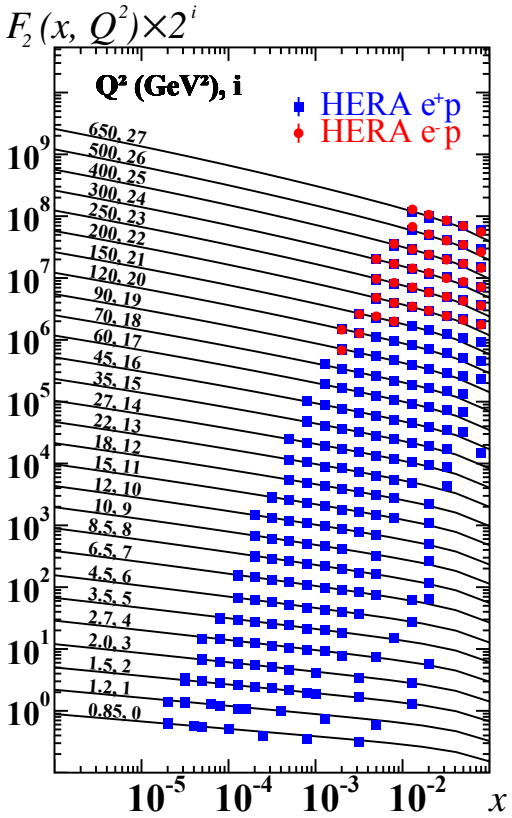
But there should be more ....



Saturation models and non-linear evolution and HERA data

Saturation seen in the data ..... ?

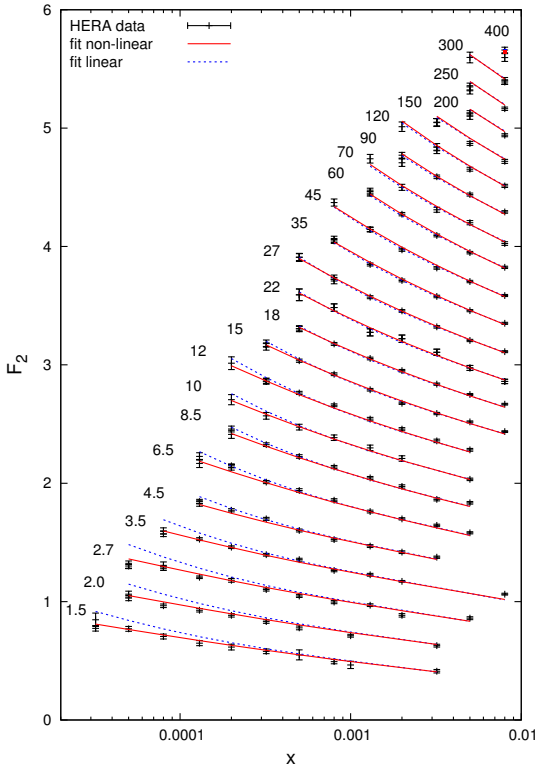
take a look at recent fits



Impact-Parameter dependent Saturation model

[Rezaeian, Siddikov, Van de Klundert, Venugopalan; PRD 87 (2013) 3, 034002]

$\leftrightarrow$

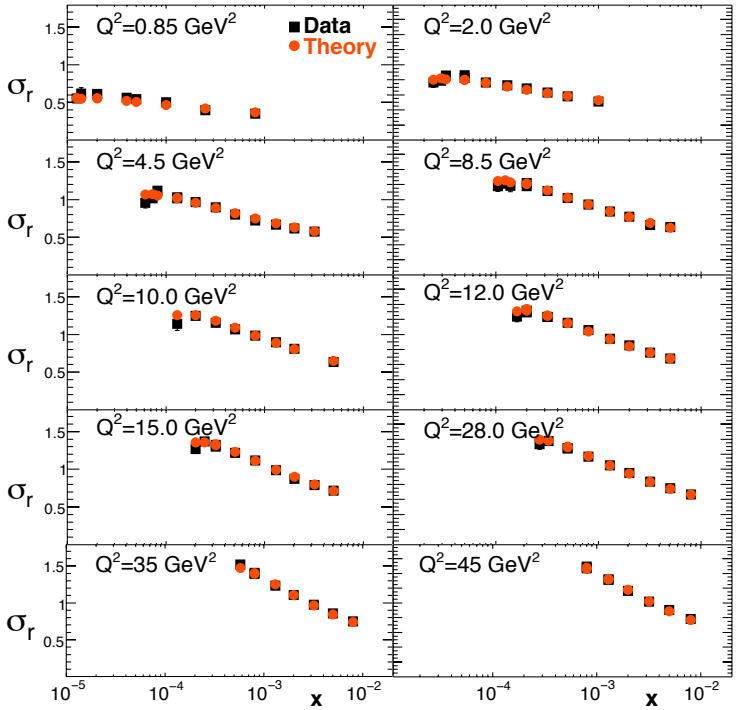


unified DGLAP/BK framework

[Kutak, Sapeta; PRD 86 (2012) 094043]

Saturation models and non-linear evolution and HERA data

to be confronted with success of linear DGLAP evolution .....



non-linear small x evolution (rcBK)

[Alabacete, Armesto, Milhano, Quiroga, Salgado; EPJ C71 (2011) 1705]

$\leftrightarrow$

- theoretical formulation based on expansion in powers $x = \frac{Q^2}{s}$ (high energy factorzation)
- *e.g.* BK evolution: only leading power $\rightarrow$ breaks down for $x > 10^{-2}$
- describe rise of F_2 at small x, but miss non-trivial valence quark dynamics at large x
- DGLAP appears to be more complete, also given the description of hard processes at TEVATRON, LHC etc.

hard to establish non-linear effects, as long as linear evolution works so well

Where does DGLAP naturally break down?

- theoretical formulation based on expansion in powers of $\frac{Q_0^2}{Q^2}$ (collinear factorizaton)
- DGLAP evolution: only leading power
  $\longrightarrow$ DIS on proton: breaks down for $Q_0^2 \simeq 1 - 2\text{GeV}^2$
- in general Q_0 related to scale of proton $\sim \Lambda_{\text{QCD}}$; in presence of saturation effects: $Q_0 \simeq Q_s(x)$.
- Q_s Increases for high parton densities
  1. small x
  2. large A

  sizeable power corrections, $Q_s > Q$ $\longrightarrow$ breakdown of collinear factorization

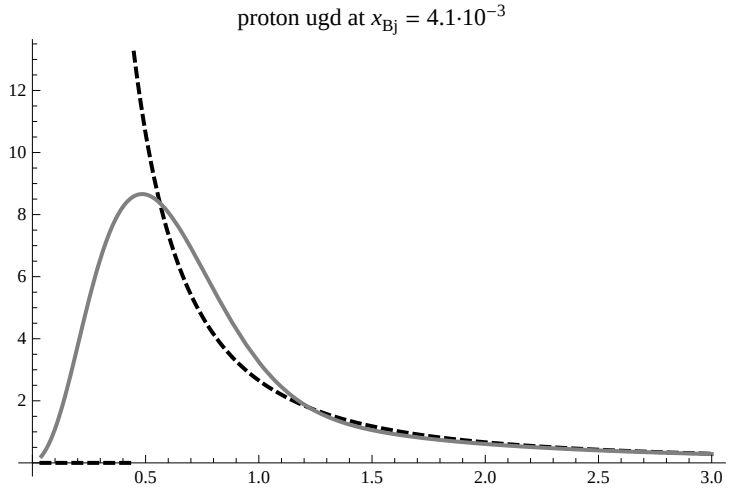
Where does DGLAP naturally break down?

- theoretical formulation based on expansion in powers of $\frac{Q_0^2}{Q^2}$ (collinear factorizaton)
- DGLAP evolution: only leading power
  ➡ DIS on proton: breaks down for $Q_0^2 \simeq 1 - 2\text{GeV}^2$
- in general Q_0 related to scale of proton $\sim \Lambda_{\text{QCD}}$; in presence of saturation effects: $Q_0 \simeq Q_s(x)$.
- Q_s Increases for high parton densities
  1. small x
  2. large A

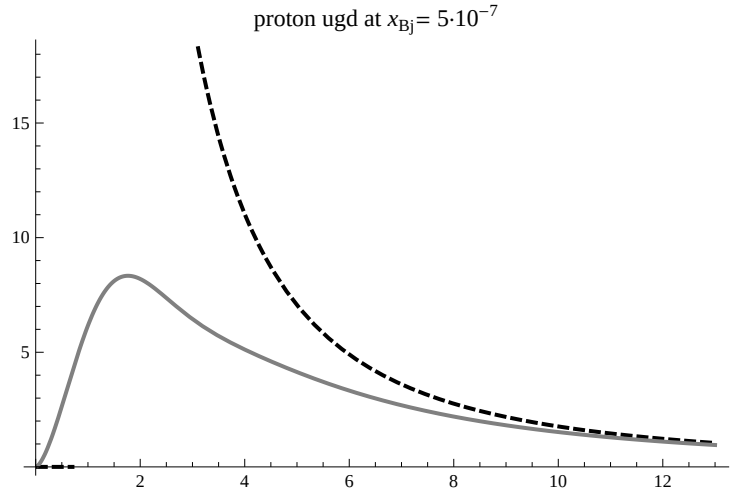
  sizeable power corrections, $Q_s > Q$ ➡ breakdown of collinear factorization

rcBK unintegrated gluon [k_T-dep.]

[Alabacete, Armesto, Milhano, Quiroga, Salgado; EPJ C71 (2011) 1705]



decrease x ⬇ = larger energy



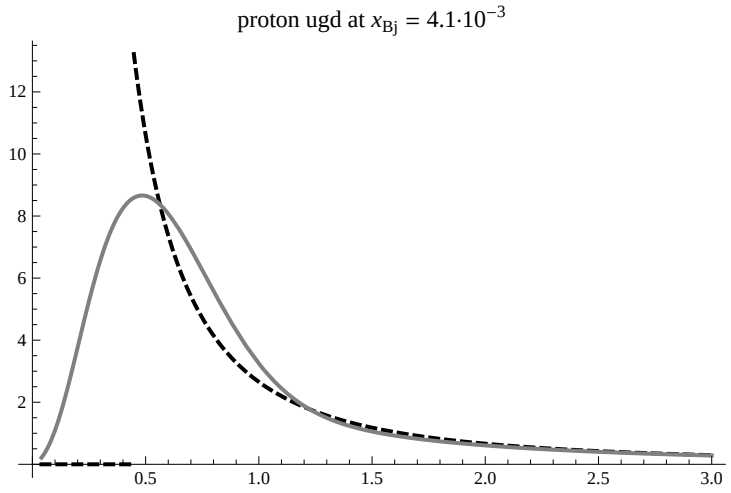
Where does DGLAP naturally break down?

- theoretical formulation based on expansion in powers of $\frac{Q_0^2}{Q^2}$ (collinear factorizaton)
- DGLAP evolution: only leading power
  ➡ DIS on proton: breaks down for $Q_0^2 \simeq 1 - 2\text{GeV}^2$
- in general Q_0 related to scale of proton $\sim \Lambda_{\text{QCD}}$; in presence of saturation effects: $Q_0 \simeq Q_s(x)$.
- Q_s Increases for high parton densities
  1. small x
  2. large A

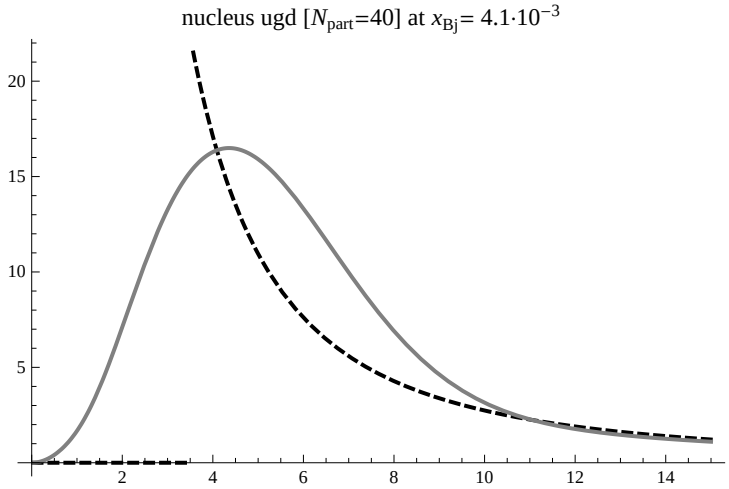
  sizeable power corrections, $Q_s > Q$ ➡ breakdown of collinear factorization

rcBK unintegrated gluon [k_T-dep.]

[Alabacete, Armesto, Milhano, Quiroga, Salgado; EPJ C71 (2011) 1705]



increase density ⬇ DIS on nucleus



Effects appear to be big, but .....

- DGLAP only sensitive to the μ-dependence of integrated distribution

$$f_k(x,\mu_1) = \int^{\mu_1^2} \frac{d\boldsymbol{k}^2}{\boldsymbol{k}^2} F_k(x,\boldsymbol{k}^2) \to \int^{\mu_2^2} \frac{d\boldsymbol{k}^2}{\boldsymbol{k}^2} F_k(x,\boldsymbol{k}^2) = f_k(x,\mu_2)$$

  need μ_f evolution to see breakdown due to Q_s-peak

- DGLAP fits initial condition at $Q^2 \simeq 1-2$ GeV with 20-30 free parameters
  parametrization of pdfs $\quad xf_q(x) = Ax^B(1-x)^C(1+\epsilon\sqrt{x}+Dx+Ex^2)$

  3 active flavors + anti-quarks + gluon ➡ a 7-dim. pdf vector
  $f = (u, d, s, \bar{u}, \bar{d}, \bar{s}, g)$ with $20-30$ parameters

  Q^2 dependence through solution of matrix valued DGLAP equation

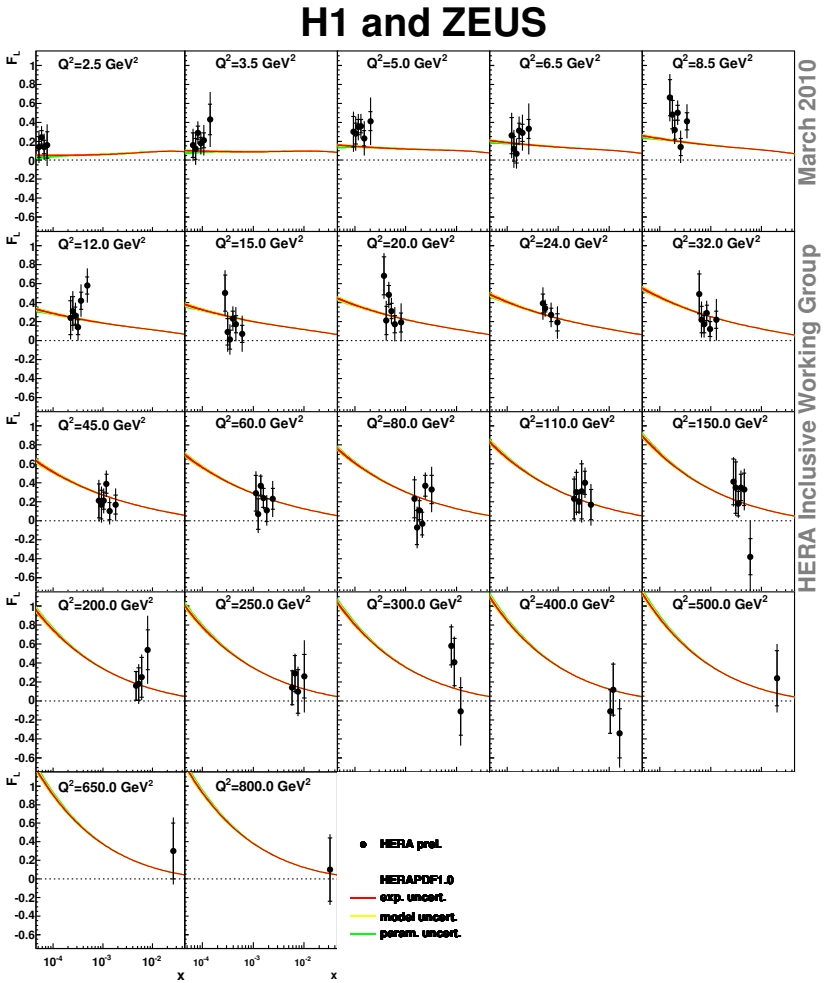
- F_2 very inclusive observable ➡ probe pdfs/unintegrated gluon not directly, but in convolution with coefficents/photon wave function

- pdfs/unintegrated gluon not physical, scheme dependence in their definition

Solution 1: ➡ New Collider experiments

- stronger experimental constrains: measure more than one structure functions *i.e.*. F_2 and F_L
- F_L from y-slope ➡ measurement at different center of mass energies; HERA with large errors
- DIS on the nucleus ➡ enhance density effects

- next generation Electron-Ion-Colliders: **LHeC** (CERN), **MEIC** (JLab), **eRHIC** (BNL) plan precise (F_2, F_L) and DIS on nucleus



Solution 2: theory efforts ...

as we require deviation from DGLAP evolution ....

Guiding question:

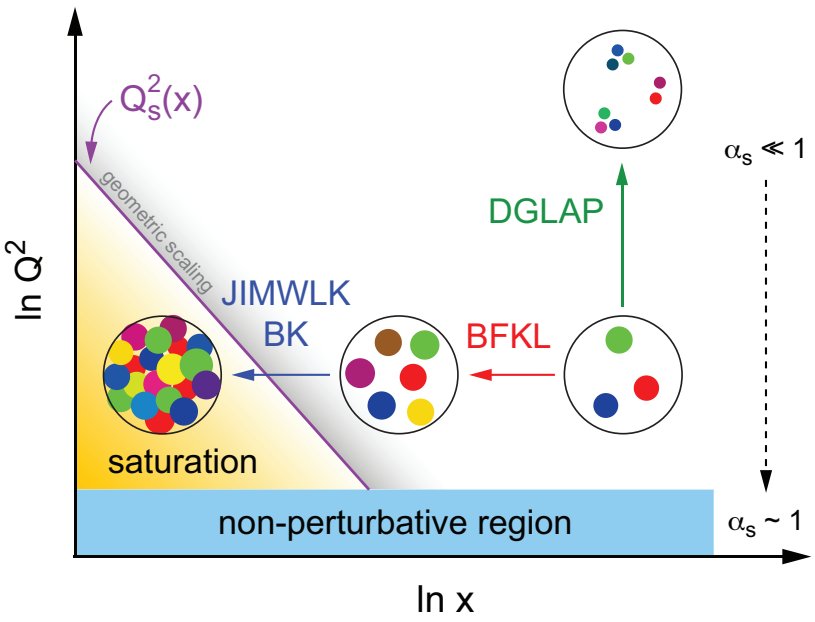
- How can we increase sensitivity of DGLAP evolution to density/small x effects

Where is improvement possible?

DGLAP fitting uses 7-dim. pdf vector $f(x) = (u(x), d(x), s(x), \bar{u}(x), \bar{d}(x), \bar{s}(x), g(x))$ with $20 - 30$ parameters for initial cond. at $Q_0^2 \simeq 1 - 2\text{GeV}^2$

needed for a global fit ...

- a lot of freedom to parametrize distribution at initial scale ...
- .... but expect non-linear effects just right there



Can we find a more restrictive parametrization at the initial scale ?

Collinear factorization .... the transition function Γ

Start: factorization into bare coefficent & parton density

e.g. [Collins, Soper, Sterman; ASDHEP 5 (1988) 1]

$$F_2(x,Q^2) = \sum_{k=q,g} \hat{C}_{2,k} \otimes \hat{f}_k$$

unphysical & contain divergencies

convolution in x space $\otimes = \int_0^1 dz_1 dz_2 \delta(z_1 z_2 - x)$

extract divergencies through transition function Γ_{lk}

$\hat{C}_{2,k} = \sum_{l=q,g} C_{2,l} \otimes \Gamma_{lk}$ $\qquad$ $f_l = \sum_{k=q,g} \Gamma_{lk} \otimes \hat{f}_k$

F_2 in terms of finite quantities

$$F_2(x,Q^2) = \sum_{k=q,g} C_{2,k} \otimes f_k$$

Scale & Scheme Dependence

transition function Γ_{lk}

- separates long range from short range scales at factorization scale μ_f

  ➡ μ_f dependence of perturbative coefficients & non-perturbative pdfs

  ➡ Renormalization group ($\equiv$ DGLAP) equation

$$\frac{d}{d\ln\mu^2} f_k\left(x,\mu^2\right) = \sum_{l=q,g} \left[P_{kl}(z)\otimes f_l\right]\left(x,\mu^2\right)$$

Scale & Scheme Dependence

transition function Γ_{lk}

- separates long range from short range scales at factorization scale μ_f

  ➡ μ_f dependence of perturbative coefficients & non-perturbative pdfs

  ➡ Renormalization group ($\equiv$ DGLAP) equation

$$\frac{d}{d\ln\mu^2} f_k\left(x,\mu^2\right) = \sum_{l=q,g} \left[P_{kl}(z)\otimes f_l\right]\left(x,\mu^2\right)$$

- only uniquely definite up to finite terms ➡ scheme dependence

$$C_{2,k} \to \sum_{m=q,g} \bar{C}_{2,m}\otimes Z^{-1}_{mk} \qquad \Gamma_{lk} \to \sum_{m=q,g} Z_{lm}\otimes \bar{\Gamma}_{mk}$$

$$P_{lk} \to \sum_{m,n=q,g} Z_{lm}\otimes \bar{P}_{mn} Z^{-1}_{nk} - 4\pi\beta(\alpha_s)\sum_{m,n} Z_{lm}\otimes \frac{d}{d\alpha_s} Z^{-1}_{mk}$$

physical observables invariant

fixed perturbative order: spurious higher order terms

scheme depedence & small x analysis

- transition function Γ_{lk}:
  - factorization scheme & scale dependent;
  - popular **choice**: $\mu_f = Q$ & $\overline{MS}$ scheme

→ quark and gluon distribution not physical; a theory definition

- observable (F_2, F_L) independent of μ_f & scheme
  - finite perturbative order: up to higher order terms

small x: possible enhanced by $\left(\alpha_s \ln \frac{1}{x}\right)^n$; differences can be sizeable

scheme depedence & small x analysis

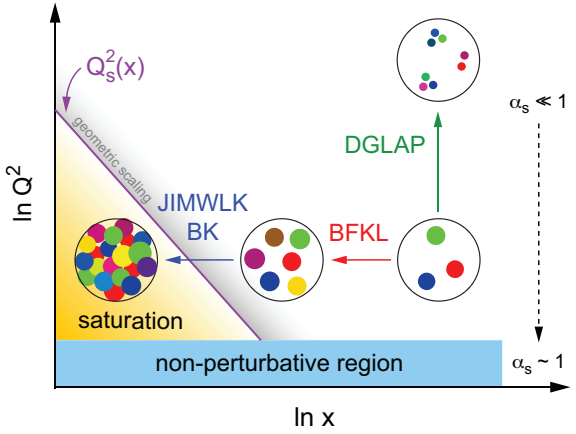
- transition function Γ_{lk}:
  - factorization scheme & scale dependent;
  - popular **choice**: $\mu_f = Q$ & $\overline{MS}$ scheme

→ quark and gluon distribution not physical; a theory definition

- observable (F_2, F_L) independent of μ_f & scheme
  - finite perturbative order: up to higher order terms

small x: possible enhanced by $\left(\alpha_s \ln \frac{1}{x}\right)^n$; differences can be sizeable

- Proton at Small x & search for effects beyond DGLAP ....



- what is still DGLAP evolution?
- what is only a clever scheme choice?

physical evolution equations

idea: don't care about pdfs

evolve observable itself

[Furmanski, Petronzio, ZP C 11, 293(1982)], [Catani, ZP C 75, 665 (1997)], [Blümlein, Ravindran, van Neerven, NPB 586, 349 (2000)]

$$\text{“}Q^2 \frac{d}{dQ^2} F(x, Q^2) = K \otimes F(x, Q^2)\text{”}$$

observable itself!

evolution kernels K

- physical
- no factorization scheme ambiguity; only renormalization scale

equivalent to [Catani, ZP C 75, 665 (1997)]

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

physical evolution at small x

idea:

- physical evolution kernel $\equiv$ DGLAP evolution without factorization scheme & scale ambiguities
- parametrize *e.g.* DIS data at initial scale Q_0 $\rightarrow$ few parameter fit & evolve directly observables

$\Downarrow$

increase sensitivity to possibly hidden non-linear dynamics

from pdfs to observables

make use of two renormalization group equations

$$\frac{d}{d\ln\mu^2} a_s(\mu^2) = \beta(a_s) = -a_s^2\beta_0 - a_s^3 b_1 + \ldots$$ running coupling $a_s \equiv \frac{\alpha_s}{4\pi}$

$$\frac{d}{d\ln\mu^2} f_k(x,\mu^2) = \sum_{l=q,g} P_{kl} \otimes f_l(x,\mu^2)$$ DGLAP – matrix valued

need to invert matrix valued DGLAP Eqs.

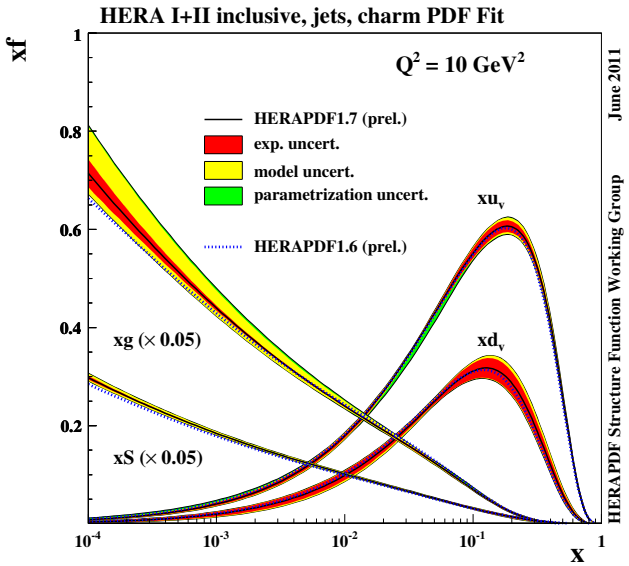
→ # observables $\equiv$ # active flavors + gluon

at small x:

- seaquark $\simeq$ singlet $\gg$ valence quarks $\simeq$ non-singlets

  flavor singlet $\Sigma = \sum_f \left(q_f + \bar{q}_f\right)$

  DGLAP evolution in terms of (Σ, g) alone (7 dim. $\to$ 2 dim.)



A possible sets of observables

next generation Electron Ion Colliders: (F_2, F_L) with high precision;

➡ use to substitute doublet (Σ, g)

$$\begin{pmatrix} F_2 \\ F_L \end{pmatrix} = \underbrace{\begin{pmatrix} C_{2q} C_{2g} \\ C_{Lq} C_{Lg} \end{pmatrix}}_{\text{coeff. matrix } C} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}$$

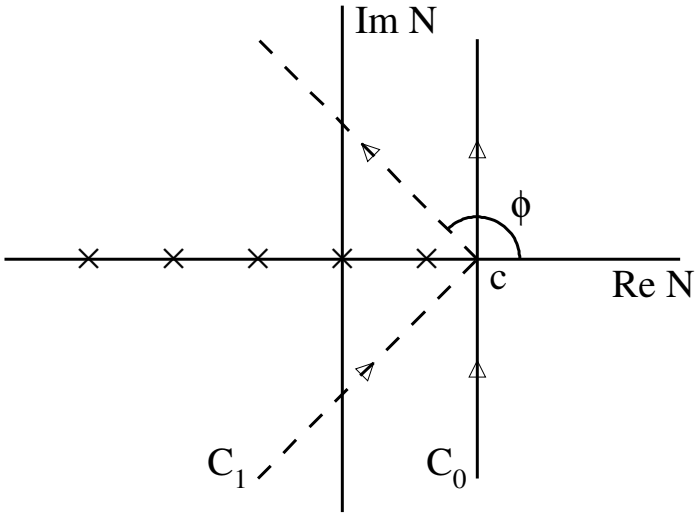
theory: both structure functions large at small x

experiment: F_L from the $y-$slope of the reduced $\gamma^* p-$X-sec. ➡ need measurement at different ep center of mass energies

Analysis in Mellin space

In moment space, $a(N) = \int_0^1 dx x^{N-1} a(x)$, convolutions turn into products

$$c(x) = a \otimes b \qquad \Leftrightarrow \qquad c(N) = a(N) \cdot b(N)$$



- turns analysis into linear algebra
- inverse Mellin transform: numerically

$$a(x) = \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} x^{-N} a(N)$$

Physical evolution kernels – master formula

For a suitable doublet of observables determine (with $a_s = \frac{\alpha_s}{4\pi}$):

$$d_{\ln Q^2} \begin{pmatrix} F_A \\ F_B \end{pmatrix} = d_{\ln Q^2} \left[C \cdot \begin{pmatrix} \Sigma \\ g \end{pmatrix} \right]$$

$$= \left[\beta \frac{dC}{da_s} + C \cdot P \right] \cdot \begin{pmatrix} \Sigma \\ g \end{pmatrix}$$

$$= \left[\beta \frac{dC}{da_s} + C \cdot P \right] C^{-1} \begin{pmatrix} F_A \\ F_B \end{pmatrix} \equiv K \cdot \begin{pmatrix} F_A \\ F_B \end{pmatrix}$$

master formula

$$K = \left[\beta \frac{dC}{da_s} + C \cdot P \right] C^{-1} = a_s K^{(0)} + a_s^2 K^{(1)} + a_s^3 K^{(2)} + \ldots$$

- kernel K independent of factorization scheme & scale order by order in perturbation theory

  [Blümlein, Ravindran, van Neerven, NPB 586, 349 (2000)]

- finite order: dependence on renormalization scale & scheme remains → use for α_s determination

Physical anomalous dimensions for the doublet (F_2, F_L)

leading order coefficents of F_L vanish → need to use rescaled version

$$\tilde{F}_L = \frac{F_L}{a_s C_{Lq}^{(1)}} \qquad \text{or} \qquad \tilde{F}_L = \frac{F_L}{a_s C_{Lg}^{(1)}}$$

possible as $C_{Lq,g}^{(1)}$ scheme independent

LO kernels (quark convention)

$$K_{22}^{(0)} = P_{\mathsf{qq}}^{(0)} - \frac{C_{\mathsf{Lq}}^{(1)} P_{\mathsf{qg}}^{(0)}}{C_{\mathsf{Lg}}^{(1)}} \qquad K_{2L}^{(0)} = \frac{C_{\mathsf{Lq}}^{(1)} P_{\mathsf{qg}}^{(0)}}{C_{\mathsf{Lg}}^{(1)}}$$

$$K_{L2}^{(0)} = \frac{C_{\mathsf{Lg}}^{(1)} P_{\mathsf{gq}}^{(0)}}{C_{\mathsf{Lq}}^{(1)}} - \frac{C_{\mathsf{Lq}}^{(1)} P_{\mathsf{qg}}^{(0)}}{C_{\mathsf{Lg}}^{(1)}} - P_{\mathsf{gg}}^{(0)} + P_{\mathsf{qq}}^{(0)} \qquad K_{LL}^{(0)} = \frac{C_{\mathsf{Lq}}^{(1)} P_{\mathsf{qg}}^{(0)}}{C_{\mathsf{Lg}}^{(1)}} + P_{\mathsf{gg}}^{(0)}$$

LO kernels (gluon convention)

non-diagonal terms differ → see *e.g.*
[Blümlein, Ravindran, van Neerven, NPB 586, 349 (2000)]

NLO Kernels for the doublet (F_2, F_L)

$$K_{22}^{(1)} = P_{qq}^{(1)} - \frac{C_{L,q}^{(1)} P_{qg}^{(1)}}{C_{L,g}^{(1)}} + \frac{C_{2,g}^{(1)} C_{L,q}^{(1)} P_{qq}^{(0)}}{C_{L,g}^{(1)}} + C_{2,g}^{(1)} P_{gq}^{(0)} + \left[\frac{C_{L,g}^{(2)} C_{L,q}^{(1)}}{\left(C_{L,g}^{(1)}\right)^2} - C_{2,g}^{(1)} \left(\frac{C_{L,q}^{(1)}}{C_{L,g}^{(1)}} \right)^2 \right.$$
$$\left. - \frac{C_{L,q}^{(2)}}{C_{L,g}^{(1)}} \right] P_{qg}^{(0)} - \frac{C_{2,g}^{(1)} C_{L,q}^{(1)} P_{gg}^{(0)}}{C_{L,g}^{(1)}} + \beta_0 \left(\frac{C_{2,g}^{(1)} C_{L,q}^{(1)}}{C_{L,g}^{(1)}} - C_{2,q}^{(1)} \right)$$

$$K_{2L}^{(1)} = \frac{C_{L,q}^{(1)} P_{qg}^{(1)}}{C_{L,g}^{(1)}} - \frac{C_{2,g}^{(1)} C_{L,q}^{(1)} P_{qq}^{(0)}}{C_{L,g}^{(1)}} + \frac{C_{2,g}^{(1)} C_{L,q}^{(1)} P_{gg}^{(0)}}{C_{L,g}^{(1)}} - \frac{\beta_0 C_{2,g}^{(1)} C_{L,q}^{(1)}}{C_{L,g}^{(1)}}$$
$$+ \left[C_{2,g}^{(1)} \left(\frac{C_{L,q}^{(1)}}{C_{L,g}^{(1)}} \right)^2 + \frac{C_{2,q}^{(1)} C_{L,q}^{(1)}}{C_{L,g}^{(1)}} - \frac{C_{L,g}^{(2)} C_{L,q}^{(1)}}{\left(C_{L,g}^{(1)}\right)^2} \right] P_{qg}^{(0)}$$

NLO Kernels for the doublet (F_2, F_L)

$$K_{L2}^{(1)} = \left(\frac{C_{L,g}^{(1)}}{C_{L,q}^{(1)}} - \frac{C_{L,q}^{(1)}}{C_{L,g}^{(1)}}\right) P_{gq}^{(1)} - P_{gg}^{(1)} + P_{qq}^{(1)} + \left(\frac{C_{2,g}^{(1)} C_{L,q}^{(1)}}{C_{L,g}^{(1)}} - C_{2,q}^{(1)} + \frac{C_{L,q}^{(2)}}{C_{L,q}^{(1)}}\right) P_{qq}^{(0)}$$
$$+ \left(\frac{C_{L,g}^{(2)}}{C_{L,q}^{(1)}} - \frac{C_{2,q}^{(1)} C_{L,g}^{(1)}}{C_{L,q}^{(1)}} + C_{2,g}^{(1)}\right) P_{gq}^{(0)} + \left(C_{2,q}^{(1)} - \frac{C_{2,g}^{(1)} C_{L,q}^{(1)}}{C_{L,g}^{(1)}} - \frac{C_{L,q}^{(2)}}{C_{L,q}^{(1)}}\right) P_{gg}^{(0)}$$
$$+ \left[\frac{C_{2,q}^{(1)} C_{L,q}^{(1)}}{C_{L,g}^{(1)}} - C_{2,g}^{(1)} \left(\frac{C_{L,q}^{(1)}}{C_{L,g}^{(1)}}\right)^2 - \frac{2C_{L,q}^{(2)}}{C_{L,g}^{(1)}} + \frac{C_{L,g}^{(2)} C_{L,q}^{(1)}}{\left(C_{L,g}^{(1)}\right)^2}\right] P_{qg}^{(0)} + \beta_0 \left(\frac{C_{L,g}^{(2)}}{C_{L,g}^{(1)}} - \frac{C_{L,q}^{(2)}}{C_{L,q}^{(1)}}\right)$$

$$K_{LL}^{(1)} = \frac{C_{L,q}^{(1)} P_{qg}^{(1)}}{C_{L,g}^{(1)}} + P_{gg}^{(1)} - \frac{C_{2,g}^{(1)} C_{L,q}^{(1)} P_{qq}^{(0)}}{C_{L,g}^{(1)}} + \left[C_{2,g}^{(1)} \left(\frac{C_{L,q}^{(1)}}{C_{L,g}^{(1)}}\right)^2 - \frac{C_{L,g}^{(2)} C_{L,q}^{(1)}}{\left(C_{L,g}^{(1)}\right)^2} + \frac{C_{L,q}^{(2)}}{C_{L,g}^{(1)}}\right] P_{qg}^{(0)}$$
$$+ \frac{C_{2,g}^{(1)} C_{L,q}^{(1)} P_{gg}^{(0)}}{C_{L,g}^{(1)}} - C_{2,g}^{(1)} P_{gq}^{(0)} - \beta_0 \frac{C_{L,g}^{(2)}}{C_{Lg}^{(1)}} \,.$$

Alternative: scaling violations

instead of F_L, can also measure scaling violations

$$F_D = -\frac{\beta_0}{2\beta(a_s(Q^2))}\frac{dF_2}{d\ln Q^2};$$

➡ use doublet (F_2, F_D) to substitute (Σ, g)

$$\begin{pmatrix} F_2 \\ F_D \end{pmatrix} = \underbrace{\begin{pmatrix} C_{2q} C_{2g} \\ C_{Dq} C_{Dg} \end{pmatrix}}_{\text{coeff. matrix } C} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}$$

experiment: need to be careful with binning to avoid correlation in data

Kernels for the doublet (F_2, F_D)

scaling violations $F_D = -\frac{\beta_0}{2\beta(a_s(Q^2))}\frac{dF_2}{d\ln Q^2}$

LO kernels

$$K_{22}^{(0)} = 0\,, \qquad K_{2D}^{(0)} = 2\,,$$

$$K_{D2}^{(0)} = \frac{1}{2}\left[P_{gq}^{(0)}P_{qg}^{(0)} - P_{gg}^{(0)}P_{qq}^{(0)}\right]\,, \qquad K_{2D}^{(0)} = P_{gg}^{(0)} + P_{qq}^{(0)}$$

NLO kernels

lengthy but straight forward ➡ see *e.g.*

[Blümlein, Ravindran, van Neerven, NPB 586, 349 (2000)],

[MH, Stratmann arXiv:1311.2825]

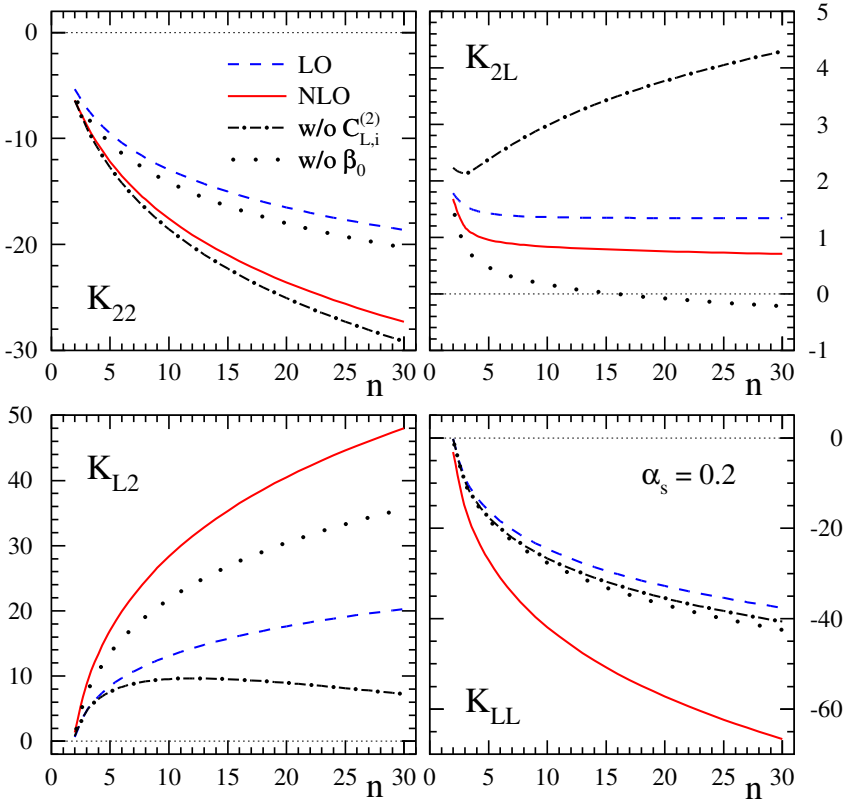
So far:

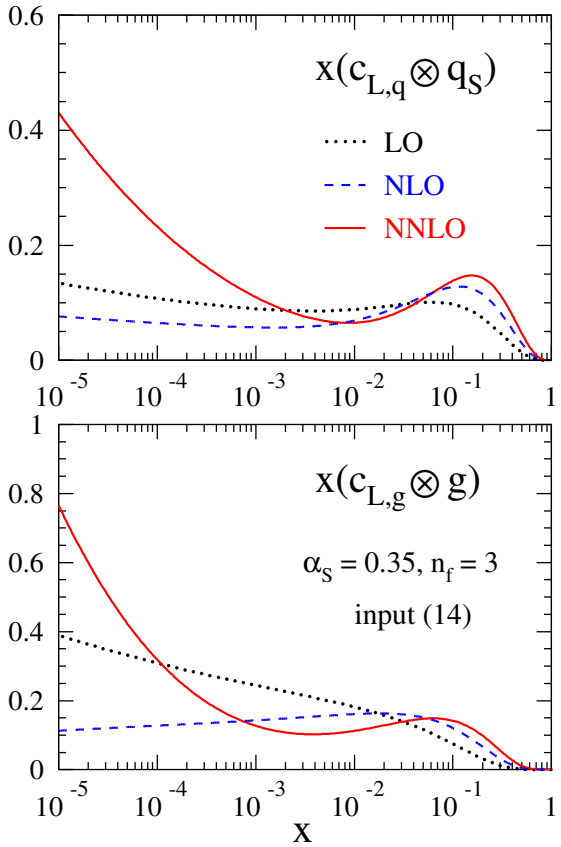
mainly a theory concept

This study:

practical implementation $\equiv$ do some numerics

a study of the kernels





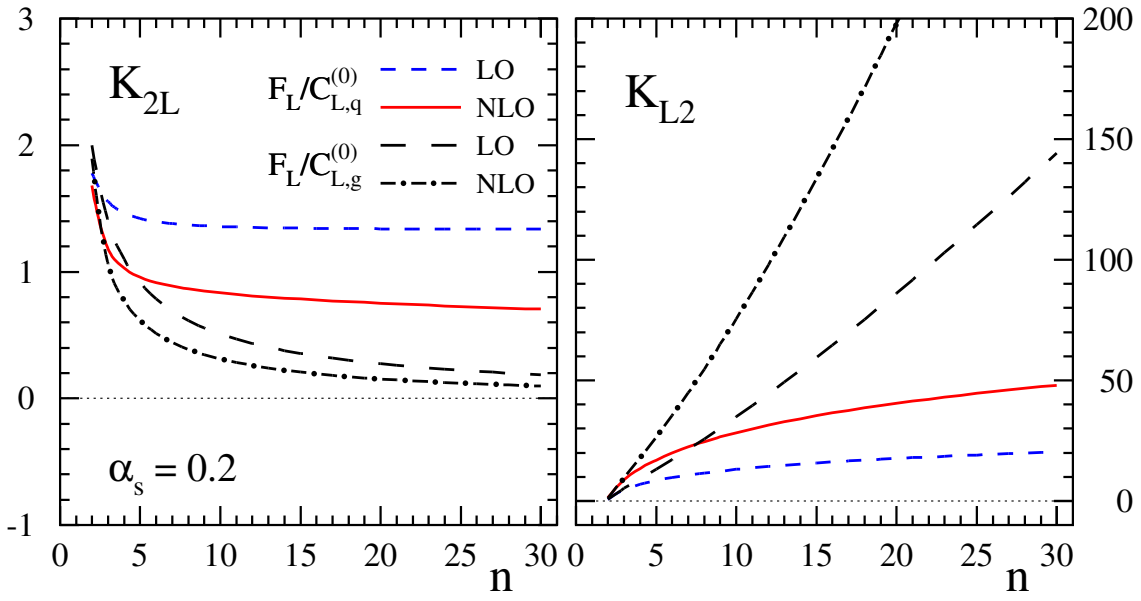
- NLO corrections to kernels are very large (& significantly larger than corrections to DGLAP splitting kernels)

[Moch, Vogt, Vermaseren; NPB 691 129 (2004)]

➡ reason: large NLO corrections to F_L coefficients

study of the kernels

diagonal kernels agree, huge differences for non-diagonal kernels for "gluon" & "quark" convention for $\tilde{F}_L$



reason: $C^{(1)}_{L,q} \sim \frac{1}{n}$ $\quad C^{(1)}_{L,g} \sim \frac{1}{n^2}$

overall result agrees ➡ no new scheme dependence

solution – leading order

matrix valued equation:

$$\frac{d}{d\ln Q^2}\begin{pmatrix} F_A \\ F_B \end{pmatrix} = K \cdot \begin{pmatrix} F_A \\ F_B \end{pmatrix}$$

diagonalize evolution kernel (complete analogy to DGLAP equation)

LO: eigenvalues of matrix K

$$\lambda_\pm = \frac{1}{2}\left[K_{22}^{(0)} + K_{LL}^{(0)} \pm \sqrt{(K_{22}^{(0)} - K_{LL}^{(0)})^2 + 4K_{2L}^{(0)}K_{L2}^{(0)}}\right],$$

agree for both conventions for (F_2, F_L) & also with leading order DGLAP eigenvalues (scheme independent!)

$$\begin{pmatrix} F_A(Q^2) \\ F_B(Q^2) \end{pmatrix} = L^{(0)} \cdot \begin{pmatrix} F_A(Q_0^2) \\ F_B(Q_0^2) \end{pmatrix} \qquad L^{(0)} = \left[\left(\frac{a_s}{a_0}\right)^{-\frac{\lambda_-}{\beta_0}} e_- + (+) \leftrightarrow (-)\right]$$

$a_0 \equiv a_s(Q_0^2)$ $\qquad$ $e_\pm$ projectors on eigenspaces

next-to-leading order — truncated solution

$$\begin{pmatrix} F_A(Q^2) \\ F_B(Q^2) \end{pmatrix} = \left(L^{(0)} + L^{(1)} \right) \cdot \begin{pmatrix} F_A(Q_0^2) \\ F_B(Q_0^2) \end{pmatrix}$$

NLO correction

[Glück, Reya, Vogt; Z. Phys. C 48, 471 (1990)], [A. Vogt, CPC 170, 65 (2005)]

$$L^{(1)} = \left\{ \left(\frac{a_s}{a_0} \right)^{-\frac{\lambda_-}{\beta_0}} \left[(a_0 - a_s)\, e_- R_i^{(1)} e_- \right.\right.$$
$$\left.\left. - \left(a_0 - a_s \left(\frac{a_s}{a_0} \right)^{\frac{\lambda_- - \lambda_+}{\beta_0}} \right) \beta_0 \frac{e_- R_i^{(1)} e_+}{\lambda_+ - \lambda_- - \beta_0} \right] + \quad (+) \leftrightarrow (-) \right\}$$

NLO through subtracted & rescaled NLO Kernel $R^{(1)} = \dfrac{K^{(1)}}{\beta_0} - \dfrac{\beta_1}{\beta_0} \dfrac{K^{(0)}}{\beta_0}$

Numerical implementation

- initial condition: to test framework

  ➡ toy input at $Q_0^2 = 2\text{GeV}^2$,
  (PEGASUS [A. Vogt, CPC 170, 65 (2005)] default intial parton distributions)

  fix $n_f = 3$ and $\alpha_s(Q_0^2) = 0.35$

$$
\begin{aligned}
xu_v(x, Q_0^2) &= 5.10722x^{0.8}(1-x)^3 \\
xd_v(x, Q_0^2) &= 3.064320x^{0.8}(1-x)^4 \\
xg(x, Q_0^2) &= 1.70000x^{-0.1}(1-x)^5 \\
x\bar{d}(x, Q_0^2) &= 0.1939875x^{-0.1}(1-x)^6 \\
x\bar{u}(x, Q_0^2) &= (1-x)x\bar{d}(x, Q_0^2) \\
xs(x, Q_0^2) &= x\bar{s}(x, Q_0^2) = 0.2(\bar{u}+\bar{d})(x, Q_0^2)
\end{aligned}
$$

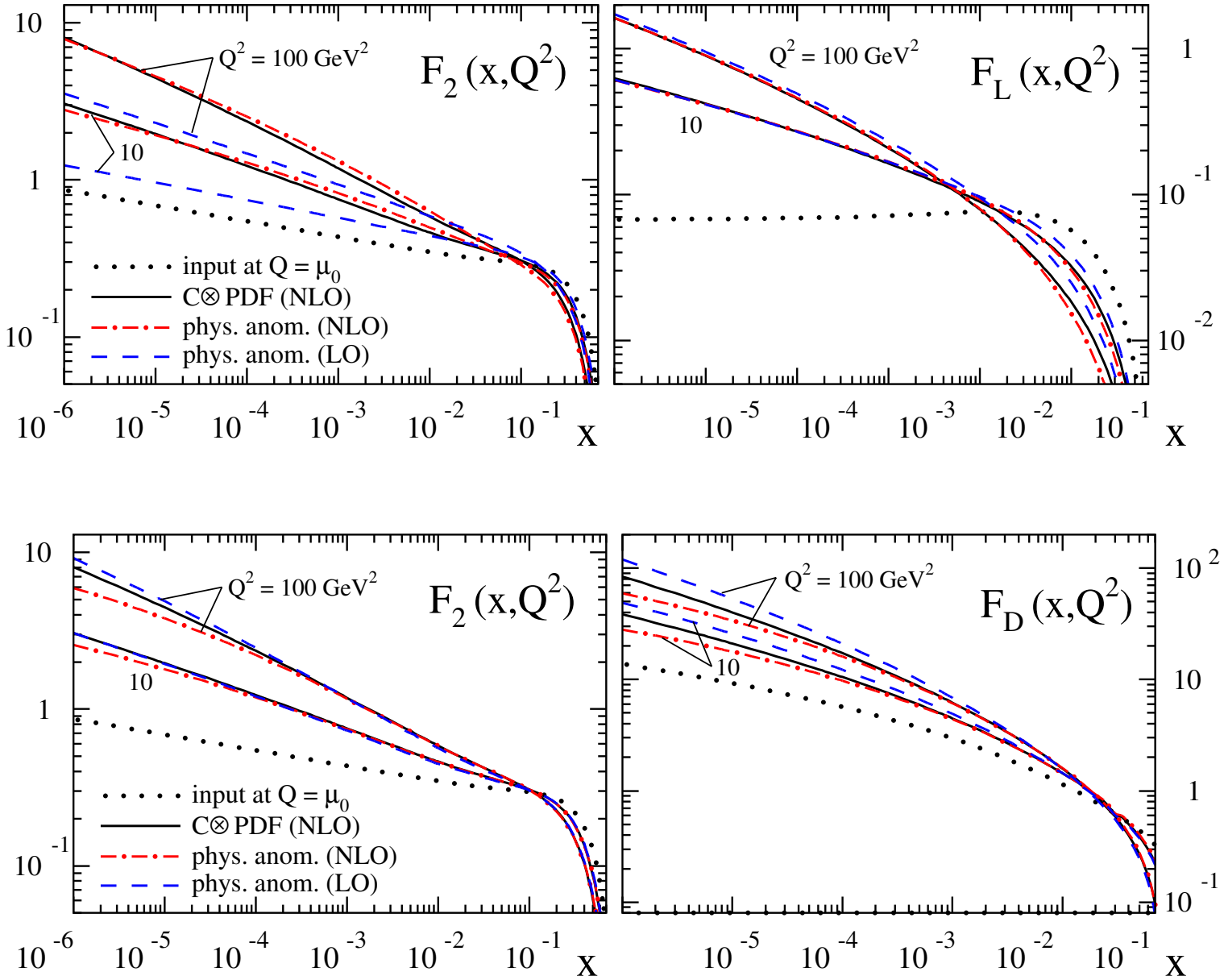
- structure functions at input scale from pdfs

$$F_I(x, Q_0^2) = \sum_k C_{I,k}(Q_0^2) \cdot f_k(x, Q_0^2)$$

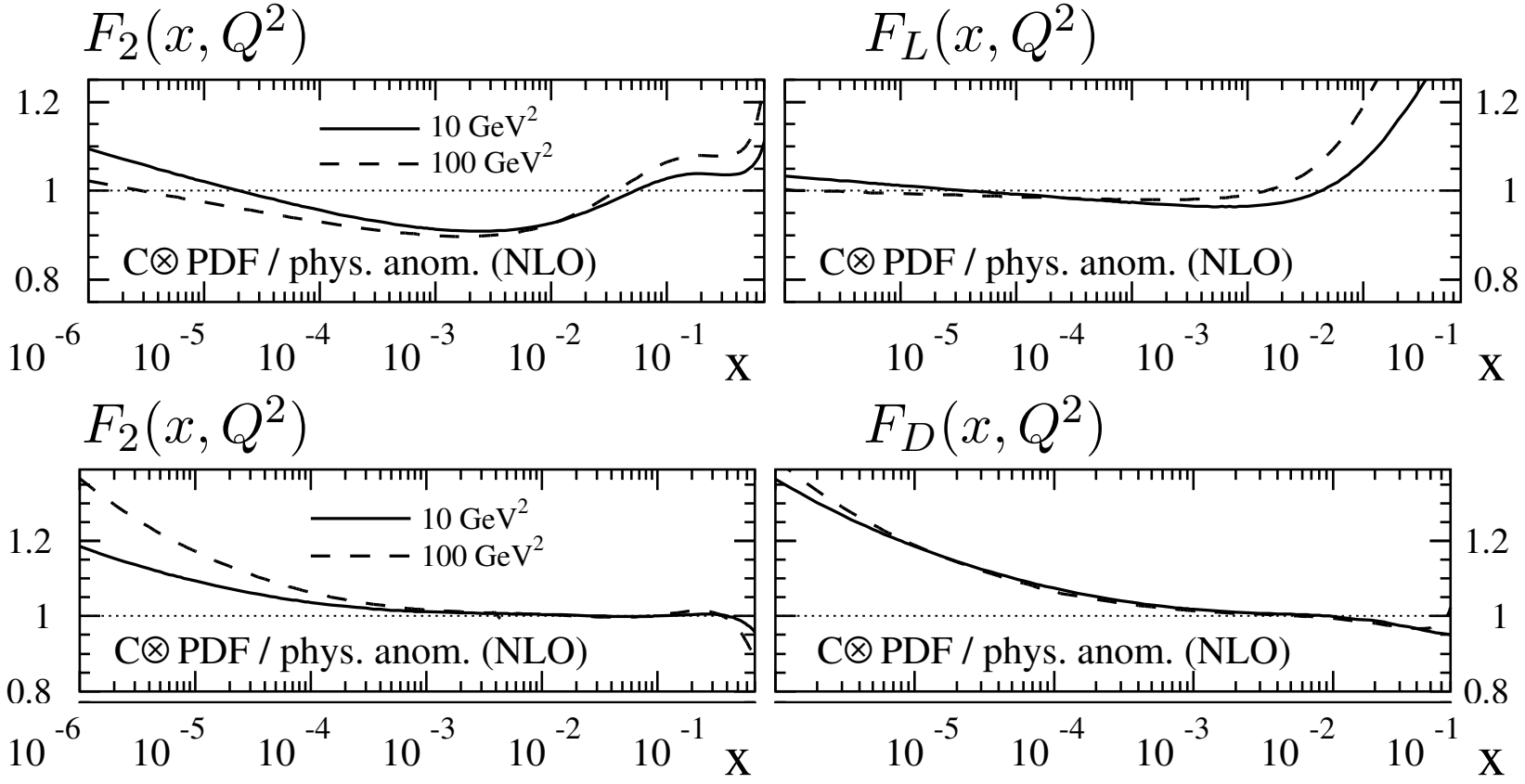
Numerical implementation

- parallel implementation for conventional 'coeff $\otimes$ pdf' and physical evolution kernels
- two independent codes (FORTRAN, MATHEMATICA)
- pdf evolution cross-checked with PEGASUS [A. Vogt, CPC 170, 65 (2005)]
- $\boldsymbol{C}^{(0)}$, $\boldsymbol{C}^{(1)}$ and $\boldsymbol{P}^{(0)}$, $\boldsymbol{P}^{(1)}$ taken from [Floratos, Kounas, Lacaze (1981)], $C^{(2)}_{Lq,g}$ in the parametrized version of [van Neerven, Vogt (1999), (2000)]

results for (F_2, F_L) and (F_2, F_D)



more interesting relative deviations – pdfs versus physical evolution



- (F_2, F_L): $\Delta F_2 \sim 10\%$ for full range; large x: huge deviations for F_L
- (F_2, F_D): Differences increase dramatically towards small x

Technical explanation for differences

pdf implementation (schematic)

$$F^{\text{convent.}} = (1 + a_s C) \left(\frac{a_s}{a_0}\right)^{R^{(0)}} \left[\text{"1"} + (a_0 - a_s) R^{(1)}_{\mathbf{P}}\right] f(Q_0^2)$$

implementation of physical evolution (schematic)

$$F^{\text{phys. evolv.}} = \left(\frac{a_s}{a_0}\right)^{R^{(0)}} \left[\text{"1"} + (a_0 - a_s) R^{(1)}_{\mathbf{K}}\right] (1 + a_0 C) f(Q_0^2)$$

contains spurious NNLO terms

- convent. DGLAP: $\mathcal{O}(a_s a_0)$, $\mathcal{O}(a_s^2)$ – final scale Q^2
- phys. evolution: $\mathcal{O}(a_s a_0)$, $\mathcal{O}(a_0^2)$ – intital scale Q_0^2
- NLO corrections to coefficents are sizeable

Avoid those terms: the "higer order zero" (HOzero) solution

possible to avoid terms beyond NLO entirely
(originally [Glück, Grassie, Reya, PRD 30, 1447 (1984)]; [Glück, Reya, Vogt, PRD 46, 1973 (1992)])

at least for the theoretical study, evolution of experimental data can be more involved

schematic:

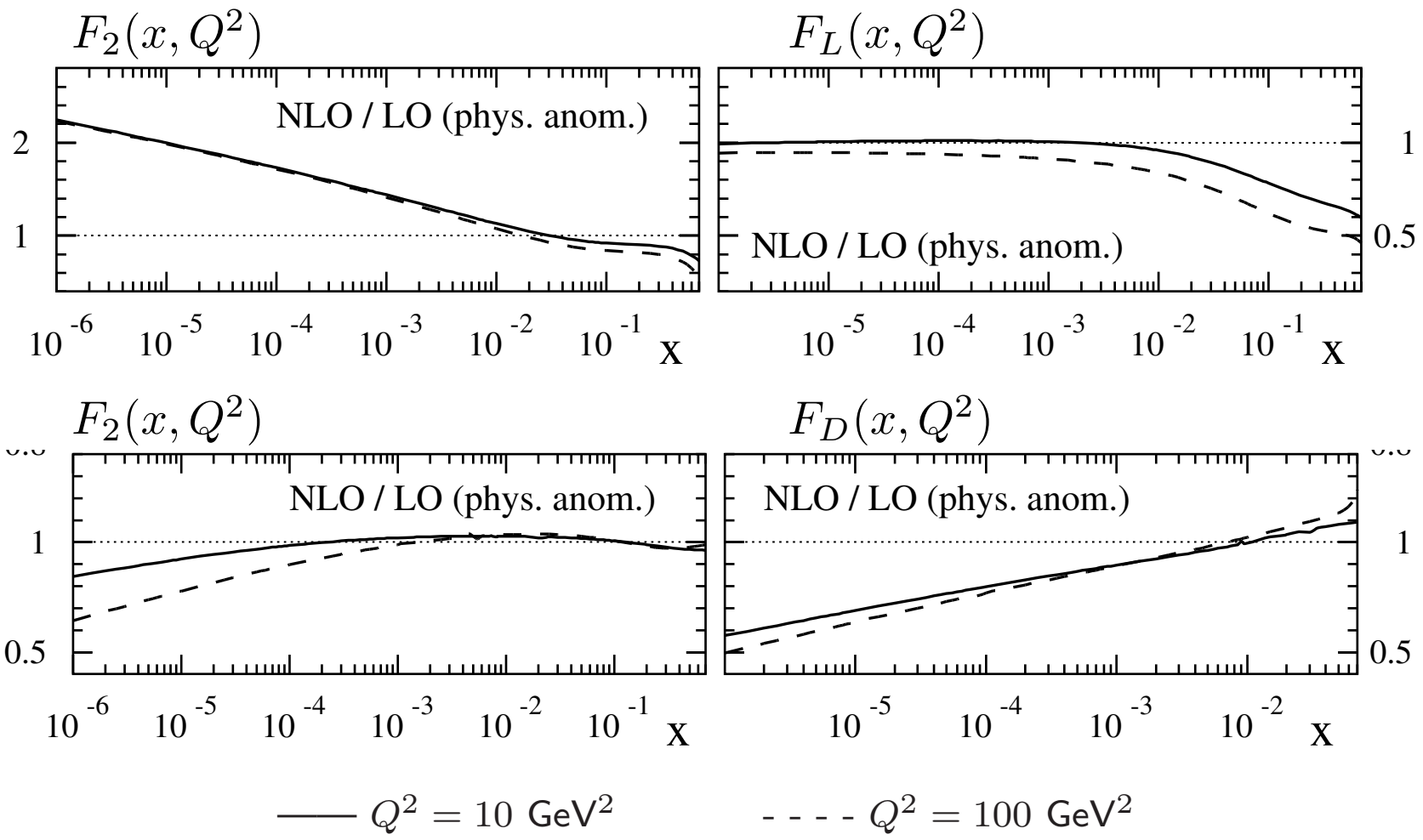
$$F_{\text{HOzero}}^{\text{convent.}} = \left(\frac{a_s}{a_0}\right)^{R^{(0)}} \left[\text{"1"} + (a_0 - a_s) R_{\mathbf{P}}^{(1)} + a_s C\right] f(Q_0^2)$$

$$F_{\text{HOzero}}^{\text{phys. evolv.}} = \left(\frac{a_s}{a_0}\right)^{R^{(0)}} \left[\text{"1"} + (a_0 - a_s) R_{\mathbf{K}}^{(1)} + a_0 C\right] f(Q_0^2)$$

both solutions agree exactly over the entire phase space (for $\mu_f^2 = Q^2$)

➡ no intrinsic problem, but in certain region of phase space higher order corrections are sizeable

physical evolution: convergence of higher order corrections



NLO sizeable/huge in the small x region $x < 10^{-2} - 10^{-3}$

Physical evolution at Next-to-Next-to-Leading Order

- "new" effects due to physical anomalous dimensions first time at NLO ➡ 'NLO' effective 'LO'
- NLO corrections large ➡ convergence of the perturbative series?

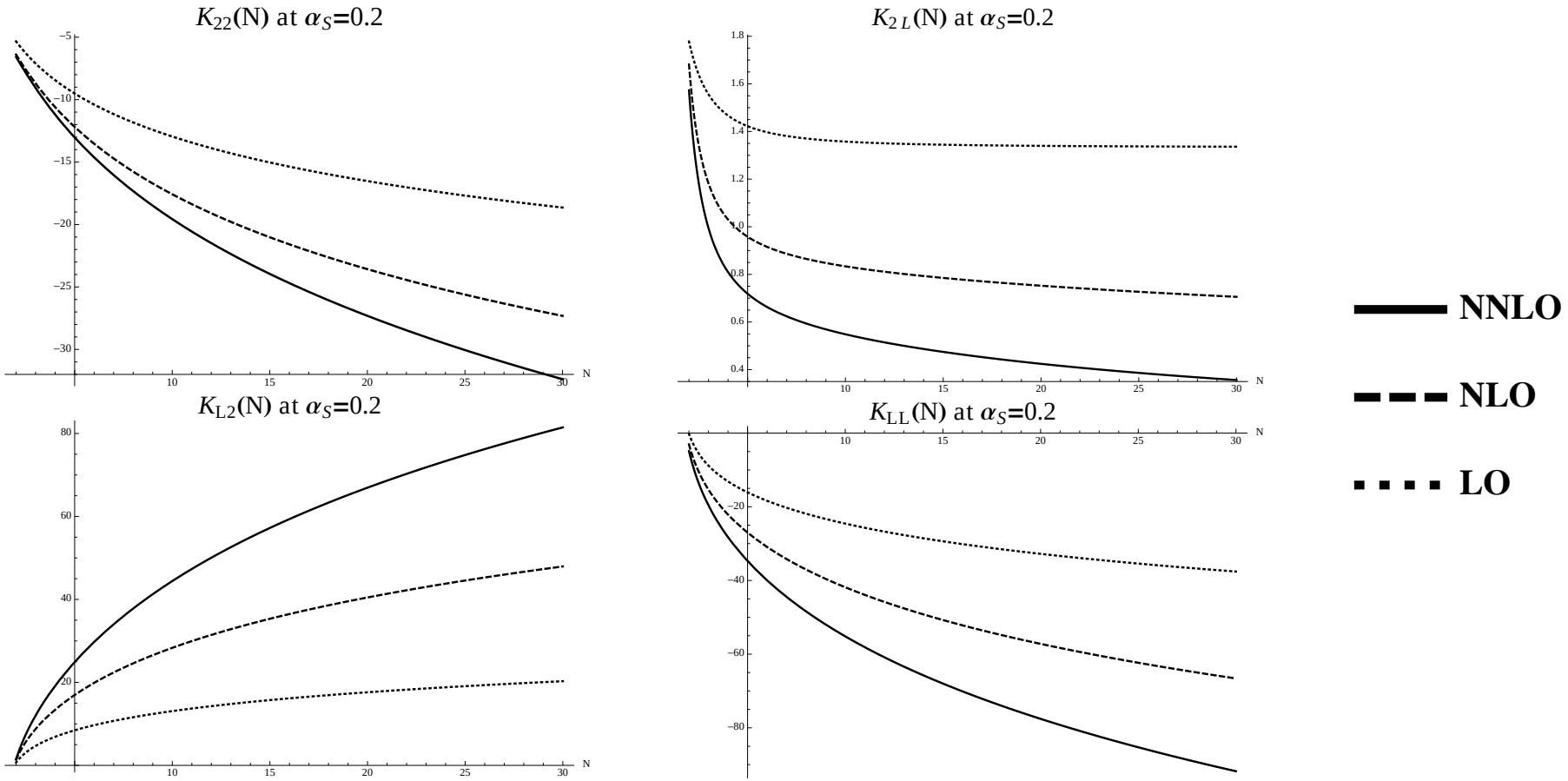
literature:

- 3-loop splitting functions [Moch, Vermaseren, Vogt, Nucl.Phys.B688:101-134,2004; Nucl.Phys. B691 (2004) 129-181]
- 2-loop coefficient for F_2 [van Neerven, Vogt, Nucl.Phys.B568:263-286,2000; Nucl.Phys. B588 (2000) 345-373]
- 3-loop coefficient for F_L [Moch, Vermaseren, Vogt, Phys.Lett. B606 (2005) 123-129]

results very lengthy: use x-space parametrization and transform them to moment space

NNLO evolution kernels

For the first time calculated! – very lengthy expressions, generated automatically



NNLO corrections even larger than NLO

Truncated Solution at NNLO

requires careful formulation using evolution matrices [R. K. Ellis, Z. Kunszt and E. M. Levin, Nucl. Phys. B 420, 517 (1994)]; [A. Vogt, CPC 170, 65 (2005)]

$$L^{(0)} = \left[\left(\frac{a_s}{a_0} \right)^{-\frac{\lambda_-}{\beta_0}} e_- + (+) \leftrightarrow (-) \right]$$

$$L^{(1)} = a_s U_1 L^{(0)} - a_0 L U_1$$

$$L^{(2)} = a_s^2 U_2 L - a_s a_0 U_1 L U_1 + a_0^2 L U_1^2 - a_0^2 U_2$$

evolution matrices through recursion relation

$$U_k = -\frac{1}{k} \left[e_- \tilde{R}_k e_- + e_+ \tilde{R}_k e_+ \right] + \frac{e_+ \tilde{R}_k e_-}{\lambda_-/\beta_0 - \lambda_+/\beta_0 - k} + \frac{e_- \tilde{R}_k e_+}{\lambda_+/\beta_0 - \lambda_-/\beta_0 - k}$$

$$\tilde{R}_k = R_k + \sum_{i=1}^{k-1} R_{k-1} U_i \qquad R_k = \frac{1}{\beta_0} K^{(k)} - \sum_{i=1}^{k} \frac{\beta_k}{\beta_0} R_{k-1}$$

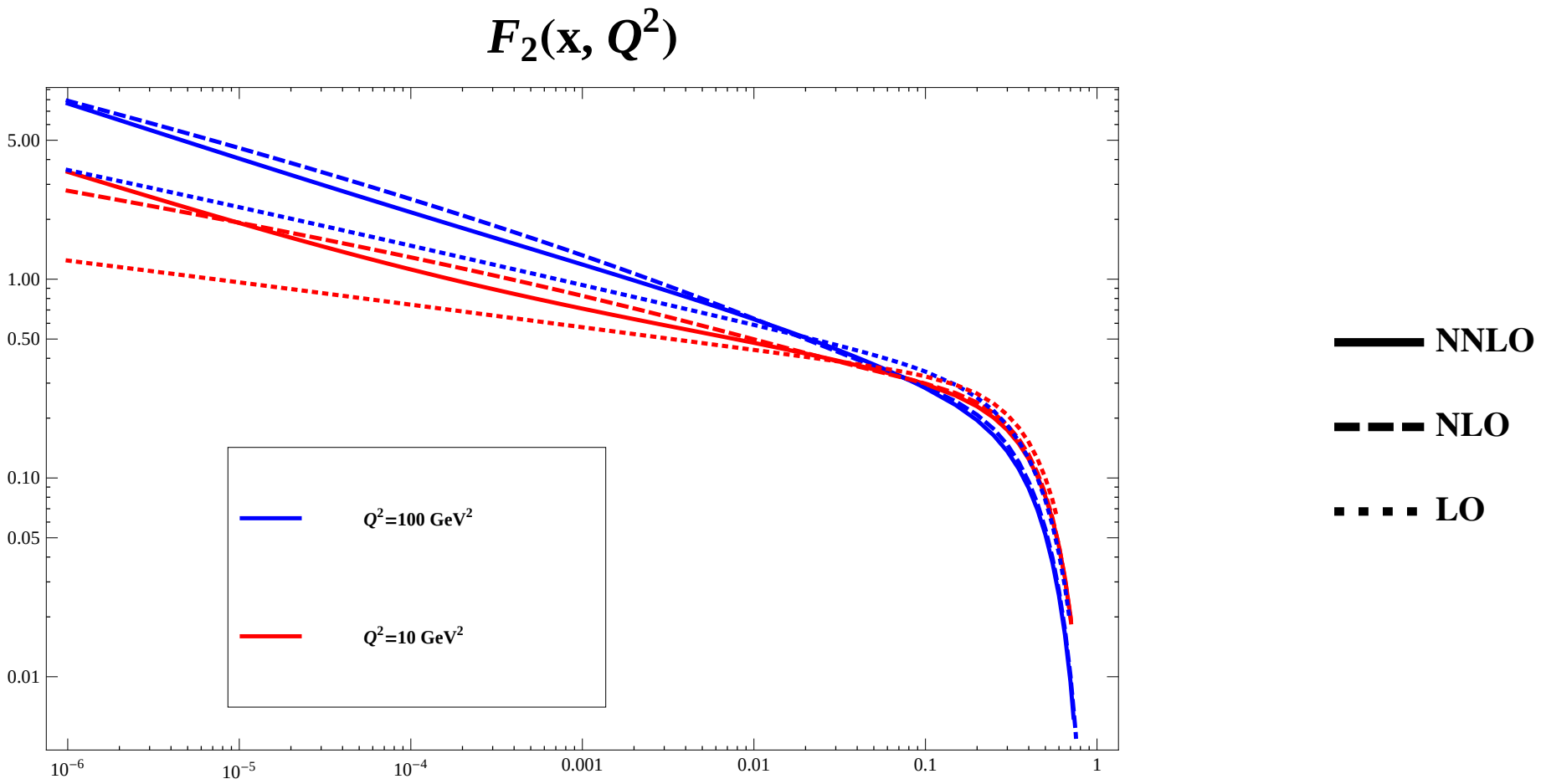
plug directly into *e.g.* MATHEMATICA: more than a minute for one point in N-space

Higher order zero solution at NNLO

analogy to NLO: can avoid spurious terms beyond NNLO entirely for the toy model

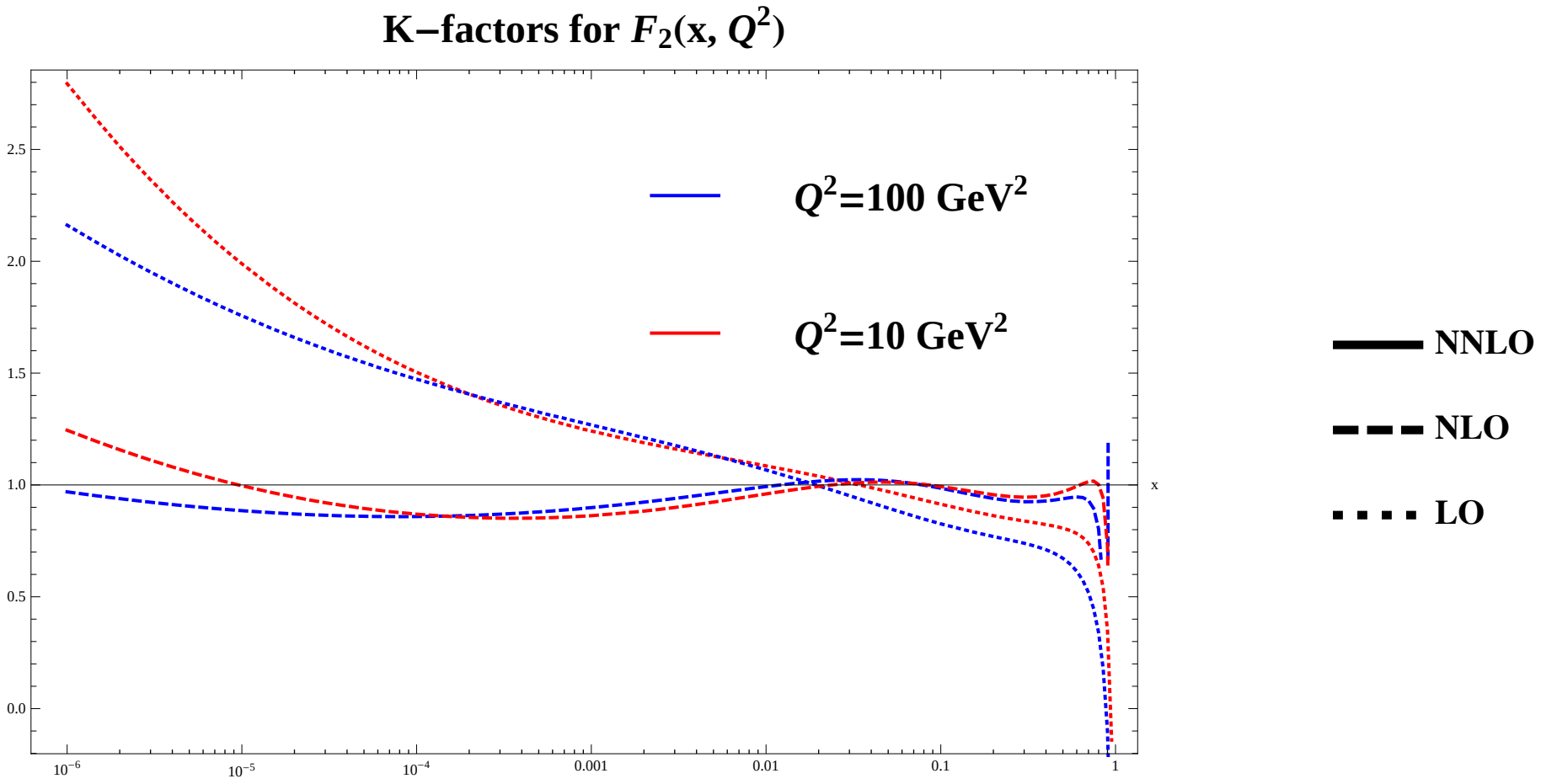
✓ find again: evolution in terms of $\overline{\text{MS}}$ splitting function and physical anomalous dimensions agrees exactly for "higher order zero" solution

Results including NNLO



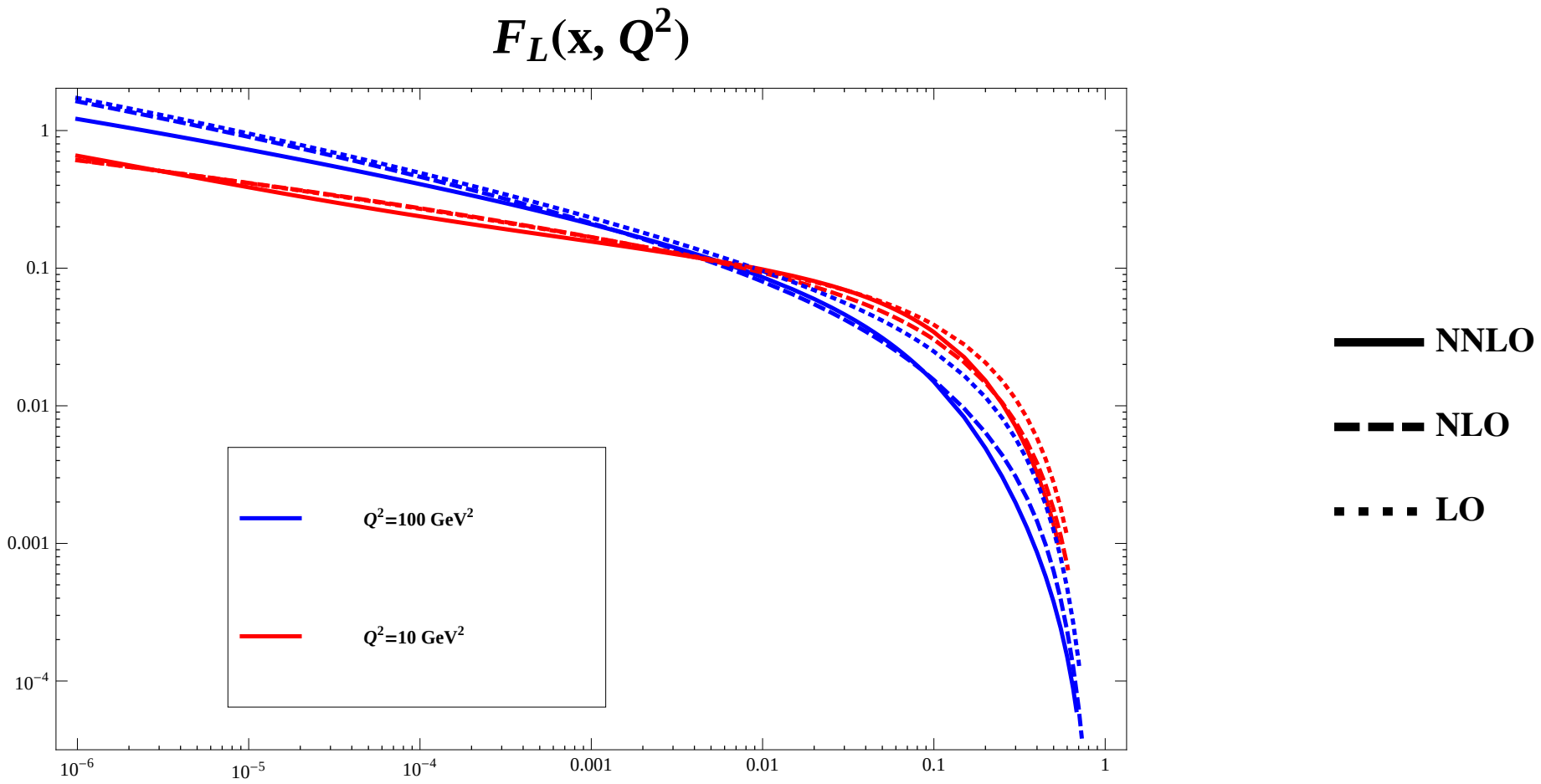
evolution of the doublet (F_2, F_L)

Results including NNLO



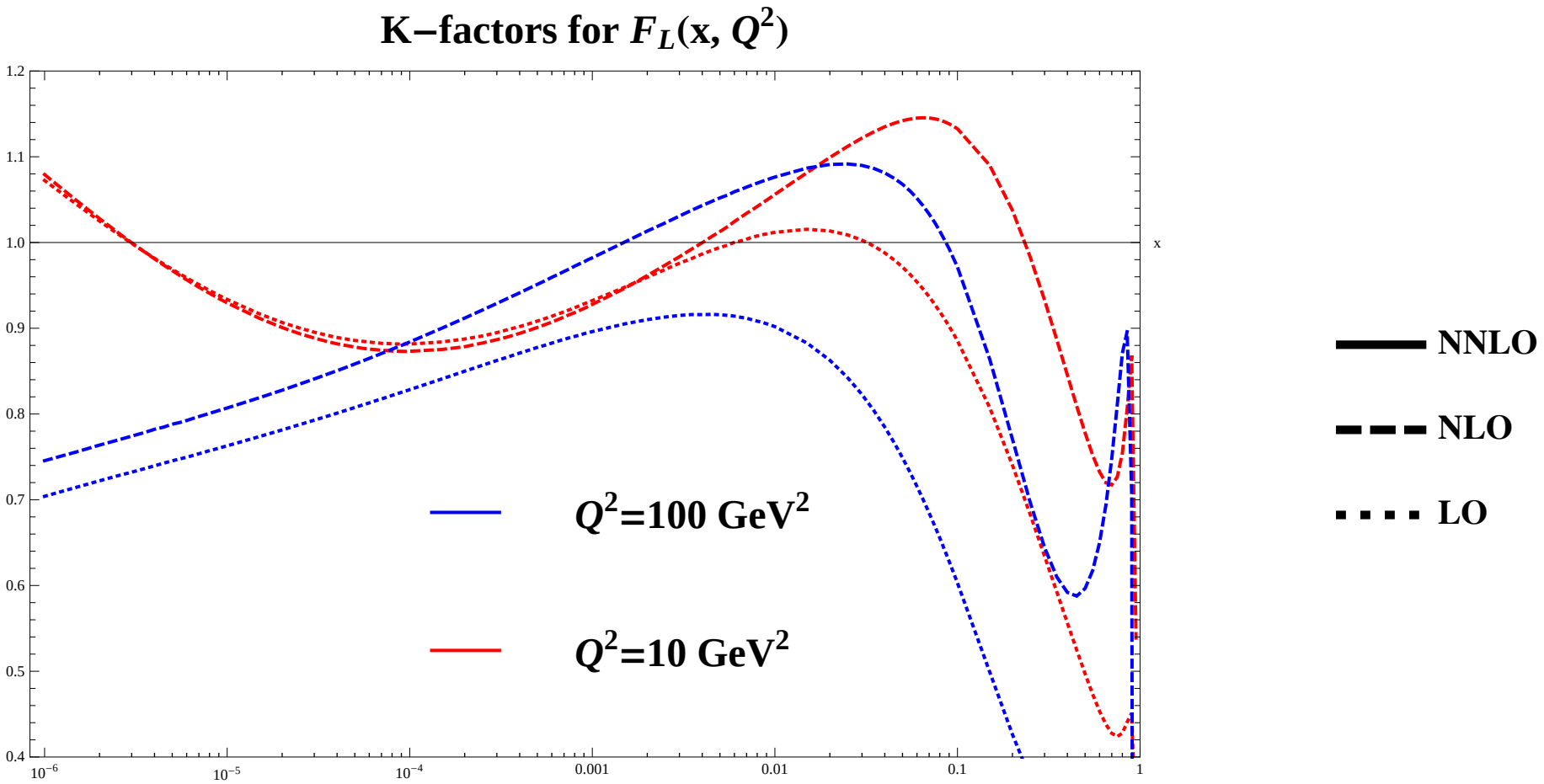
evolution of the doublet (F_2, F_L); ratios NNLO/NLO and NNLO/LO

Results including NNLO



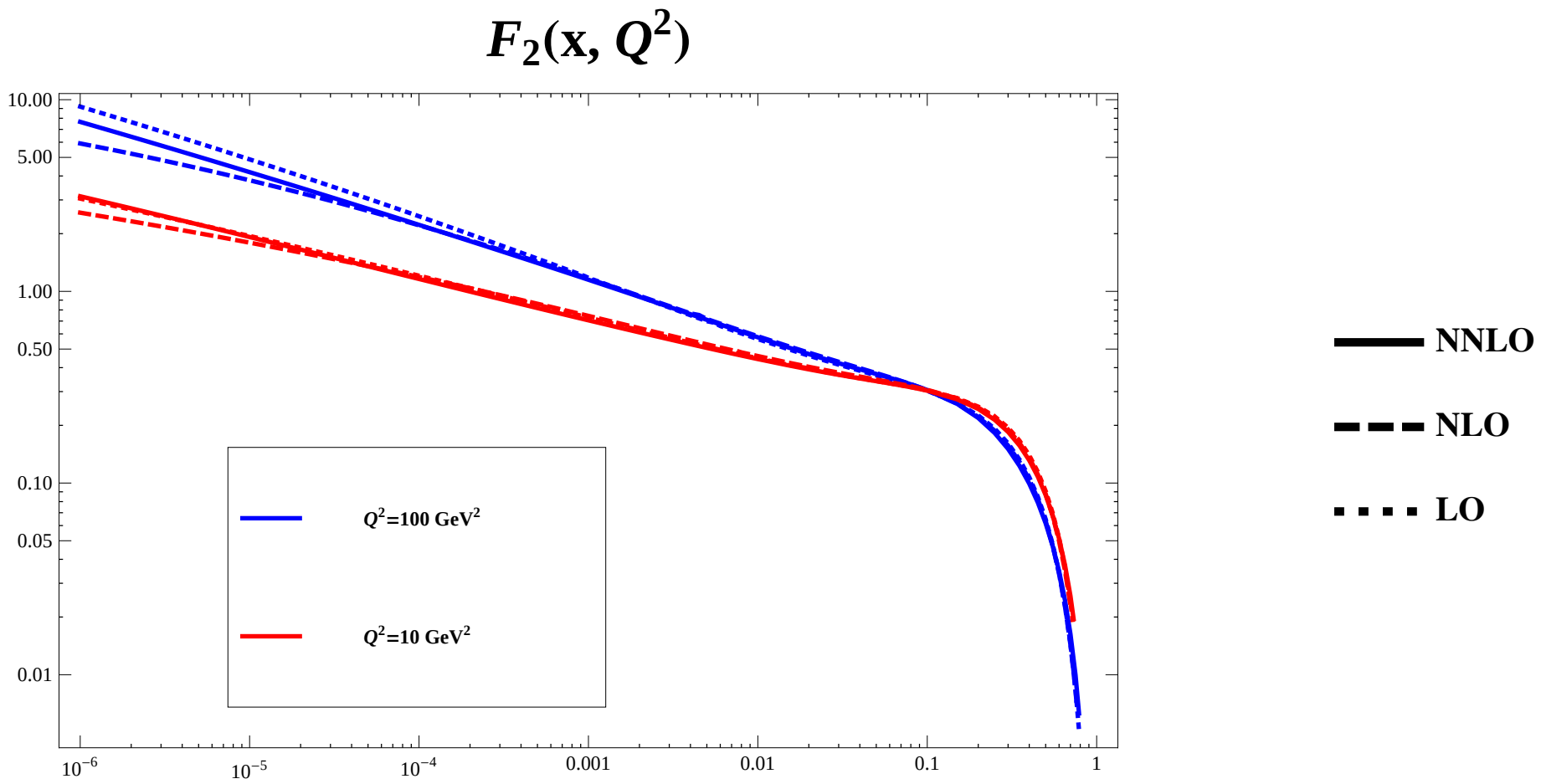
evolution of the doublet (F_2, F_L)

Results including NNLO



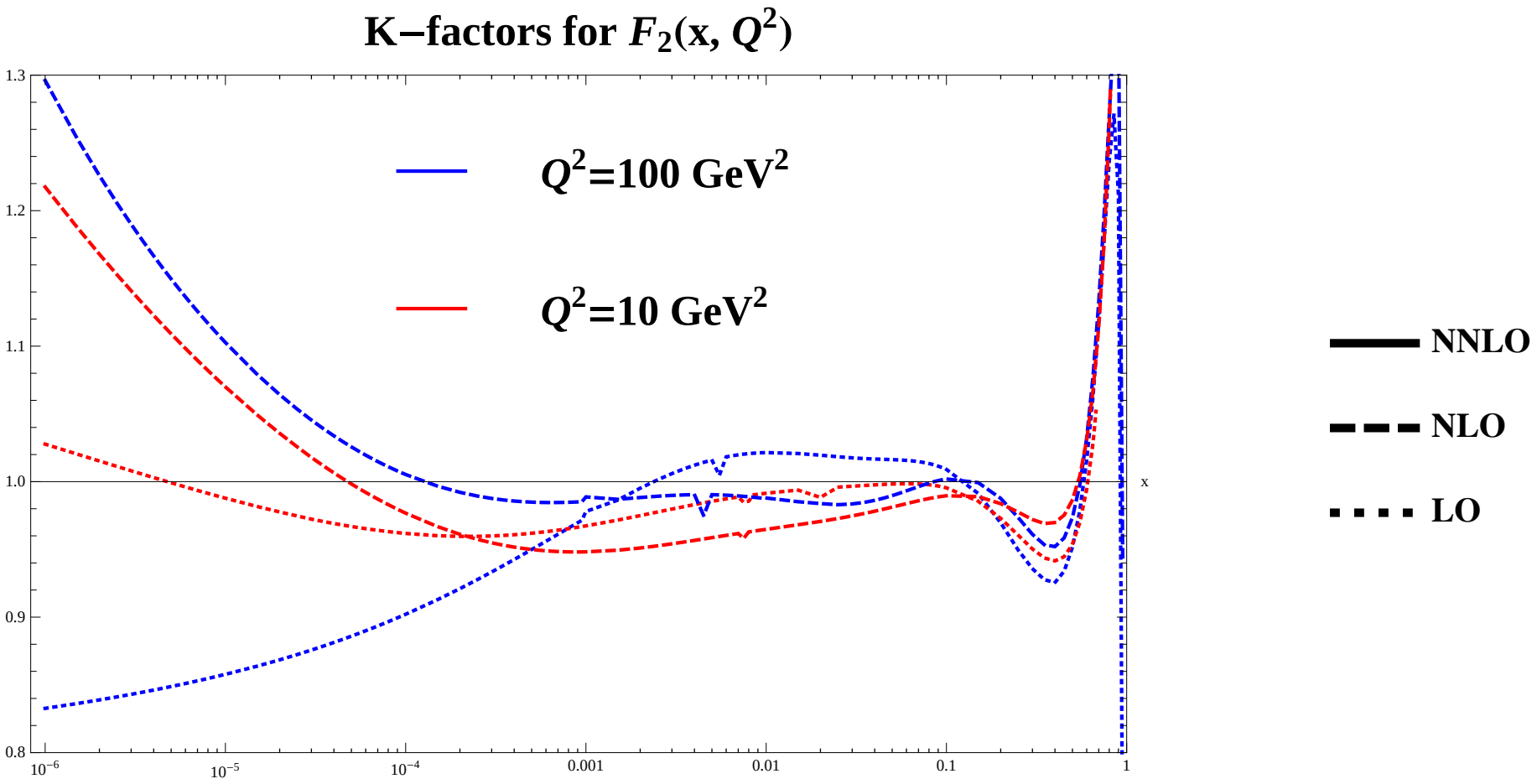
evolution of the doublet (F_2, F_L); ratios NNLO/NLO and NNLO/LO

Results including NNLO



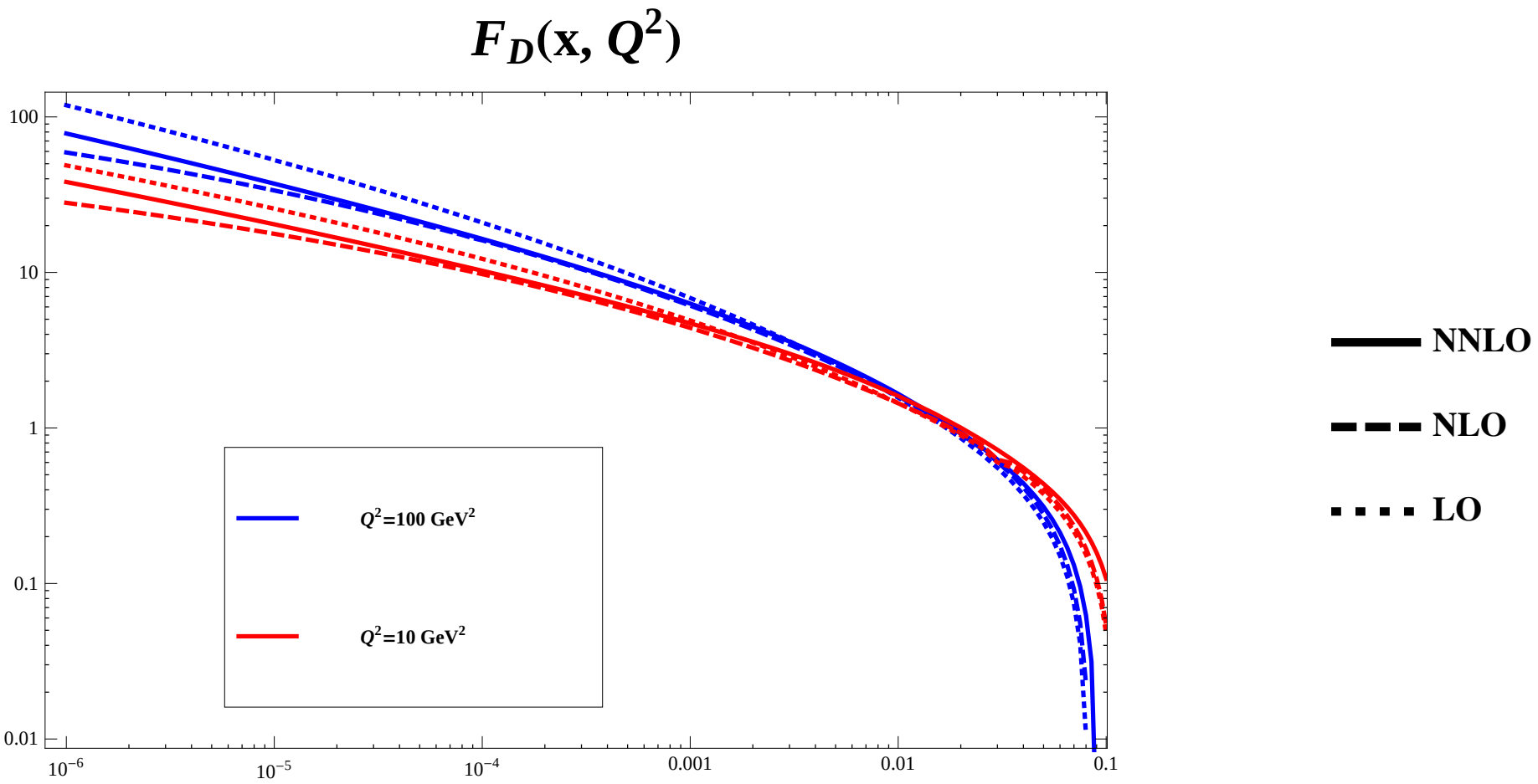
evolution of the doublet (F_2, F_D)

Results including NNLO



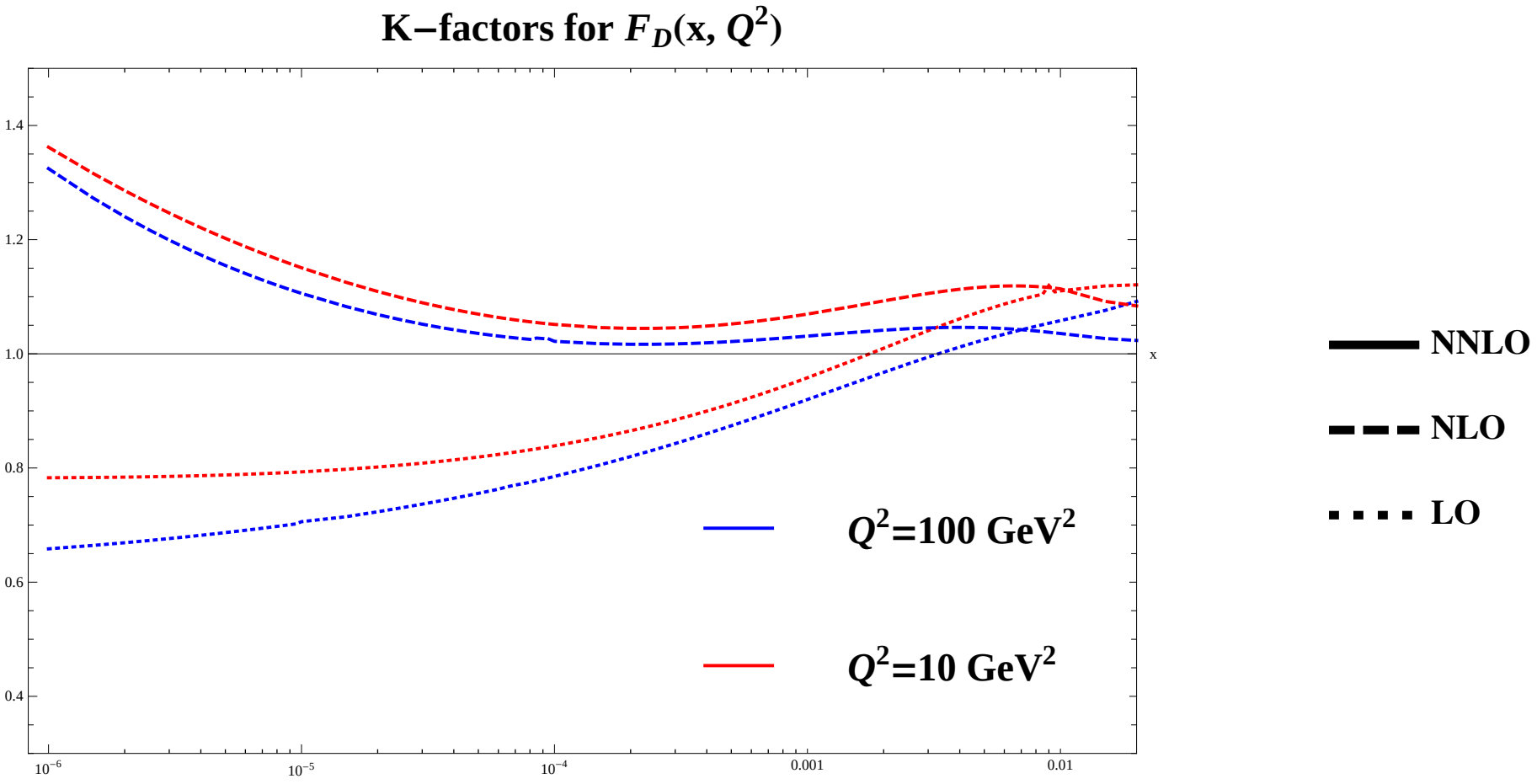
evolution of the doublet (F_2, F_D); ratios NNLO/NLO and NNLO/LO

Results including NNLO



evolution of the doublet (F_2, F_D)

Results including NNLO



evolution of the doublet (F_2, F_D); ratios NNLO/NLO and NNLO/LO

physical anomalous dimensions - a short summary

- Physical evolution kernels work ➡ can be applied to the analysis of small x data
- NLO and NNLO corrections numerically sizable ➡ absorb large higher order QCD corrections to the hard scattering Wilson coefficients
- NNLO suggests convergence of perturbative series
- to be done for phenomenology: heavy quarks
- expected instability of DGLAP at ultra-small $x \simeq 10^{-6}$ possible cure ➡ BFKL resummation of physical anomalous dimensions $\sum_k (a_s \ln 1/x)^k$ following *e.g.* [Catani, Hautmann; NPB 427 (1994) 475]

physical anomalous dimensions - outlook

direct evolution of (F_2, F_L) and/or $(F_2.F_D)$

✓ scheme and scale ambiguities removed

✓ initial conditions with less parameters

Two possible applications:

- search for deviations from linear DGLAP evolution
- use factorization scheme independence for precise determination of renormalization scheme ambiguity *i.e.* the QCD running coupling

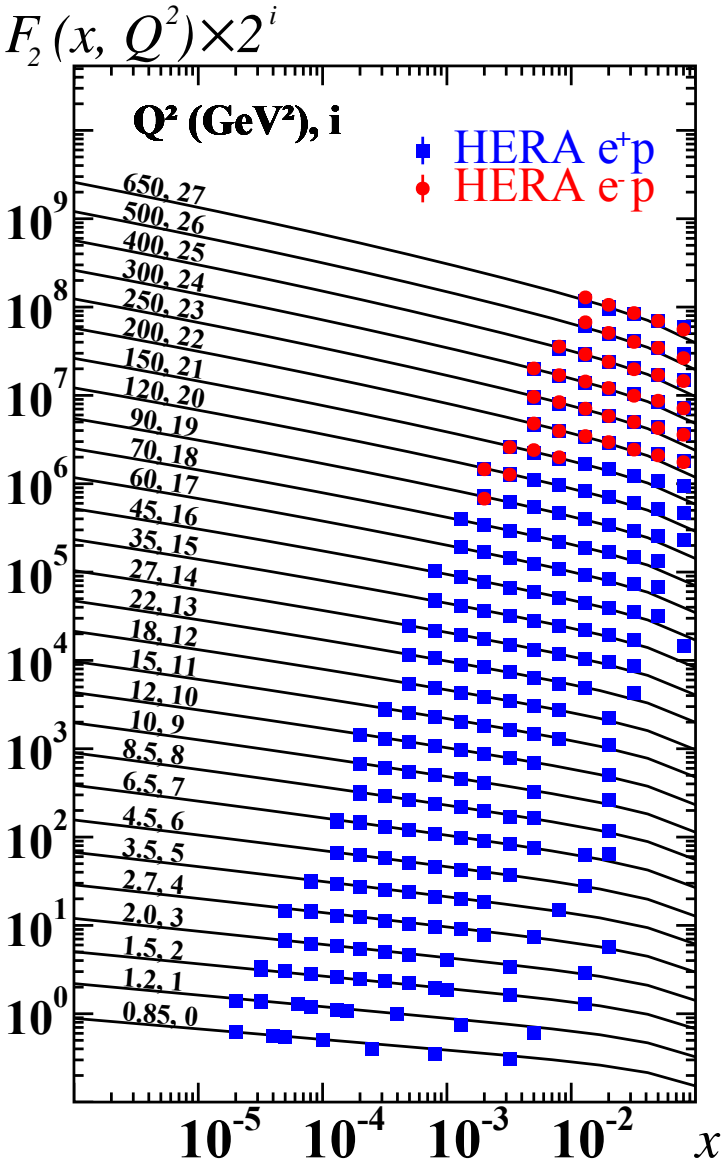
Outlook – First excercise: IP-sat model

saturation models with matching to DGLAP evolution [J. Bartels, K. J. Golec-Biernat and H. Kowalski, Phys. Rev. D66, 014001 (2002).];[H. Kowalski and D. Teaney, Phys. Rev. D68, 114005 (2003)];[H. Kowalski, L. Motyka and G. Watt, Phys. Rev. D74, 074016 (2006)];
with impact parameter dependence [H. Kowalski, T. Lappi and R. Venugopalan, Phys. Rev. Lett. 100, 022303 (2008)]; [H. Kowalski, T. Lappi, C. Marquet and R. Venugopalan, Phys. Rev. C78, 045201 (2008)]

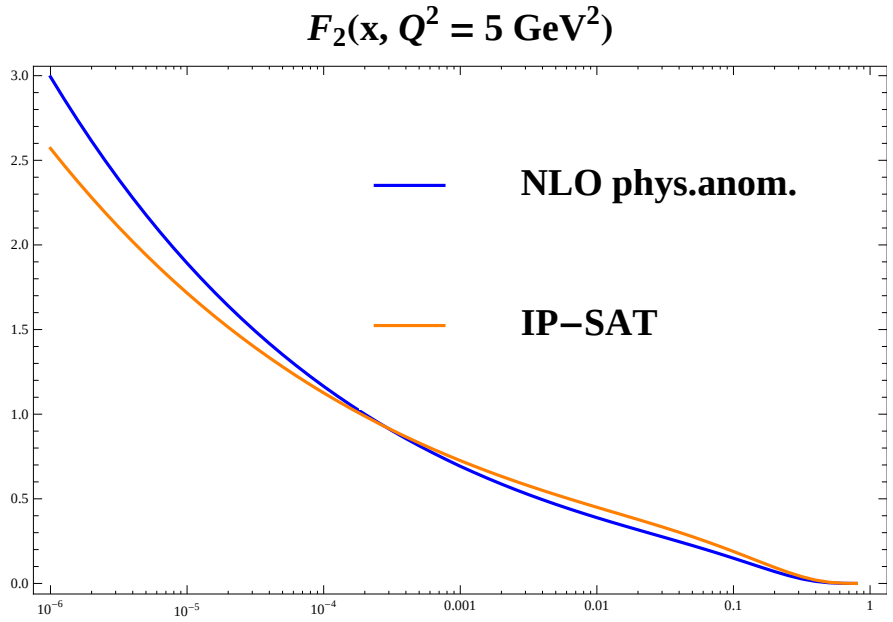
recent fit with combined HERA data:
A. H. Rezaeian, M. Siddikov, M. Van de Klundert and R. Venugopalan, Phys. Rev. D 87, no. 3, 034002 (2013)

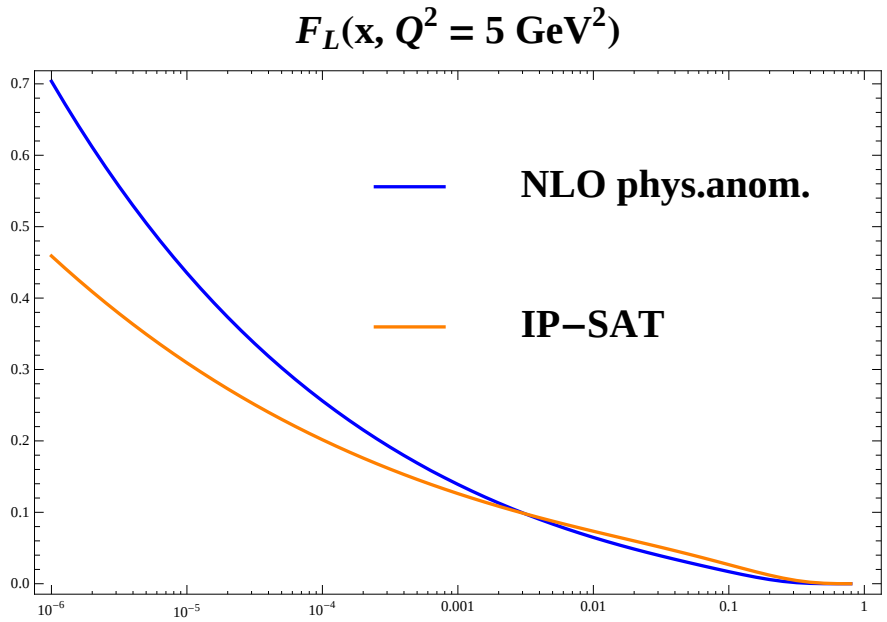
allows to calculate (F_2, F_L) even in the un-described large x region at fixed Q^2 values [thanks to A. H. Rezaeian, M. Siddikov!]

➡ take IP-sat at $Q_0^2 = 2\text{GeV}^2$, evolve with DGLAP and compare at higher Q^2 values



Some very preliminary results ....





largest difference for F_L – bascially no saturation effects in ep for EIC kinematics

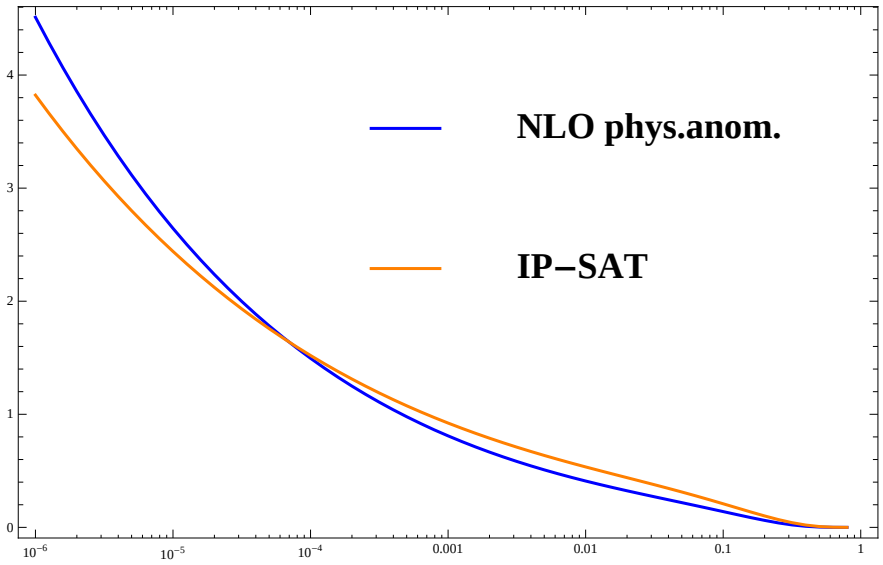
caveats:

- IPsat contains DGLAP resummation itself
- possible effects due to unphysical extrapolation to large x?
- quark masses

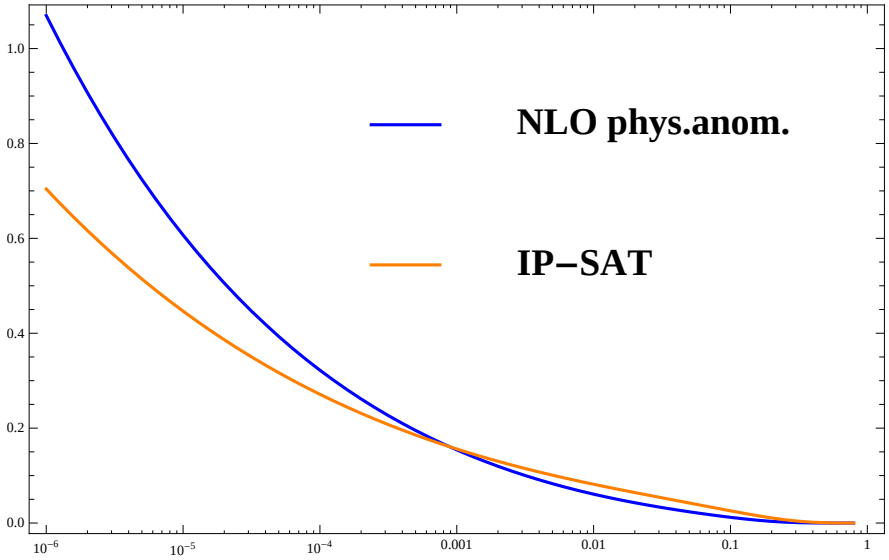
maybe more interesting: same exercise for large nucleus ...

Backup

$F_2(x, Q^2 = 10\ \mathrm{GeV}^2)$

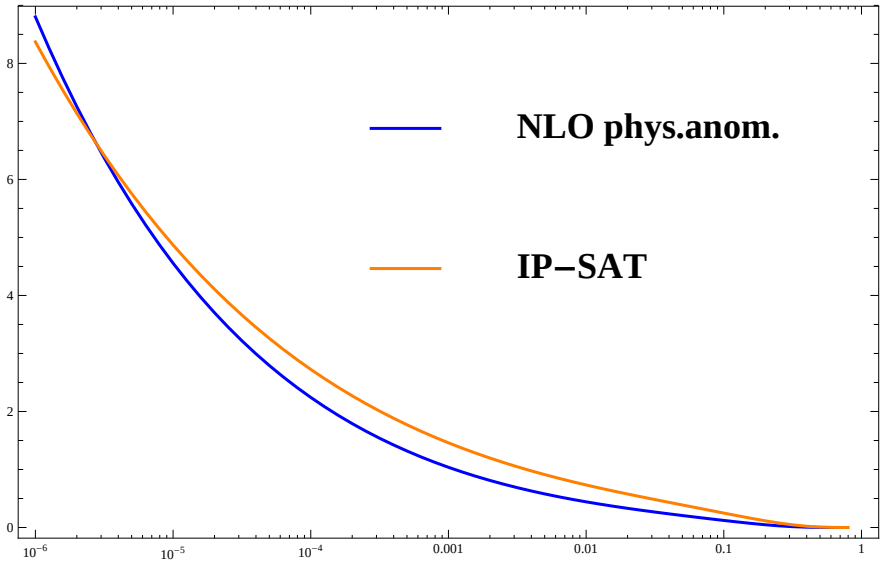
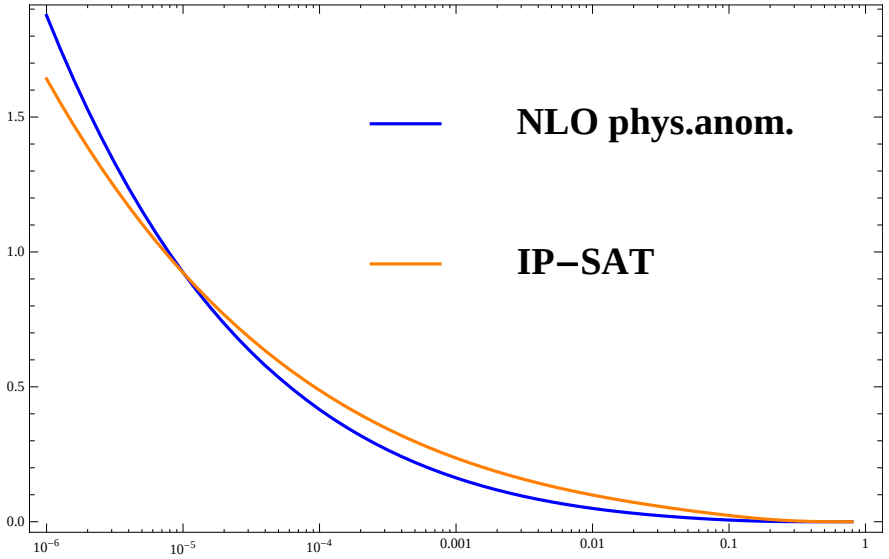


$F_L(x, Q^2 = 10\ \mathrm{GeV}^2)$



Backup

$F_2(x, Q^2 = 50\ \mathrm{GeV}^2)$



$F_L(x, Q^2 = 50\ \mathrm{GeV}^2)$