A non-perturbative renormalization scheme in position space

Piotr Korcyl

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work done in collaboration with Krzysztof Cichy (DESY, Zeuthen; now at Frankfurt Univ.) and Karl Jansen (DESY, Zeuthen)

+ some new results (preliminary)
Motivation

Renormalization constants can contribute significantly to the error budget of some physical quantities computed on the lattice.

One would like to upgrade existing schemes or propose new (hopefully more precise) schemes

- schemes which need separate simulations
  - SF scheme
  - $\chi$SF scheme
  - ...

- schemes which do not need separate simulations
  - RI-MOM scheme and its upgraded versions
  - X-space scheme / coordinate scheme
  - ...
Motivation: digression

\( \chi \text{SF} \)

\( \chi \text{SF} \) is constructed out of the regular SF framework by changing the boundary conditions, so that the \( \mathcal{O}(a) \) improvement is automatic. Similarly to SF, one can define correlation functions with source at one boundary and an insertion of the current in the bulk. [Sint, 2011]

![Graph showing comparison of \( Z_A \) calculated from the Ward identity (Nucl.Phys. B865 (2012) 397) and from \( \chi \text{SF} \) (left) and check of the \( \mathcal{O}(a) \) improvement (right). [Mattia dalla Brida, PhD thesis & Latt14]](image-url)
Motivation

Our main aim: investigate whether it is possible (and practical) to use the X-space renormalization scheme to extract renormalization constants for ETMC $N_f = 2$ ensembles.

Method of:

Compare values of RCs with 2 papers:

Applied to Möbius domain-wall fermions by M. Tomii and S. Hashimoto, Latt14
In the remainder of the talk:

1. Method and analysis procedure
2. Setup
3. Example – extraction of $Z_P$
4. Renormalization constants – summary, comparison with RI-MOM
5. Non-perturbative running of the renormalization constants
6. Conclusions and prospects
Consider the correlation functions of flavor non-singlet bilinear quark operators of the form $\langle O_\Gamma(x) O_\Gamma(0) \rangle$, where:

$$ O_\Gamma(x) = \bar{\psi}(x) \Gamma \psi(x), \quad \Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5\}. $$

Impose the following coordinate space conditions in the chiral limit:

$$ \lim_{a \to 0} \left| \langle O^X_\Gamma(x) O^X_\Gamma(0) \rangle \right|_{x^2 = x_0^2} = \langle O_\Gamma(x_0) O_\Gamma(0) \rangle^\text{free, massless}_\text{cont}. $$

The renormalized operator is

$$ O^X_\Gamma(x, x_0) = Z^X_\Gamma(x_0) O_\Gamma(x), $$

$x_0$ is the renormalization point, which must satisfy:

$$ a \ll x_0 \ll \Lambda_{\text{QCD}}^{-1} $$

to keep the discretization & non-perturbative effects under control.
The renormalization condition should be understood in the following way: For every finite value of the lattice spacing $a$ and quark mass $\mu$ (i.e. for each ensemble) and for each correlator type, we impose the condition:

$$(Z_X^\Gamma(x_0; a, \mu))^2 \langle O_\Gamma(x_0)O_\Gamma(0) \rangle_{\text{lat}} = \langle O_\Gamma(x_0)O_\Gamma(0) \rangle_{\text{cont}}^{\text{free, massless}},$$

from which the renormalization constant at this value of $a$, $Z_X^\Gamma(x_0; a, \mu)$ can be calculated.

Comments:

- we use correlation functions which are of primary interest in any simulation
- we use gauge-invariant objects: no gauge fixing required
- the presented application deals with ensembles at maximal twist, i.e. chiral extrapolation performed in $a\mu$
Choose ensemble and compute the correlation function of interest (for all $\mathcal{X} \equiv (x, y, z, t) \equiv (x_1, x_2, x_3, x_4)$).

1. For each configuration, average over equivalent sites
2. For each $\mathcal{X}$, take the ensemble average
3. Correct for tree-level discretization effects
4. Apply some cuts to eliminate points with large remaining discretization effects
5. Average over remaining points with equal $\mathcal{X}^2 = \sum_{\mu} x_\mu x_\mu$
6. Compute the $\mathcal{X}^2$ dependence of RCs in the $\mathcal{X}$-space scheme $Z_{\Gamma}^X(x_0; a, \mu)$
7. Choose the interval to extract the final value of RC: $a \ll \mathcal{X} \ll \Lambda_{\text{QCD}}^{-1}$
8. (comparison with other methods) Convert to the $\overline{\text{MS}}$ scheme, evolve to $\mu = 2$ GeV
Ensembles

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\kappa$</th>
<th>$(L/a)^3 \times T/a$</th>
<th>$a\mu$</th>
<th>Label</th>
<th>lattice spacing [fm]</th>
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<tr>
<td>3.9</td>
<td>0.160856</td>
<td>$24^3 \times 48$</td>
<td>0.0040</td>
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<td>0.079(1)</td>
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<td>0.0150</td>
<td>B150.24</td>
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<tr>
<td>4.05</td>
<td>0.157010</td>
<td>$32^3 \times 64$</td>
<td>0.0030</td>
<td>C30.32</td>
<td>0.063(1)</td>
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<tr>
<td></td>
<td></td>
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<td>0.0060</td>
<td>C60.32</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>0.0080</td>
<td>C80.32</td>
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<tr>
<td>4.20</td>
<td>0.154073</td>
<td>$48^3 \times 96$</td>
<td>0.0020</td>
<td>D20.48</td>
<td>0.051(1)</td>
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<tr>
<td>4.35</td>
<td>0.151740</td>
<td>$32^3 \times 64$</td>
<td>0.00175</td>
<td>E17.32</td>
<td>0.042(1)</td>
</tr>
</tbody>
</table>

Details:

- tree-level Symanzik improved gauge action and 2 flavours of twisted mass fermions
- fermions: periodic boundary conditions in spatial directions, antiperiodic in time
Details of analysis

- Ensemble: \( \beta = 4.35, \frac{L}{a} = 32, a\mu = 0.00175, a \approx 0.042 \text{ fm} \).
- Flavour non-singlet charged pseudoscalar correlator.

**Stage 1.** For each configuration, average over equivalent sites:

- permutations of \((x, y, z, t)\): \((y, z, x, t), (z, x, y, t), (x, z, y, t), (y, x, z, t), (z, y, x, t)\)
- permutations of \((L - x, y, z, t)\): 6 of them
- permutations of \((x, L - y, z, t)\): 6 of them
- permutations of \((x, y, L - z, t)\): 6 of them
- permutations of \((L - x, L - y, z, t)\): 6 of them
- permutations of \((L - x, y, L - z, t)\): 6 of them
- permutations of \((x, L - y, L - z, t)\): 6 of them
- permutations of \((L - x, L - y, L - z, t)\): 6 of them

+ all of the above with \(t \rightarrow T - t\)

In total: max. 96 equivalent sites (fewer if some coordinates zero or equal)

**Stage 2.** For each \(X\), take the ensemble average.
Let’s see the two correlators we are dealing with:

the free PP correlator computed in the continuum and the interacting correlator from the lattice.
By naively averaging over all points with equal $X^2$ we average points with different cut-off effects.
Therefore we first need to account for the cut-off effects.

Stage 3. Correct for tree-level discretization effects.
Tree-level correction

To subtract the free-field discretization errors, we computed a correction factor $\Delta_\Gamma$, defined as the ratio of the free correlator on the lattice over the continuum one computed in the infinite volume and in the chiral limit:

$$\Delta_\Gamma = \frac{\langle O_\Gamma(x)O_\Gamma(0) \rangle_{\text{free}}^{\text{lat}}}{\langle O_\Gamma(x)O_\Gamma(0) \rangle_{\text{free}}^{\text{cont}}} = \frac{\langle O_\Gamma(x)O_\Gamma(0) \rangle_{\text{free}}^{\text{lat}}}{\frac{c}{\pi^4(x^2)^3}}$$

where $c = 3$ for PP and SS correlators, $c = 6$ for VV and AA correlators.

Thus, we obtained the corrected correlation functions, $C'_{\Gamma\Gamma}(x)$:

$$C'_{\Gamma\Gamma}(x) = \frac{C_{\Gamma\Gamma}(x)}{\Delta_\Gamma(x)}$$
Tree-level corrected correlator

\[ \log C(X/a) \]

\[ (X/a)^2 \]

Free continuum theory
\[ \beta = 4.35, \ L/a = 32, \ a\mu = 0.00175 \]

\[ P^+P^+ \text{ correlator} \]
\[ \text{corr., all points} \]

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Stage 4. Apply some cuts to eliminate points with large remaining discretization effects.

- If the position component along one or more of the lattice axes is zero, one can expect enhanced discretization effects – breaking of Lorentz symmetry.
- The least affected points are along the hypercubic diagonal (more precisely: direction \((1,1,1,1)\)).
- Choose the following criterion: compute the angle between the given position vector of point \(X\) and the hypercubic diagonal – call it \(\theta(X)\).

Remove points with \(\theta(X) > \theta_{\text{max}}\)

- We chose \(\theta_{\text{max}} = \frac{\pi}{6}\) (30 degrees).
- We call the remaining points democratic
The role of democracy

Free continuum theory

$L/a=32$, $T/a=64$, $a\mu=0.012500$
Corrected correlator

log C(X/a) vs (X/a)^2

Free continuum theory

\beta = 4.35, L/a = 32, a \mu = 0.00175

P^+P^+ correlator
corr., democratic

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In order to extract the renormalization constants, we impose the X-space renormalization condition directly on the corrected correlation functions $C'_Γ(x)$, namely:

$$\left. \frac{\langle O^X_Γ(x)O^X_Γ(0) \rangle'}{\langle O_Γ(x)O_Γ(0) \rangle_{\text{cont}}^{\text{free}}} \right|_{x^2 = x^2_0} = \left. \frac{\langle O^X_Γ(x)O^X_Γ(0) \rangle}{\langle O_Γ(x)O_Γ(0) \rangle_{\text{lat}}^{\text{free}}} \right|_{x^2 = x^2_0} = 1$$

Since:

$$O^X_Γ(x) = Z^X_Γ(x_0)O_Γ(x),$$

we have:

$$Z^X_Γ(x_0) = \sqrt{\frac{C_Γ(x_0)}{C'_Γ(x_0)}} = \sqrt{\frac{C_Γ(x_0)}{C_Γ(x_0)}}.$$
**Stage 6.** Compute the $X^2$ dependence of RCs in the X-space scheme.
Stage 6. Compute the $X^2$ dependence of RCs in the X-space scheme

$\beta=4.35$, $L/a=32$, $\alpha_\mu=0.00175$
Stage 7. Choose the interval to extract the final value of RC: 

\[ a \ll X \ll \Lambda_{\text{QCD}}^{-1} \]

Chosen intervals:

- \((X/a)^2 = [7, 13]\), i.e. [700,950] MeV for B-ensembles
- \((X/a)^2 = [7, 15]\), i.e. [800,1200] MeV for C-ensembles
- \((X/a)^2 = [10, 18]\), i.e. [900,1250] MeV for D-ensemble
- \((X/a)^2 = [10, 21]\), i.e. [1000,1500] MeV for E-ensemble

The result of this stage is the value of the chosen RC in the X-space scheme, at some energy scale \(1/X\) of \(\approx 1\) GeV.

However, this value is still subject to some remaining fluctuations.

Therefore, averaging within the chosen interval is recommended. To perform this averaging, one needs to know the running of the RC with the scale.
Stage 8. Convert to the $\overline{\text{MS}}$ scheme, evolve to $\mu = 2 \text{ GeV}$

This also serves the purpose of averaging with respect to $X^2$:

- $Z_V, Z_A$: scale independent in the $\overline{\text{MS}}$ scheme, BUT scale dependent in the X-space scheme:
  \[
  Z^X_{V,A}(1/X) \xrightarrow{\text{conversion}} Z^\overline{\text{MS}}_{V,A}(\mu = 2e^{-\gamma_E}/X) = Z^\overline{\text{MS}}_{V,A}(2 \text{ GeV})
  \]

- $Z_P, Z_S$: scale dependent in the $\overline{\text{MS}}$ scheme and also in the X-space scheme:
  \[
  Z^X_{P,S}(1/X) \xrightarrow{\text{conversion}} Z^\overline{\text{MS}}_{P,S}(\mu = 2e^{-\gamma_E}/X) \xrightarrow{\text{evolution}} Z^\overline{\text{MS}}_{P,S}(2 \text{ GeV})
  \]

N$^4$LO (4-loop) calculation in:
### Explicit example

<table>
<thead>
<tr>
<th>$(X/a)^2$</th>
<th>$1/X$ [MeV]</th>
<th>$Z_p^X(1/X)$</th>
<th>$\overline{\text{MS}}$ scale [MeV]</th>
<th>$Z_p^{\overline{\text{MS}}}(\mu = 2$ GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1490(35)</td>
<td>0.4850(8)</td>
<td>1670(40)</td>
<td>0.4921(9)(29)(2)</td>
</tr>
<tr>
<td>12</td>
<td>1360(30)</td>
<td>0.4911(6)</td>
<td>1520(35)</td>
<td>0.5101(7)(32)(7)</td>
</tr>
<tr>
<td>13</td>
<td>1300(30)</td>
<td>0.4697(10)</td>
<td>1460(35)</td>
<td>0.4932(11)(33)(11)</td>
</tr>
<tr>
<td>15</td>
<td>1210(30)</td>
<td>0.4734(8)</td>
<td>1360(30)</td>
<td>0.5075(9)(36)(20)</td>
</tr>
<tr>
<td>16</td>
<td>1170(30)</td>
<td>0.4549(17)</td>
<td>1320(30)</td>
<td>0.4924(18)(36)(24)</td>
</tr>
<tr>
<td>18</td>
<td>1110(25)</td>
<td>0.4574(10)</td>
<td>1240(30)</td>
<td>0.5047(11)(40)(35)</td>
</tr>
<tr>
<td>20</td>
<td>1050(25)</td>
<td>0.4589(13)</td>
<td>1180(30)</td>
<td>0.5156(14)(44)(47)</td>
</tr>
<tr>
<td>21</td>
<td>1025(25)</td>
<td>0.4425(14)</td>
<td>1150(25)</td>
<td>0.5017(16)(45)(51)</td>
</tr>
</tbody>
</table>

Error for X-space scheme: statistical

Errors for $\overline{\text{MS}}$:

- 1st = statistical,
- 2nd = uncertainty in lattice spacing,
- 3rd = uncertainty of $\Lambda_{\overline{\text{MS}}, N_f=2}$
  - $310(25)$ MeV [ALPHA],
  - $315(30)$ MeV [ETMC],
\[ Z_P^{\overline{\text{MS}}}(\mu = 2 \text{ GeV}) = 0.503(1)(3)(2) = 0.503(6) \]
$Z_S$

\begin{align*}
\beta &= 4.35, \ L/a = 32, \ a\mu = 0.00175 \\
Z_P &= 0.5012 (6)(36)(20)(563) = 0.5012 (565) \\
\beta &= 4.35, \ L/a = 32, \ a\mu = 0.00175 \\
Z_P &= 0.5751 (11)(40)(19)(322) = 0.5751 (325) \\
\beta &= 4.35, \ L/a = 32, \ a\mu = 0.00175 \\
Z_P &= 0.5346 (6)(38)(21)(260) = 0.5346 (264) \\
\beta &= 4.35, \ L/a = 32, \ a\mu = 0.00175 \\
Z_P &= 0.5033 (10)(35)(17)(118) = 0.5033 (124)
\end{align*}
$Z_V$

$Z_A = 0.7952 (11)(3)(11)(1495) = 0.7952 (1496)$

$\beta = 4.35, L/a = 32, a\mu = 0.00175$

$Z_A = 0.8844 (15)(3)(12)(761) = 0.8844 (762)$

$\beta = 4.35, L/a = 32, a\mu = 0.00175$

$Z_A = 0.7716 (11)(3)(11)(409) = 0.7716 (409)$

$\beta = 4.35, L/a = 32, a\mu = 0.00175$

$Z_A = 0.7677 (13)(3)(10)(108) = 0.7677 (109)$

$\beta = 4.35, L/a = 32, a\mu = 0.00175$
## Comparison with RI-MOM

<table>
<thead>
<tr>
<th></th>
<th>$Z_P$</th>
<th>$Z_S$</th>
<th>$Z_V$</th>
<th>$Z_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B40.24</td>
<td>0.458(2)(5)(21)</td>
<td>0.765(3)(8)(38)</td>
<td>0.626(1)(1)(6)</td>
<td>0.729(2)(1)(7)</td>
</tr>
<tr>
<td>B64.24</td>
<td>0.462(2)(5)(23)</td>
<td>0.776(3)(4)(41)</td>
<td>0.629(1)(1)(6)</td>
<td>0.725(2)(1)(7)</td>
</tr>
<tr>
<td>B85.24</td>
<td>0.463(2)(5)(22)</td>
<td>0.778(4)(9)(41)</td>
<td>0.629(2)(1)(6)</td>
<td>0.725(2)(1)(7)</td>
</tr>
<tr>
<td>B100.24</td>
<td>0.493(5)(6)(31)</td>
<td>0.835(10)(10)(51)</td>
<td>0.673(7)(1)(7)</td>
<td>0.766(8)(1)(7)</td>
</tr>
<tr>
<td>B150.24</td>
<td>0.460(2)(5)(21)</td>
<td>0.807(7)(9)(45)</td>
<td>0.648(3)(1)(6)</td>
<td>0.725(4)(1)(7)</td>
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<tr>
<td>chiral</td>
<td>0.460(2)(5)(25)</td>
<td>0.750(5)(10)(47)</td>
<td>0.616(2)(1)(7)</td>
<td>0.729(3)(1)(8)</td>
</tr>
<tr>
<td>[1]</td>
<td>0.437(7)</td>
<td>0.713(10)</td>
<td>0.624(4)</td>
<td>0.746(6)</td>
</tr>
<tr>
<td>[2]</td>
<td>0.457(10)(16)</td>
<td>0.726(5)(11)</td>
<td>0.627(1)(3)</td>
<td>0.758(1)(1)</td>
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<tr>
<td>C30.32</td>
<td>0.469(1)(3)(7)</td>
<td>0.705(3)(5)(12)</td>
<td>0.647(1)(1)(2)</td>
<td>0.731(2)(1)(2)</td>
</tr>
<tr>
<td>C60.32</td>
<td>0.475(2)(3)(8)</td>
<td>0.709(3)(5)(13)</td>
<td>0.648(2)(1)(2)</td>
<td>0.732(2)(1)(2)</td>
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<td>C80.32</td>
<td>0.480(2)(3)(7)</td>
<td>0.724(4)(5)(14)</td>
<td>0.652(2)(1)(2)</td>
<td>0.743(3)(1)(2)</td>
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<td>chiral</td>
<td>0.463(2)(5)(11)</td>
<td>0.692(6)(9)(20)</td>
<td>0.644(2)(1)(3)</td>
<td>0.723(3)(1)(4)</td>
</tr>
<tr>
<td>[1]</td>
<td>0.477(6)</td>
<td>0.699(6)</td>
<td>0.659(3)</td>
<td>0.772(6)</td>
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<tr>
<td>[2]</td>
<td>0.497(8)(15)</td>
<td>0.691(9)(16)</td>
<td>0.662(1)(3)</td>
<td>0.773(1)(1)</td>
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<tr>
<td>D20.48</td>
<td>0.514(5)(4)(5)</td>
<td>0.727(7)(5)(7)</td>
<td>0.701(6)(1)(1)</td>
<td>0.785(7)(1)(2)</td>
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<td>[2]</td>
<td>0.501(8)(10)</td>
<td>0.695(10)(13)</td>
<td>0.686(1)(1)</td>
<td>0.789(1)(2)</td>
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<tr>
<td>E17.32</td>
<td>0.503(1)(3)(2)</td>
<td>0.681(2)(5)(3)</td>
<td>0.692(1)(1)(1)</td>
<td>0.768(1)(1)(1)</td>
</tr>
</tbody>
</table>

Errors: 1st = statistical, 2nd = uncertainty in lattice spacing, 3rd = uncertainty of $\Lambda_{\overline{\text{MS}}, N_f=2} = 310(25)$ MeV.
Comparison with RI-MOM

<table>
<thead>
<tr>
<th></th>
<th>$Z_P/Z_S$</th>
<th>$Z_V/Z_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B40.24</td>
<td>0.599(4)</td>
<td>0.861(4)</td>
</tr>
<tr>
<td>B64.24</td>
<td>0.595(5)</td>
<td>0.869(4)</td>
</tr>
<tr>
<td>B85.24</td>
<td>0.597(5)</td>
<td>0.872(5)</td>
</tr>
<tr>
<td>B100.24</td>
<td>0.585(13)</td>
<td>0.880(18)</td>
</tr>
<tr>
<td>B150.24</td>
<td>0.574(8)</td>
<td>0.900(9)</td>
</tr>
<tr>
<td>chiral</td>
<td>0.611(6)</td>
<td>0.847(6)</td>
</tr>
<tr>
<td>[1]</td>
<td>0.613(13)</td>
<td>0.836(12)</td>
</tr>
<tr>
<td>[2]</td>
<td>0.639(3)(1)</td>
<td>0.827(2)(5)</td>
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<td>C30.32</td>
<td>0.669(5)</td>
<td>0.886(4)</td>
</tr>
<tr>
<td>C60.32</td>
<td>0.672(6)</td>
<td>0.886(5)</td>
</tr>
<tr>
<td>C80.32</td>
<td>0.668(6)</td>
<td>0.879(5)</td>
</tr>
<tr>
<td>chiral</td>
<td>0.672(9)</td>
<td>0.890(8)</td>
</tr>
<tr>
<td>[1]</td>
<td>0.682(12)</td>
<td>0.854(10)</td>
</tr>
<tr>
<td>[2]</td>
<td>0.682(2)(1)</td>
<td>0.856(2)(5)</td>
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<td>D20.48</td>
<td>0.707(14)</td>
<td>0.893(16)</td>
</tr>
<tr>
<td>[2]</td>
<td>0.713(2)(2)</td>
<td>0.869(2)(4)</td>
</tr>
<tr>
<td>E17.32</td>
<td>0.740(3)</td>
<td>0.902(3)</td>
</tr>
</tbody>
</table>

Errors: 1st = statistical
Step scaling with X-scheme?

The running of the renormalization constants can be estimated non-perturbatively on the lattice by calculating the step scaling function, i.e. the ratio

$$\Sigma_\Gamma(\mu, 2\mu) = \lim_{a \to 0} \frac{Z_\Gamma(2\mu, a)}{Z_\Gamma(\mu, a)}$$

Example:

$$\frac{Z_\Gamma(\sqrt{X^2} = 0.0196\text{fm})}{Z_\Gamma(\sqrt{X^2} = 0.0392\text{fm})} = \frac{Z_\Gamma(\mu = 10\text{GeV})}{Z_\Gamma(\mu = 5\text{GeV})} \bigg|_{\beta=7.90'} \bigg|_{\beta=8.62'} \bigg|_{\beta=9.00'} \bigg|_{\beta=9.50}$$

On the lattice this corresponds to the ratios:

$$\frac{Z_\Gamma(X = \{1,1,1,1\})}{Z_\Gamma(X = \{2,2,2,2\})} \bigg|_{\beta=7.90'}$$

$$\frac{Z_\Gamma(X = \{3,3,3,3\})}{Z_\Gamma(X = \{6,6,6,6\})} \bigg|_{\beta=9.00'}$$

$$\frac{Z_\Gamma(X = \{2,2,2,2\})}{Z_\Gamma(X = \{4,4,4,4\})} \bigg|_{\beta=8.62'}$$

$$\frac{Z_\Gamma(X = \{4,4,4,4\})}{Z_\Gamma(X = \{8,8,8,8\})} \bigg|_{\beta=9.50}$$
Step scaling with coordinate scheme?

<table>
<thead>
<tr>
<th>STEP</th>
<th>32/64</th>
<th>24/48</th>
<th>16/32</th>
<th>8/16</th>
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<tbody>
<tr>
<td>1</td>
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<td>$\beta = 7.90$</td>
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Comments:
- simulations done with Chroma with periodic boundary conditions
- matching of volumes done through the non-perturbative definition of the coupling constant defined with Creutz ratios
- errors coming from the mismatch of the volumes not propagated yet
Step scaling with coordinate scheme: $Z_S$

**Figure**: Continuum extrapolation of $\Sigma_S(5\text{GeV}, 10\text{GeV})$ as a function of $a^2$. The two data sets correspond to two renormalization conditions differing in the tree-level side: one used the continuum and the other the lattice tree-level expression.
Step scaling with coordinate scheme: $Z_S$

<table>
<thead>
<tr>
<th>nr</th>
<th>fit type</th>
<th>ratio</th>
<th>stat.</th>
<th>$\Lambda_{\text{MS}}$</th>
<th>$r_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>cont</td>
<td>1.0740</td>
<td>–</td>
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<td>0.0010</td>
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<td>0.0098</td>
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<td>0.0001</td>
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<tr>
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<td>0.0113</td>
<td>0.0002</td>
<td>0.0001</td>
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<td>1.0857</td>
<td>0.0172</td>
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<td>0.0001</td>
</tr>
</tbody>
</table>

![Graph showing the ratio of step scaling with coordinate scheme $Z_S$]
Step scaling with coordinate scheme: $Z_A$

Figure: Continuum extrapolation of $\Sigma_A(5\text{GeV}, 10\text{GeV})$ as a function of $a^2$. The two data sets correspond to two renormalization conditions differing in the tree-level side: one used the continuum and the other the lattice tree-level expression.
Step scaling with coordinate scheme: $Z_A$

<table>
<thead>
<tr>
<th>nr</th>
<th>fit type</th>
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<th>stat.</th>
<th>$\Lambda_{\text{MS}}$</th>
<th>$r_0$</th>
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</thead>
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<tr>
<td>1</td>
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</tbody>
</table>

![Graph showing ratio versus fit type](image-url)
We have computed the values of RCs for ETMC $N_f = 2$ ensembles in the X-space scheme and converted them to the $\overline{\text{MS}}$ scheme at the reference scale of 2 GeV.


- use of “democratic” points
- use of 4-loop conversion formulae from the X-space scheme to the $\overline{\text{MS}}$ scheme
- dynamical ensembles vs. quenched

We find very good agreement in comparison with the values obtained with the RI-MOM method.

Preliminary results for the running of RC are encouraging!
The error of the X-space scheme computation is considerably larger for coarse lat.spac., but comparable for fine lat.spac. than with RI-MOM,

However, can we really compare the errors?

The main source of the error: $\mathcal{O}(a^2g^2)$ effects. Their computation in lattice perturbation theory would allow to decrease the error to approx. the error of RI-MOM. (Note that in RI-MOM the $\mathcal{O}(a^2g^2)$ effects are subtracted and this reduces the error considerably.)

The biggest advantages of the X-space method:

- gauge invariance
- considerable ease of application,
- relatively low computational cost.
Prospects

### Improvements
- $O(a^2 g^2)$ corrections?
- egalitarian method?

### Applications
- Step Scaling?
- Four-fermion operators?
- Heavy Quark Effective Theory?