Large b behaviour and CGC/saturation approach: the BFKL equation with massive gluon.

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Papers: E.L., L. Lipatov and M. Siddikov:

- *"BFKL Pomeron with massive gluons,"* Phys. Rev. D **89** (2014) 074002 [arXiv:1401.4671 [hep-ph]];
- *"Semi-classical solution to the BFKL equation with massive gluons"* (in preparation);

The main questions to answer:

1. How the correct behaviour at large b will influence the high energy behaviour of the scattering amplitude in the BFKL dynamics.

2. What is the high energy asymptotic behaviour of the BFKL scattering amplitude that provides the high energy amplitude for the electro-weak theory which can be measured experimentally

Large b dependence of the BFKL Pomeron

- $$N\left(r_1, r_2; Y, b\right) = \int \frac{d\gamma}{2\,\pi\, i}\, \phi^{(0)}_{in}(\nu)\, e^{\omega(\gamma = \frac{1}{2} + i\nu, 0)\, Y}$$

$$\times \left\{ b_\nu \left(w w^*\right)^{\frac{1}{2} + i\nu} + b_{-\nu} \left(w w^*\right)^{\frac{1}{2} - i\nu} \right\} \xrightarrow{\nu \ll 1} \frac{r_1\, r_2}{b^2}\, e^{\omega_0\, Y}$$

- $$w\, w^* = \frac{r_1^2\, r_2^2}{\left(\vec{b} - \frac{1}{2}\left(\vec{r}_1 - \vec{r}_2\right)\right)^2 \left(\vec{b} + \frac{1}{2}\left(\vec{r}_1 - \vec{r}_2\right)\right)^2}$$

$$N\left(r_1, r_2; Y, b\right) \leq 1 \quad \text{for} \quad b^2 \leq r_1\, r_2 e^{\omega_0 Y}$$

Violation of Froissart theorem: (Kovner & Wiedemann)

- $\sigma_{tot} = 2 \int d^2b \, N(r_1, r_2; Y, b) \propto s^{\omega_0} \gg Y^2 = ln^2 s$

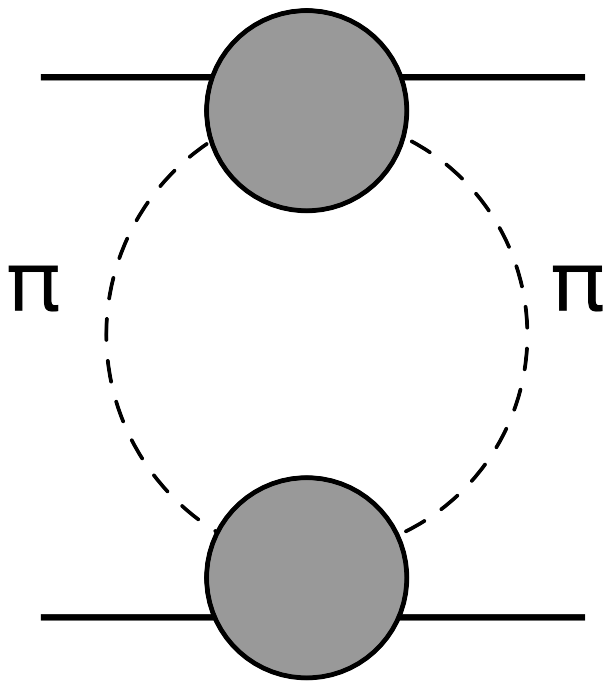
Proof:

- $\sigma_{tot} = 2\pi \int_0^{b_0^2} db^2 \, N(r_1, r_2; Y, b) + 2\pi \int_{b_0^2}^{\infty} db^2 \, N(r_1, r_2; Y, b)$
$\leq 2\pi \int_0^{b_0^2} db^2 + 2\pi \int_{b_0^2}^{\infty} db^2 \, N(r_1, r_2; Y, b) \, (N < 1)$

- Finding b_0: $\frac{r_1 \, r_2}{b^2} e^{\omega_0 \, Y} = f < 1$ $\quad \boxed{b_0^2 = r_1 \, r_2 \, e^{\omega_0 \, Y}}$

- $\sigma_{tot} \leq 2\pi \left(b_0^2 = r_1 \, r_2 \, e^{\omega_0 \, Y} \right) + 2\pi \int_{b_0^2}^{\infty} db^2 \left(\frac{r_1 \, r_2}{b^2} \right)^{2i\nu} e^{\omega_0 \, Y}$

Violation of the Froissart theorem: $\sigma \propto s^{\omega_0} \gg \ln^2 s$



$$N \propto s^{\omega_0}\, e^{-2 m_\pi b}$$

$$b_0: \quad s^{\omega_0}\, e^{-2 m_\pi b} = f < 1.$$

$$b_0 \propto \frac{\omega_0}{2\, m_\pi} \ln s$$

$$\sigma \leq \ln^2 s$$

Lessons from numerical solutions and theory considerations:

((Kovner & Wiedemann, McLerran and Iancu,Golec-Biernat & Stasto, Gotsman et al, Berger & Stasto,2011)

- **The confinement of quarks and gluon have to be included in the BFKL kernel (to include in the initial conditions is not enough);**

- **Suppressing large sizes of the produced dipoles in the decay** $one\ dipole \to two\ dipoles$ **we reproduce correct** b**-dependence;**

- **Since at large** b **the amplitude is small we do not need to take into account the non-linear corrections;**

My talk at Low x WS'2013: modified BFKL kernel

$$K\left(x_{12}, x_{20}|x_{10}\right) = \frac{x_{10}^2}{x_{12}^2\, x_{02}^2}\, e^{-B\left(x_{12}^2 + x_{02}^2\right)}$$

Gauge theory with massive gluon due to the Higgs mechanism

- It is not QCD but:
  - It has the same colour structure;
  - It is gauge invariant theory;
  - It reproduces QCD at $p_T \gg m$;
  - Can be that this theory is the effective theory which describes QCD at $p_T \geq m \gg \Lambda_{QCD}$?
    - The first answer: NO;

* The second answer: WHY NOT;

- **Indeed, trying to eliminate or average over Gribov's copies we introduce the dimensionful parameter (mass) in QCD** (J. Serreau, M. Tissier and A. Tresmontant,(2014); J. Serreau and M. Tissier, (2012),; N. Vandersickel and D. Zwanziger, Phys. Rept. (2012);).

- **The effective theory with massive gluon** (J. A. Gracey, (2014); J. M. Cornwall,(2013); N. Vandersickel and D. Zwanziger, Phys. Rept. (2012);) **describes the lattice data on the gluon and ghost propagators** (P. J. Silva, D. Dudal and O. Oliveira, *"Spectral densities from the lattice,"* arXiv:1311.3643 [hep-lat].) **with** $m = 0.54\,GeV$.

- **It has physics realization: electro-weak theory with zero Weinberg angle;**

- **It has the same problem with unitarity at high energies as QCD;**

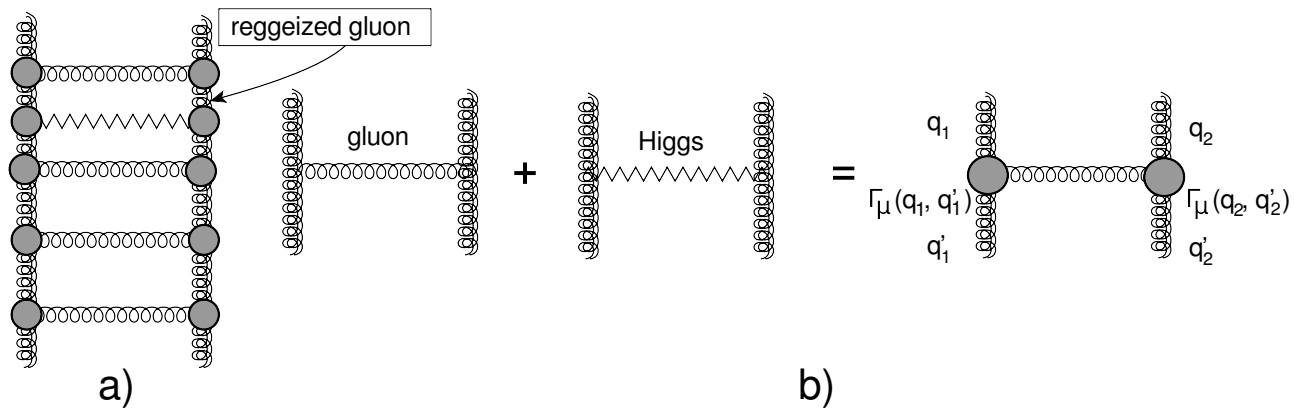
- **It has correct exponential fall down at large b;**

- **It is perfect training ground for answering the main question:**

  - **How correct b behaviour will influence the high energy asymptotic behaviour?**
  - **How much our physics observables depend on the amplitude at large b?**

Massive BFKL equation

(Fadin, Kuraev & Lipatov (1975 - 1977)



$$E = -\omega/\bar{\alpha}_S$$

$$\kappa = p^2/m^2$$

$$E\phi_E(\kappa) = \underbrace{\frac{\kappa+1}{\sqrt{\kappa}\sqrt{\kappa+4}} \ln \frac{\sqrt{\kappa+4}+\sqrt{\kappa}}{\sqrt{\kappa+4}-\sqrt{\kappa}}}_{\text{kinetic energy (T)}} \phi_E(\kappa)$$

$$\underbrace{-\int_0^\infty \frac{d\kappa'\,\phi_E(\kappa')}{\sqrt{(\kappa-\kappa')^2+2(\kappa+\kappa')+1}} + \frac{N_c^2+1}{2N_c^2}\frac{1}{\kappa+1}\int_0^\infty \frac{\phi_E(\kappa')\,d\kappa'}{\kappa'+1}}_{\text{potential energy (U)}}$$

Qualitative features of the solution

- $E\, f_E\,(r) \;=\; \mathcal{H}\, f_E\,(r)$

$$\mathcal{H} = \underbrace{T\Big(-\nabla^2\Big)}_{kinetic\quad energy} \;\underbrace{-\,2K_0\,(|r|m) \;+\; \frac{N_c^2+1}{2\,N_c^2} K_0\,(|r|m) \int \frac{d^2p'}{\pi}\frac{\phi\_E\,(p')}{p'^2\;+\;m^2}}_{potential\quad energy}$$

$\boxed{r\,m \ll 1}$ $\qquad E_{\text{BFKL}}\,(\gamma) \;=\; -\,2\psi(1) + \psi(\gamma) + \psi(1-\gamma)$

$$f_{E\,=\,E^{\text{BFKL}}(\gamma)}\,(r) \;\xrightarrow{r \ll 1/m}\; f^{\text{BFKL}}_{E\,=\,E(\gamma)}\,(r) \;\sim\; \left(r^2\right)^{\gamma-1}$$

From normalization $\int dr^2 |f^{\text{BFKL}}_{E\,=\,E(\gamma)}\,(r)\,|^2 < \infty \quad \gamma = \frac{1}{2} \pm i\nu$

$\boxed{r\,m \gg 1}$ $\quad E\,f_E\left(r\right) \;=\; T\left(-\nabla^2\right)\;f_E\left(r\right)$

$$f_{E\,=\,E^{\text{BFKL}}(\gamma)}\left(r\right) \xrightarrow{r \gg 1/m} e^{-a\,r} \text{ with}$$

$$T\left(-a^2\right) \;=\; E^{\text{BFKL}}\left(\gamma\right) \longrightarrow \text{ equation for the new scale}$$

$\boxed{\text{momentum rep.}}$ $\qquad \phi_E\left(\kappa\right) \;=\; \left(\kappa^2 \;+\; a^2\right)^{-\gamma}$

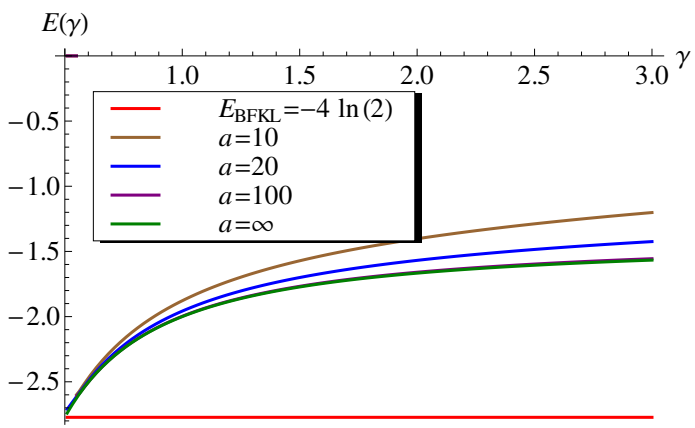
$$\phi_E\left(\kappa\right) \xrightarrow{\kappa \gg 1} \kappa^{-\gamma}; \qquad \phi_E\left(\kappa\right) \xrightarrow{\kappa \ll 1} \text{Const}$$

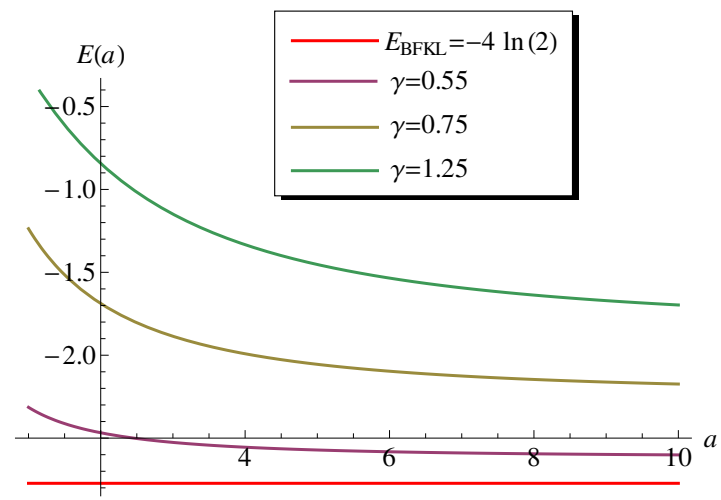
Result # 1 : $E\,(\gamma;\,\text{BFKL}\,m \neq 0) \;= E\,(\gamma;\;\text{BFKL}\,m = 0)$

1. Variational method

- $E_{\text{ground}} \equiv -\,\omega_0 \;\leq\; F[\{\phi\}] = \dfrac{\left\langle \phi^*(p) \middle| \mathcal{H} \middle| \phi(p) \right\rangle}{\left\langle \phi^*(p) \middle| \phi(p) \right\rangle}$

- **Our choice:** $\{\phi\} = \left(\kappa^2 + a^2\right)^{-\gamma}$





Can be only states with $E \;\leq\; E^{\text{BFKL}} \;=\; -4\ln 2$

2. The proof: $E \geq E^{\text{BFKL}} = -4\ln 2$

The best trial function $\phi^0_{\text{trial}} = \frac{1}{\sqrt{\kappa + a^2}}$ $E_{\text{BFKL}} = -4\ln 2$ **at any** a

- $\mathcal{H} = T(p) + U(r) = \left\{T(p) - T_0(p)\right\} + \mathcal{H}_0$
- $\mathcal{H}_0 \phi^0_{\text{trial}} = \left(T_0(p) + U(r)\right)\phi^0_{\text{trial}} = E_{\text{BFKL}}\phi^0_{\text{trial}}$

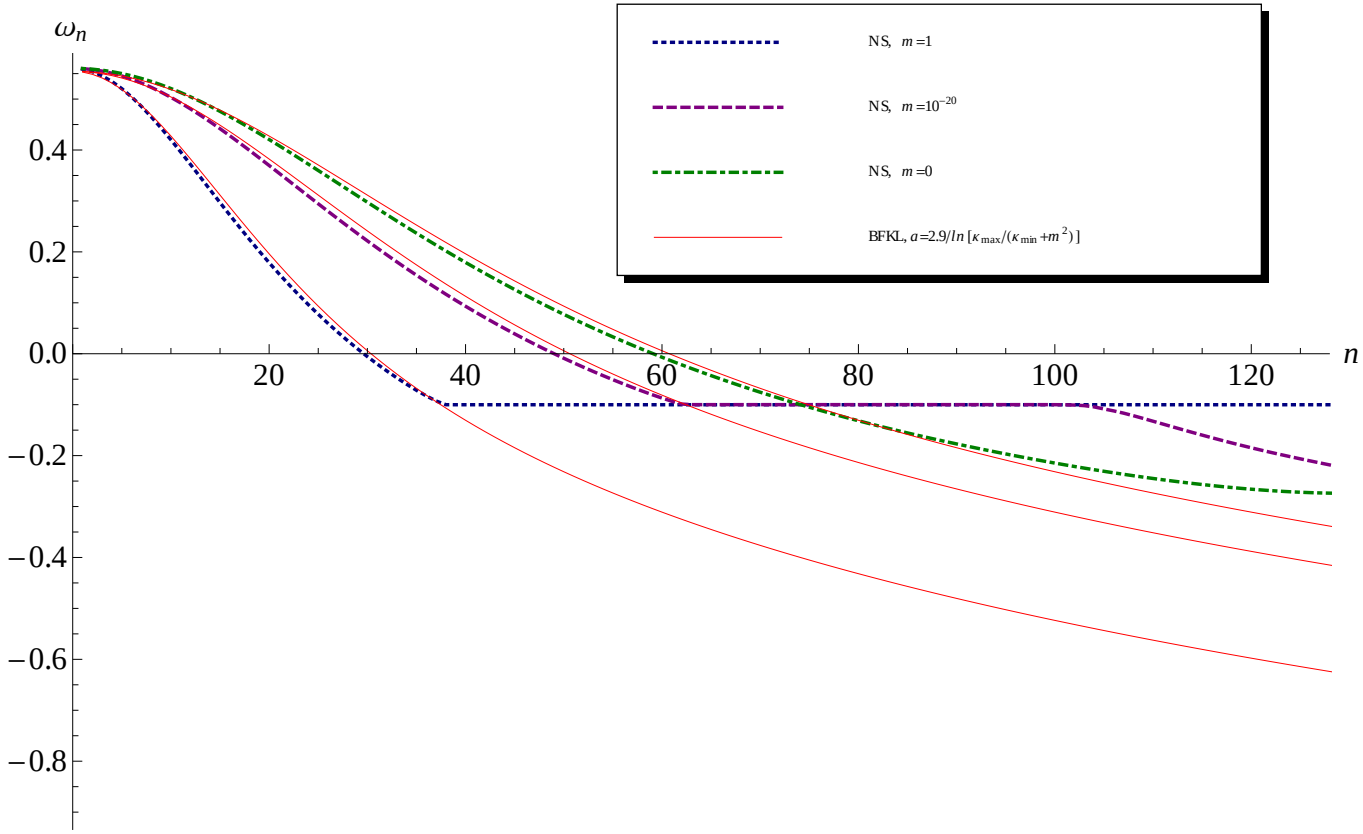
We prove $\left\{T(p) - T_0(p)\right\} \geq 0$

- $T_0(p) = E_{\text{BFKL}} - \frac{1}{\phi^0_{\text{trial}}(-i\nabla)} U(r)\,\phi^0_{\text{trial}}(r)$
- $\frac{1}{\phi^0_{\text{trial}}(p)} U(r)\,\phi^0_{\text{trial}}(r) = -\int \frac{d^2p'}{\pi} \frac{\sqrt{p^2+a^2}}{(|\vec{p}-\vec{p}\,'|^2+1)\sqrt{p'^2+a^2}}$

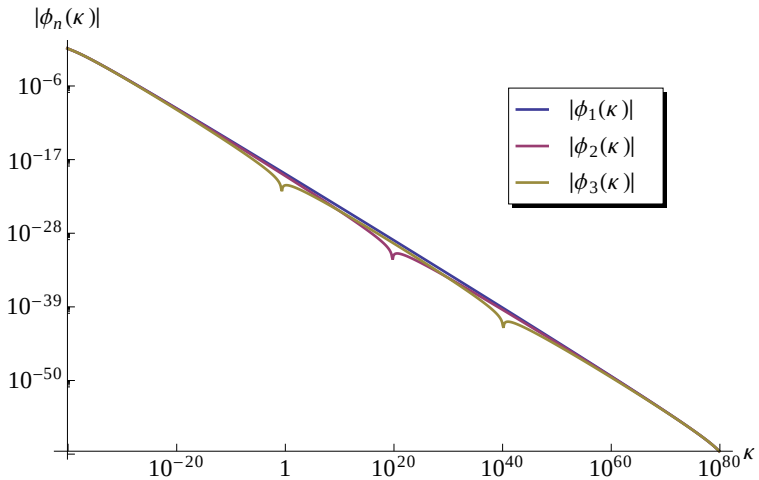
$\left\{\ldots\right\} \geq 0$ **for** $\boxed{5/2 < a < a_0^2 = 5.26}$

Solution to $T\left(-a^2\right) = E(\gamma)$ **gives** a **in this region**

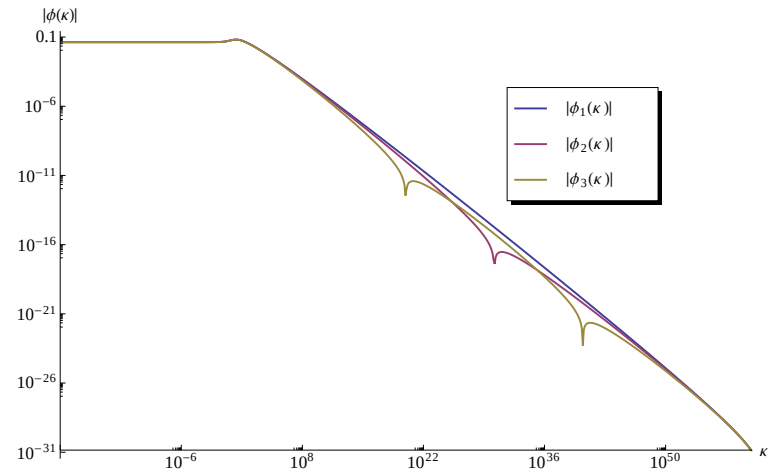
3. Numerical eigenvalues



Result # 2: Eigenfunctions



BFKL m =0



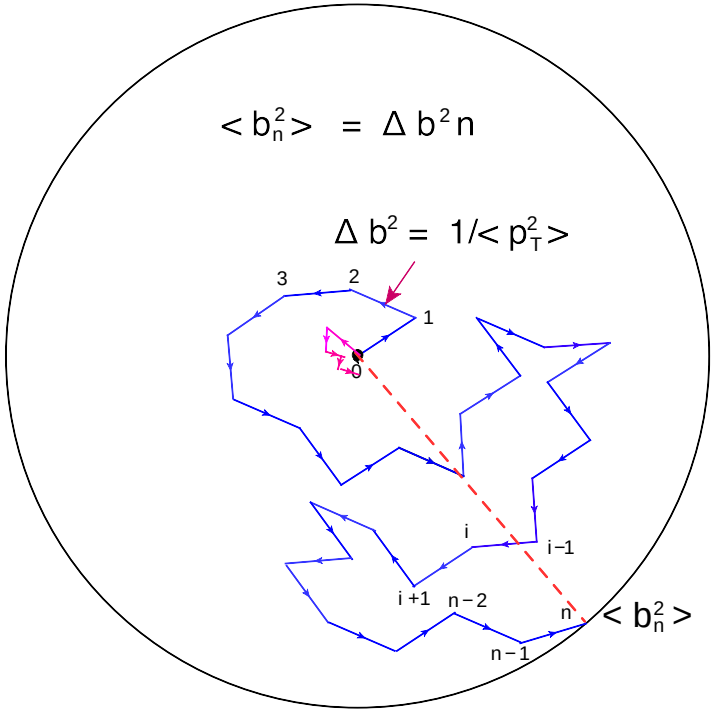
BFKL m$\neq$ 0

- $$\phi^{(approx)}(\kappa,\nu) = \frac{\alpha(\nu)}{\sqrt{\kappa+4}} \sin\left(\nu\, Ln(\kappa) + b_\phi\, \nu\right)$$

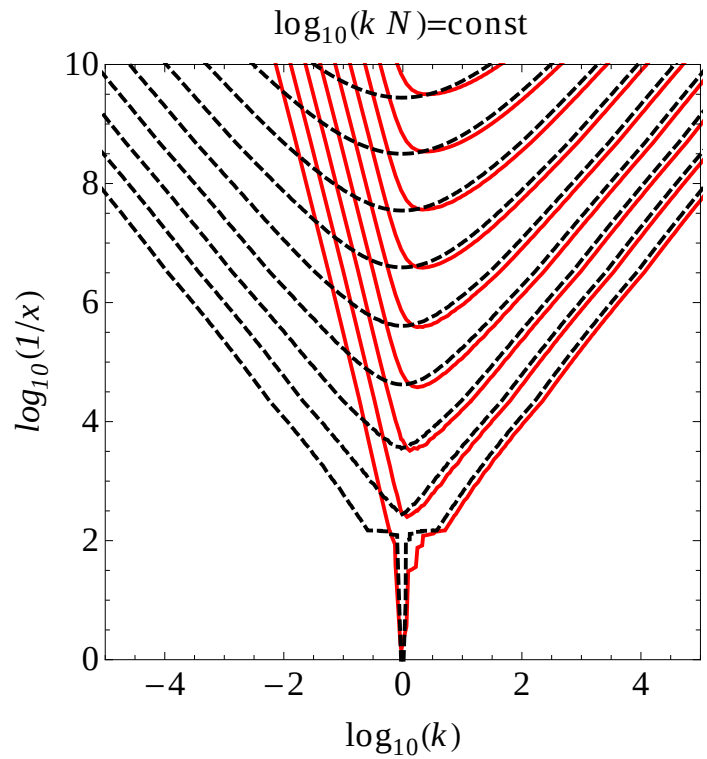
- $\gamma = \frac{1}{2} = i\nu$, $b_\phi = 1.865$ and $Ln(\kappa) = \ln\left(\frac{\sqrt{\kappa+4}+\sqrt{\kappa}}{\sqrt{\kappa+4}-\sqrt{\kappa}}\right)$

New scale: $\bar{p}_T^2 = 4\, m^2$

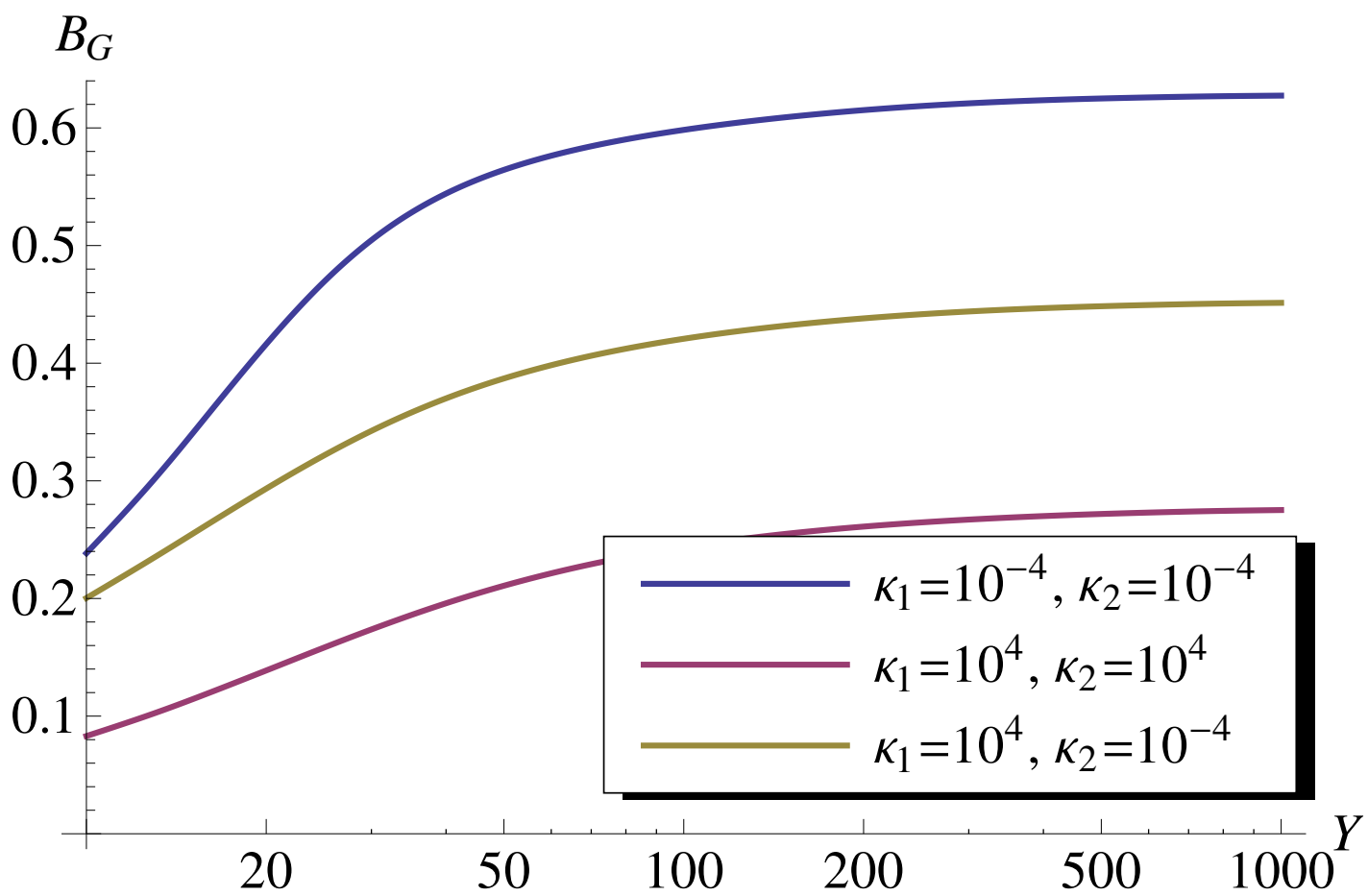
Result # 3: $\langle |b^2| \rangle$ = Const(Y)



Gribov's diffusion

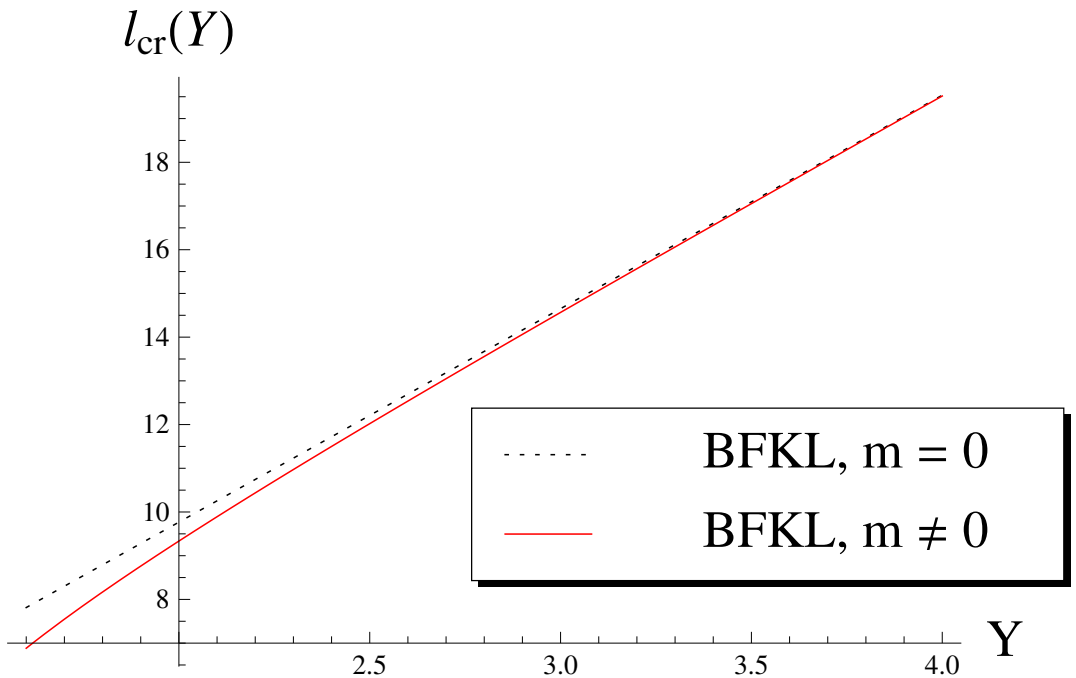


Contour plot


B_G
0.6
0.5
0.4
0.3
0.2
0.1
20
50
100
200
500
1000
Y
$\kappa_1=10^{-4}$, $\kappa_2=10^{-4}$
$\kappa_1=10^{4}$, $\kappa_2=10^{4}$
$\kappa_1=10^{4}$, $\kappa_2=10^{-4}$

$$B_G = < |b^2| >$$

Result # 4: $Q_s^2(Y,b)$



- $l_{cr}(Y) = \ln\left(Q_s^2(Y)/Q^2(0)\right)$
- $N(Y, Q_s(Y,b)) = 0.1$

- **Weak point: we have not found the b-dependence of Q_s**

Result # 5: Effective intercept in semi-classical approach

- $\Psi\left(Y,l\right) = \int \frac{d\gamma}{2\pi i}\,\phi_{in}\left(\gamma\right)\,\frac{e^{-E(\gamma,l)\,Y}}{\left(\kappa\,+\,a^2\right)^{1-\gamma}} \;=\; \int \frac{d\gamma}{2\pi i}\,\phi_{in}\left(\gamma\right)\,e^{-E(\gamma,l)\,Y\,+\,(\gamma-1)\,l}$

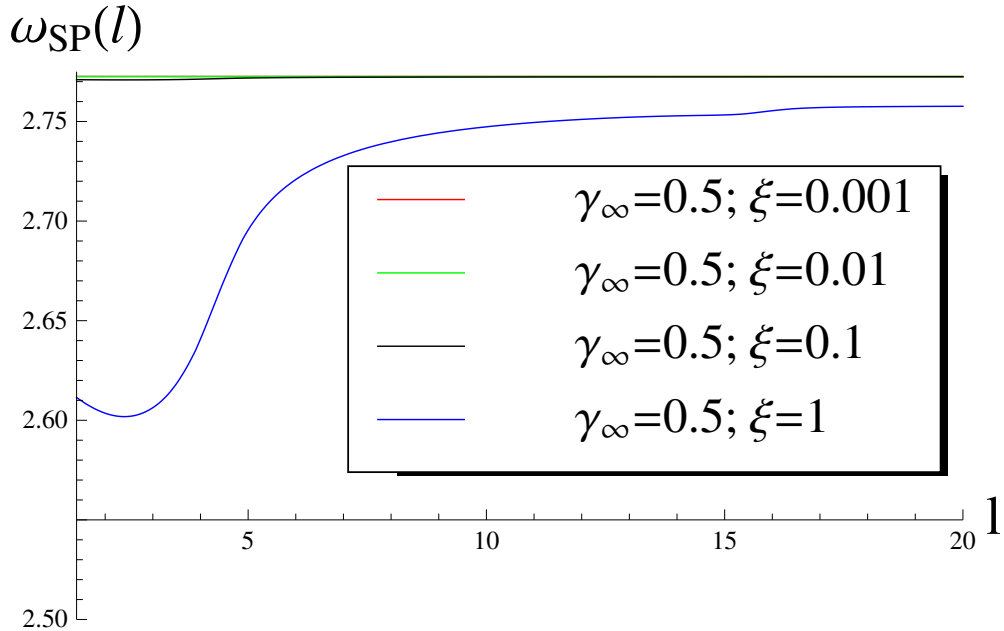
Steepest decent method (saddle point equation):

- $$\boxed{-\frac{\partial E(\gamma_{SP}(Y,l,a),l)}{\partial\gamma}\,Y\;+\;l\;=\;0}$$

- $\Psi\left(Y,l=\ln\left(\kappa+a^2\right)\right) \;=\; \phi_{in}\left(\gamma_{SP}\left(Y,l\right)\right)$

$$\times e^{-E\left(\gamma_{SP}(Y,l),l\right)\,Y+\left(\gamma_{SP}(Y,l)-1\right)l}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}\,e^{-\frac{1}{2}\frac{\partial^2 E\left(\gamma_{SP}(Y,l),l\right)}{\partial\gamma^2_{SP}}\,Y\,\left(\gamma\,-\,\gamma_{SP}\left(Y,l\right)\right)^2}$$

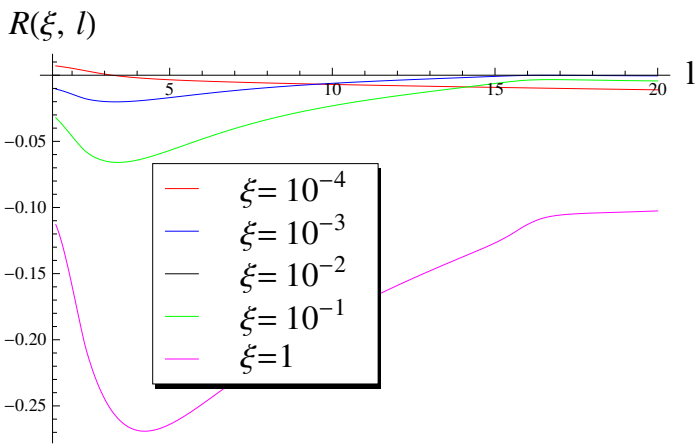
$$=\frac{\phi_{in}\left(\gamma_{SP}\left(Y,l\right)\right)}{\sqrt{2\pi\left|\frac{\partial^2 E\left(\gamma_{SP}(Y,l),l\right)}{\partial\gamma^2_{SP}}\right|\,Y}}\;e^{-\frac{1}{2}l}\;e^{\underbrace{-E\left(\gamma_{SP}\left(Y,l\right),l\right)\;Y\;+\;\left(\gamma_{SP}\left(Y,l\right)\,-\,\frac{1}{2}\right)\,l}_{\omega_{eff}\,Y}}$$

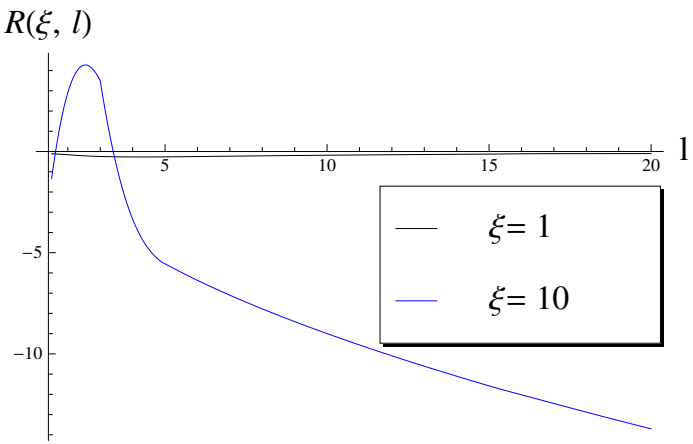

$\omega_{SP}(l)$
2.75
2.70
2.65
2.60
2.50
5
10
15
20
l
γ_∞=0.5; ξ=0.001
γ_∞=0.5; ξ=0.01
γ_∞=0.5; ξ=0.1
γ_∞=0.5; ξ=1

- $\xi = \frac{l}{Y}$

Accuracy of semiclassical approximation

- $$R = \frac{1}{6}\left|\frac{\partial^3 E\left(\gamma_{SP}\left(Y,l\right),l\right)}{\partial\gamma_{SP}^3}\right| Y \Big/ \left(\frac{1}{2}\left|\frac{\partial^2 E\left(\gamma_{SP}(Y,l),l\right)}{\partial\gamma_{SP}^2}\right| Y\right)^{3/2} \ll 1$$





Conclusions

1. $E\,(\text{BFKL } m \neq 0) = E\,(\text{BFKL } m = 0)$

2. The convenient parametrization of the wave functions is found with the new scale $\approx 4\,m^2$;

3. Average b^2 does not depend on Y at high energies;

4. $Q_s\,(\text{BFKL } m \neq 0) = Q_s\,(\text{BFKL } m = 0)$ at $Y \gg 1$;

5. Semi-classical approach works quite well leading to a possibility of analytical (semi-analytical) calculations;

Studying the influence of the second dimensional scale in CGC/saturation approach we can conclude:

- **We have not found any large influence of the correct large b behaviour on the solution to the BFKL equation.**
- **We believe that the behaviour of the amplitude at large b will not affect the main equations of the CGC/saturation approach.**
- **Much work is needed to turn this belief into proven statement.**
- **We are working on the asymptotic behaviour of the weak interaction amplitude.**

Thank You