# Far-from-equilibrium universality classes in expanding scalars

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***RIKEN / BNL***
***Lunch Time Talk***

Nov 13, 2014

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*Talk based on:*

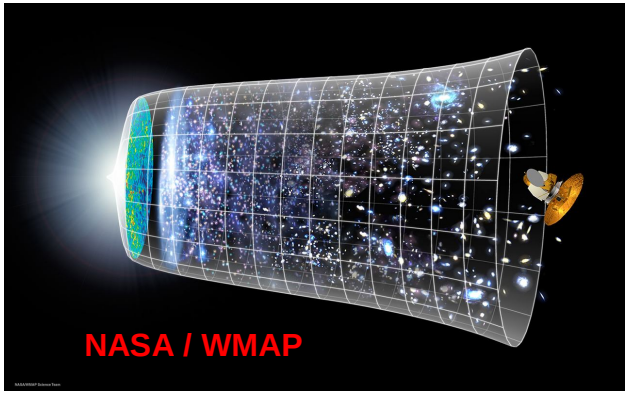
J. Berges, KB, S. Schlichting,
R. Venugopalan,

*arXiv:1408.1670;*
*in preparation,*
and our previous works

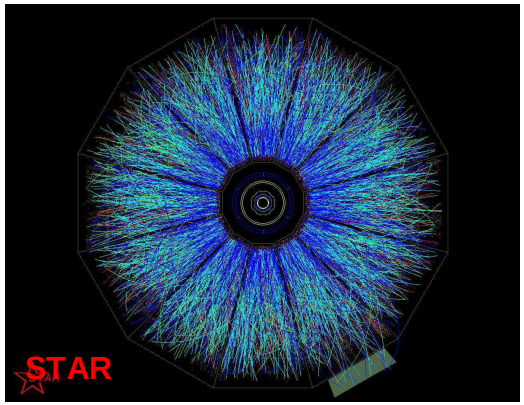
J. Berges, KB, A. Orioli,
*in preparation*

# Motivation

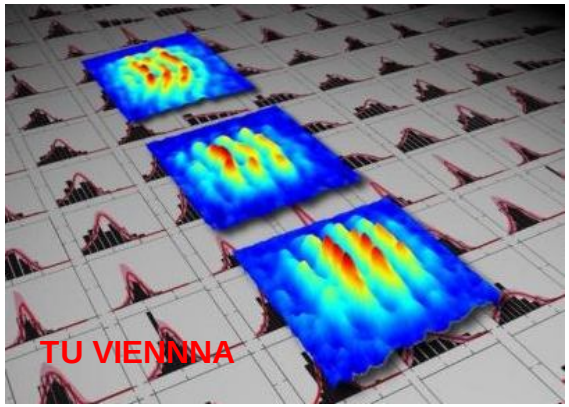
## Physical systems far from equilibrium




*Inflationary cosmology*




*Heavy-ion collisions at early stages*




*Ultracold atoms*

Very different energy scales!

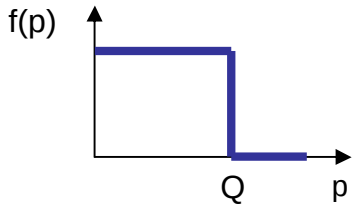
**Common properties?**

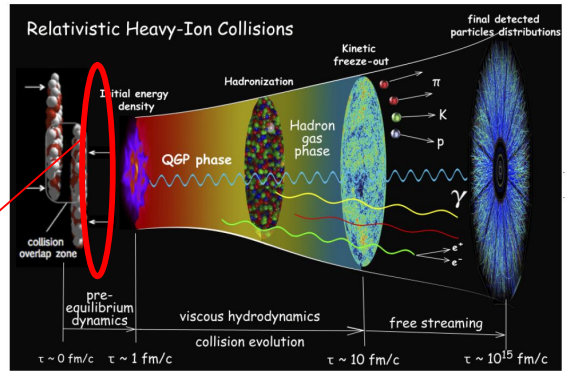
**Signatures of all of them in a single system?**

# Motivation

Example: Heavy-ion collisions at early times
(Ultrarelativistic energies, weak-coupling limit $g \ll 1$ )

Highly correlated systems …




Little Bang by P. Sorensen and C. Shen

(inspired by color glass condensate (CGC) )

$$f(p \lesssim Q) \sim 1/g^2 \qquad \text{(or } \langle A\,A \rangle \sim Q^2/g^2\text{)}$$

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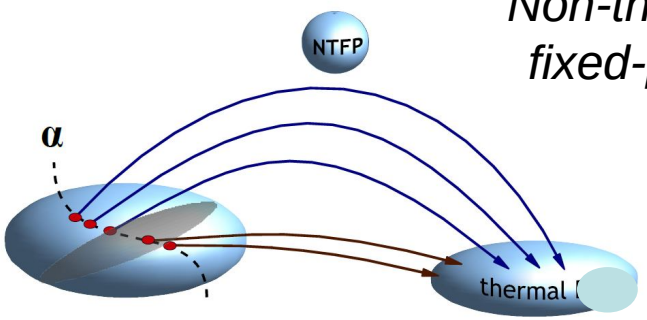
… can take ‚detour‘ during thermalization

*Non-thermal fixed-point*




Nowak, Scholle, Gasenzer

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   *From Bose gases to heavy-ion collisions*

4. Conclusion and Outlook

# Introduction

## Non-thermal fixed-points & universality classes

*At a non-thermal fixed point*

- *Memory loss* of the details of the initial conditions

- *Self-similar evolution* of distribution function f (→ critical slowing down)

$$\boxed{f(p,t) = t^{\alpha}\, f_S(t^{\beta} p)}$$ with scaling behavior of typical scales $f \sim t^{\alpha}$ , $p \sim t^{-\beta}$

- Possible: *several fixed-points* in different momentum regimes (‚*inertial range*')

- Often connected to kinetic processes (e.g. turbulence) or topological defects

*Classification: universality classes far from equilibrium*

- Non-thermal fixed-points, described by their dynamical exponents $\alpha$, $\beta$ and the scaling function $f_S(x)$, may be classified in universality classes

# Introduction

## Example: Dual cascade in non-expanding scalar theory

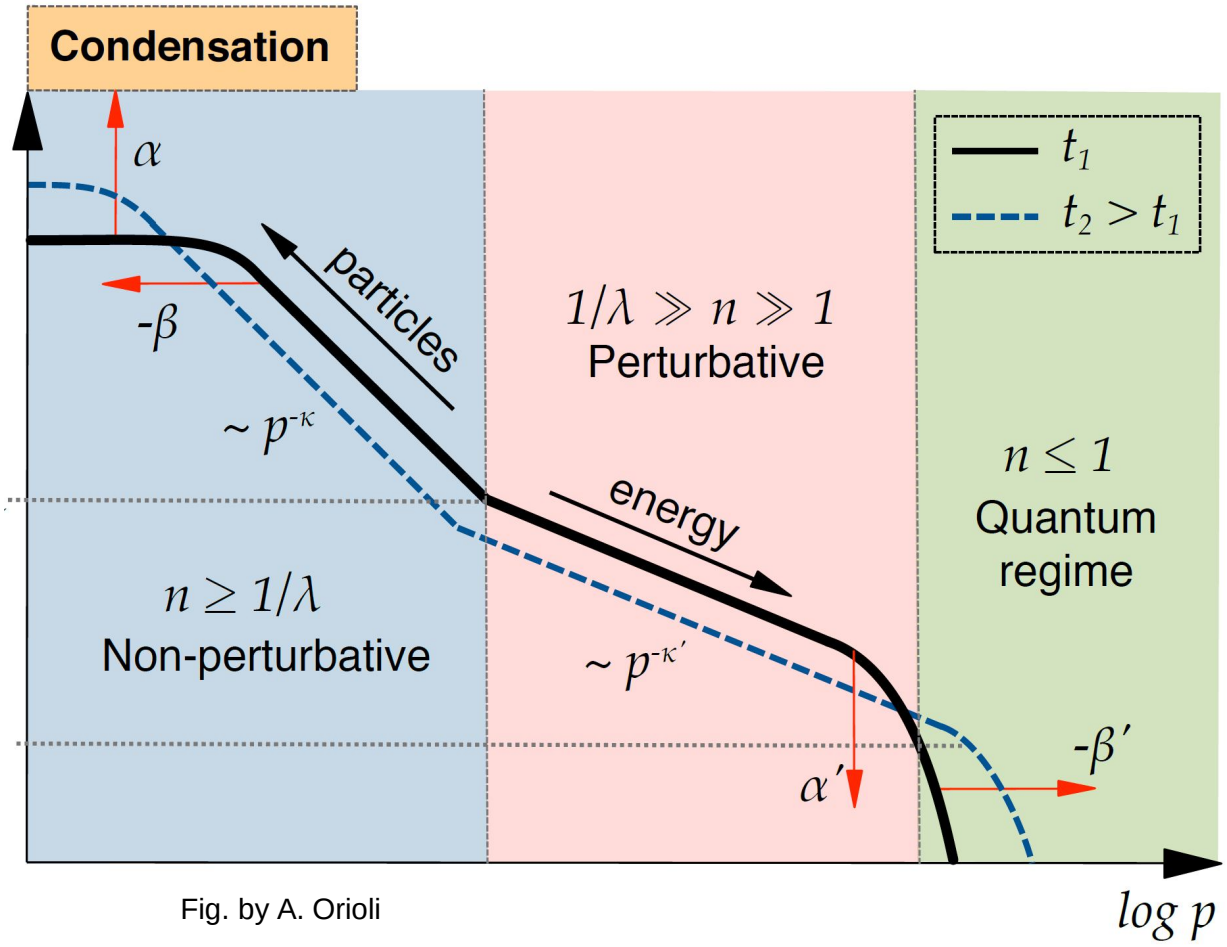



Fig. by A. Orioli

*Reminder: Self-similarity*

$$f(p,t) = t^{\alpha} f_S(t^{\beta} p)$$

### Perturbative

Micha & Tkachev,
*PRL 90, 121301 (2003);*
*PRD* 70, 043538 (2004)

### Non-perturbative

Berges, Rothkopf & Schmidt,
*PRL 101, 041603 (2008)*

Nowak, Sexty & Gasenzer
*PRB 84, 020506 (2011)*

### Bose condensation

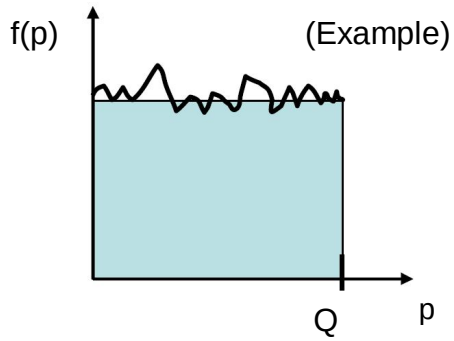
Berges & Sexty,
*PRL 108, 161601 (2012)*

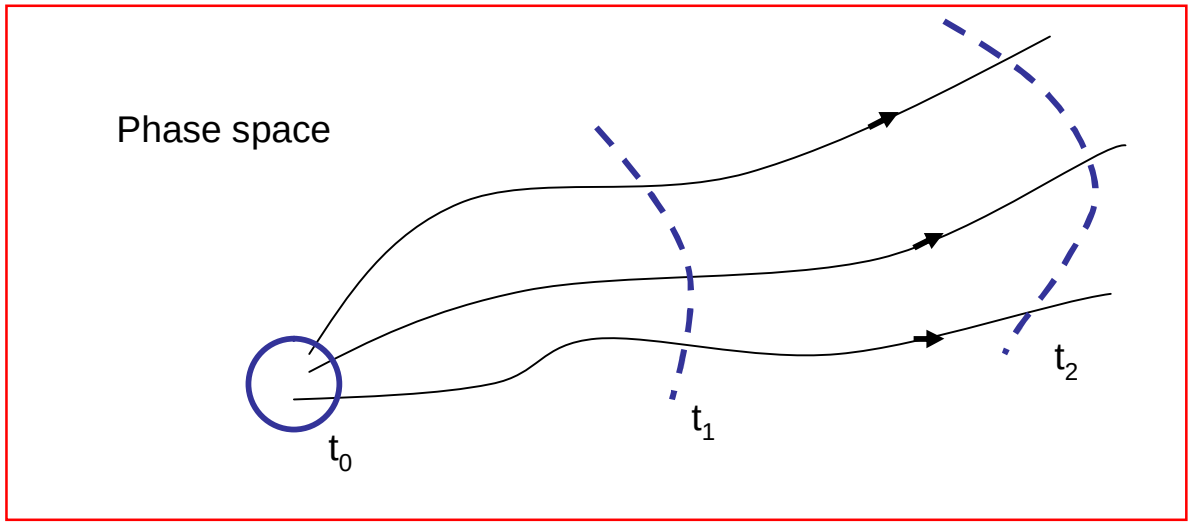
# Introduction: Classical-statistical lattice simulations

- Start with fluctuating initial conditions at initial time $t_0$
- Solve initial value problem on the lattice

  *Example: non-expanding scalars:* $\Box\varphi + \frac{\lambda}{6}\varphi^3 = 0$

- Evolve observable classically and average over set of initial conditions at time of interest t




Trajectories of classical observable $\mathcal{O}_{\rm cl}$

*Examples: Micha & Tkachev ; Smit & Tranberg; Nowak, Scholle, Sexty & Gasenzer ; Berges, KB, Schlichting & Venugopalan; Kurkela & Moore; …*

# Introduction: Classical-statistical lattice simulations

## Initialization

$$\varphi_a(\boldsymbol{x}, t_0) = \phi_a(t_0) + \int_{\boldsymbol{p}} \sqrt{f(\boldsymbol{p}, t_0) + \frac{1}{2}} \left(c_{a,\boldsymbol{p}}\, \xi(\boldsymbol{p}, (t_0))\, e^{i\boldsymbol{p}\boldsymbol{x}} + c.c.\right)$$

*(example: O(N) symmetric scalar field theory)*

Gaussian distributed complex random numbers

## Distribution function

$$\boxed{f((\boldsymbol{p}), t) + \frac{1}{2} = \sqrt{F(\boldsymbol{p})\ddot{F}(\boldsymbol{p}) - \dot{F}^2(\boldsymbol{p})}}$$

with

$$\begin{aligned} F(\boldsymbol{p}) &= \langle |\varphi(\boldsymbol{p})|^2 \rangle\,, \quad \ddot{F}(\boldsymbol{p}) = \langle |\partial_t \varphi(\boldsymbol{p})|^2 \rangle \\ \dot{F}(\boldsymbol{p}) &= \mathrm{Re}(\langle \varphi(\boldsymbol{p})\, \partial_t \varphi^*(\boldsymbol{p}) \rangle) \end{aligned}$$

## Range of validity

*Classicality condition:* $\boxed{\text{“}f(p) \gg 1\text{”}}$

→ Accurate low-energy description of quantum field theory!

Aarts & Berges, PRL 88, 041603 (2002) ; Jeon, PR C 72, 014907 (2005) ; …

# Low-momentum universality class in scalar field theories

J. Berges, KB, A. Orioli,
*in preparation*

Relativistic scalar O(N) theory (massless): $\left(\partial_\mu \partial^\mu + \frac{\lambda}{6N}\varphi_a(x)\varphi_a(x)\right)\varphi_b(x) = 0$

Non-relativistic complex scalar theory: $i\partial_t \psi(t,\boldsymbol{x}) = \left(-\frac{\nabla^2}{2m} + g|\psi(t,\boldsymbol{x})|^2\right)\psi(t,\boldsymbol{x})$

# Low-momentum universality class of scalar theories

What are the dynamical exponents of the infrared fixed-point?

**Analytical:**

- Highly over-occupied system with $\lambda f \gtrsim 1$
- Perturbative kinetic theory breaks down
- We use diagrammatic resummation from 2-particle-irreducible (2PI) effective action with 1/N expansion to NLO

Berges, Rothkopf & Schmidt,
*PRL 101, 041603 (2008)*

**Results:**

*Reminder: Self-similarity*

$$f(p,t) = t^{\alpha}\, f_S(t^{\beta} p)$$

| | relativistic | non-relativ. |
|---|---|---|
| $\alpha$ | **3** | **3/2** |
| $\beta$ | **1** | **1/2** |

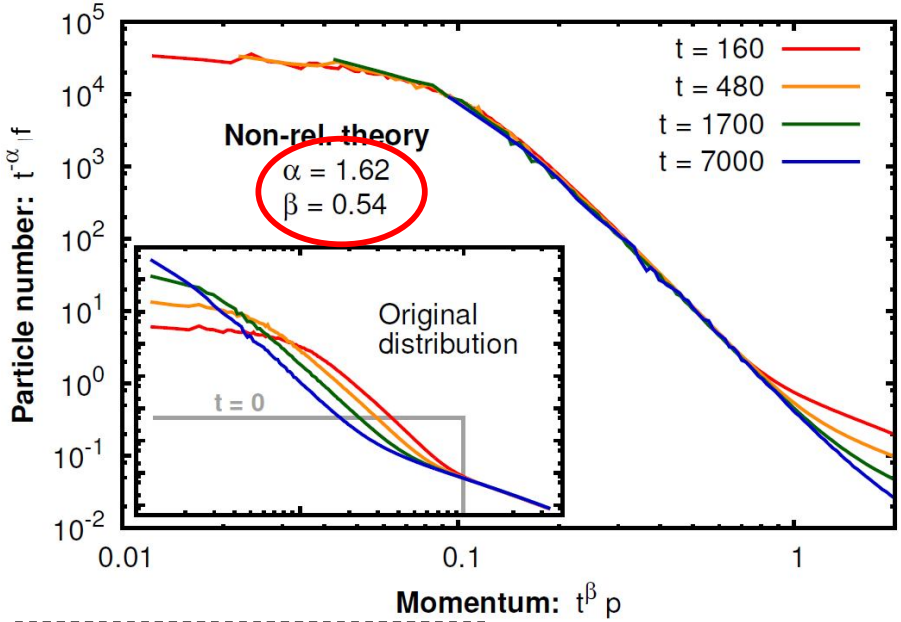
(for canoncical values of $\eta$ and z)

# Low-momentum universality class of scalar theories

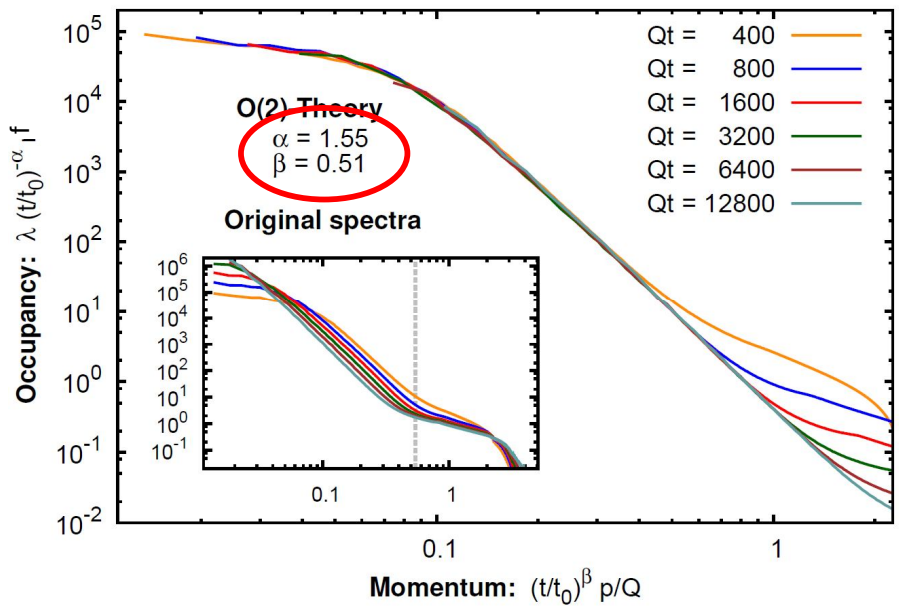
What are the dynamical exponents of the infrared fixed-point?

**Numerical**

Non-relativistic




Relativistic



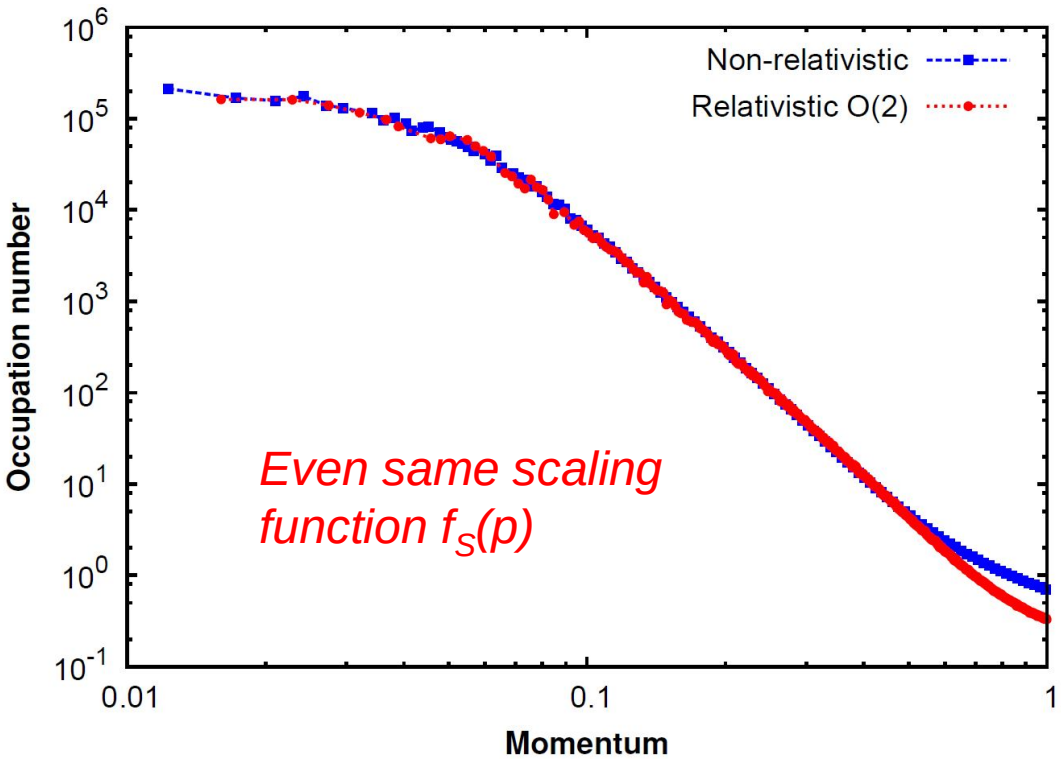

*Reminder: Self-similarity*

$$f(p,t) = t^{\alpha}\, f_S(t^{\beta} p)$$

*Both show non-relativistic values* (similar for O(4) theory)

# Low-momentum universality class of scalar theories

Universality




Conclusion:

*Relativistic and non-relativistic scalar theories lie in the same universality class!*

Reason:

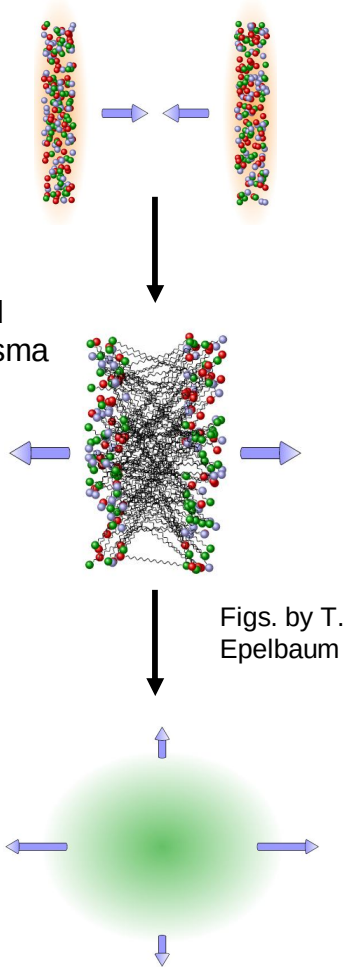
In relativistic theory, a mass is dynamically generated, which is larger than momenta within the inertial range.

Use ultracold Bose gases to simulate low-energy inflaton dynamics?

Schmiedmayer & Berges,
*Science 341, no. 6151, 1188 (2013)*

# Motivation: Longitudinally expanding systems

Early stages in heavy-ion collisions



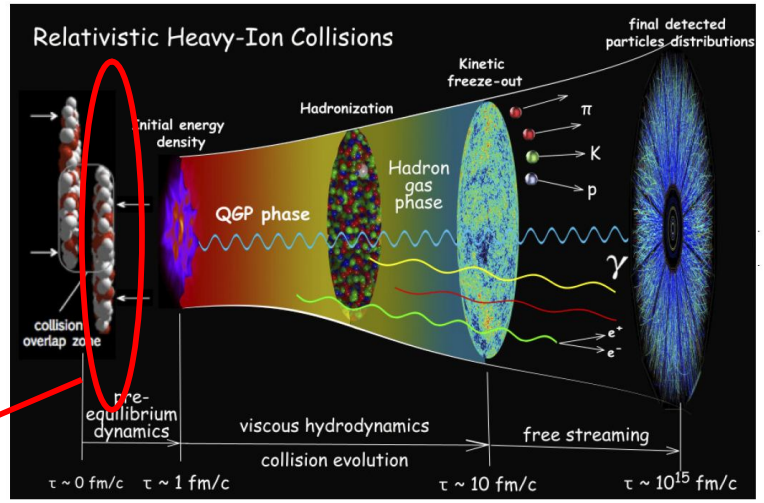
Strongly correlated mainly gluonic plasma

Hydro simulations:

- Quick thermalization
- Nearly ideal fluid

Figs. by T. Epelbaum

Thermalization process?




Little Bang by P. Sorensen and C. Shen

Ultrarelativistic energies, weak-coupling limit:

*Gluonic plasma approaches a non-thermal fixed-point!*

J. Berges, KB, S. Schlichting, R. Venugopalan,
*PRD 89, 074011 (2014) ; PRD 89, 114007 (2014)*

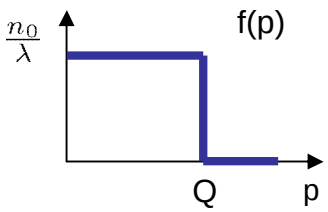
# Longitudinally expanding scalars:
## *From Bose gases to heavy-ion collisions*

J. Berges, KB, S. Schlichting, R. Venugopalan,
*arXiv:1408.1670 ; in preparation*

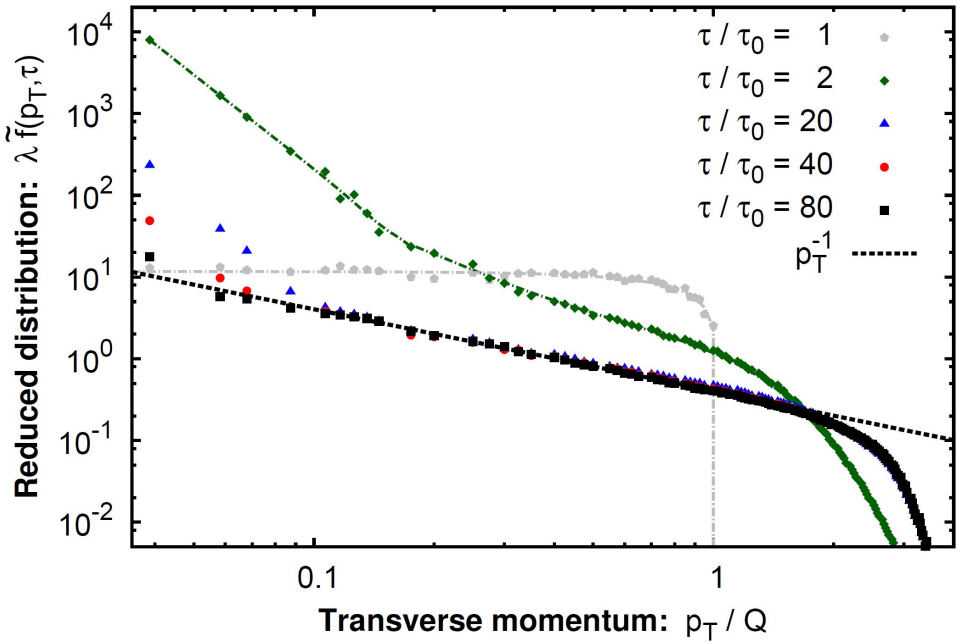
Bjorken coordinates: $\tau = \sqrt{t^2 - (x^3)^2}\,, \qquad \eta = \mathrm{artanh}\left(\frac{x^3}{t}\right)$

Metric in Bjorken coordinates: $g_{\mu\nu}(\tau) = \mathrm{diag}\,(1, -1, -1, -\tau^2)$
longitudinal expansion (in beam direction)

If not written otherwise: *O(4) theory with initial conditions (right) and $n_0$=35 at $Q\tau_0$=$10^3$*



# Longitudinally expanding scalar fields




Definition

$$\tilde{f}(p_T,\tau) = \frac{\tau}{Q\tau_0}\int \frac{dp_z}{2\pi} f(p_T,p_z,\tau)$$

Shows power-law

$$\tilde{f}(p_T,\tau) \sim 1/p_T \text{ for } p_T \lesssim 2\,Q$$

& becomes *time-independent*!

Self-similar evolution

$$f(p_T,p_z,\tau) = \tau^{\alpha} f_S\left(\tau^{\beta}p_T, \tau^{\gamma}p_z\right)$$

implies

$$\boxed{\begin{aligned} \alpha-\gamma &\approx -1 \\ \beta &\approx 0 \end{aligned}}$$

*Local conservation* of particle number and energy density in transverse momentum

$$\tau\, dn/dp_T \sim p_T\tilde{f}\ , \quad \tau\, d\epsilon/dp_T \sim p_T^2\tilde{f}$$

***Effectively no flux in $p_T$!***

# Longitudinally expanding scalar fields

## Longitudinal dynamics at $p_T \sim Q/2$

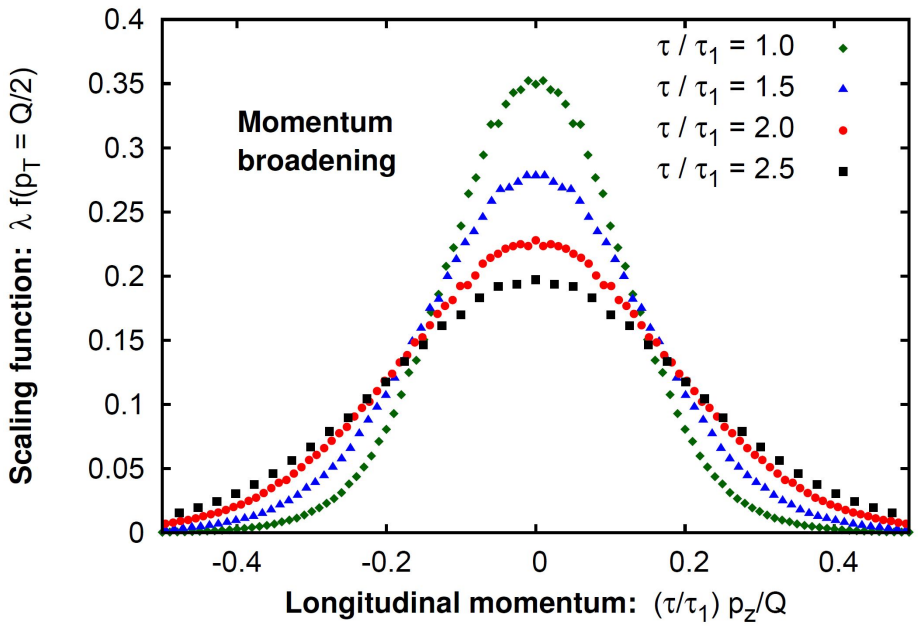
Reminder: *Self-similar evolution*

$$f(p_T, p_z, \tau) = \tau^{\alpha} f_S(\tau^{\beta} p_T, \tau^{\gamma} p_z)$$

Free evolution implies red shift $p_z \sim \tau^{-1}$

and $\alpha = 0\,, \quad \gamma = 1$

Momentum broadening hints at non-trivial scattering!



# Longitudinally expanding scalar fields

## Longitudinal dynamics at $p_T \sim Q/2$

Reminder: *Self-similar evolution*

$$f(p_T, p_z, \tau) = \tau^{\alpha} f_S(\tau^{\beta} p_T, \tau^{\gamma} p_z)$$

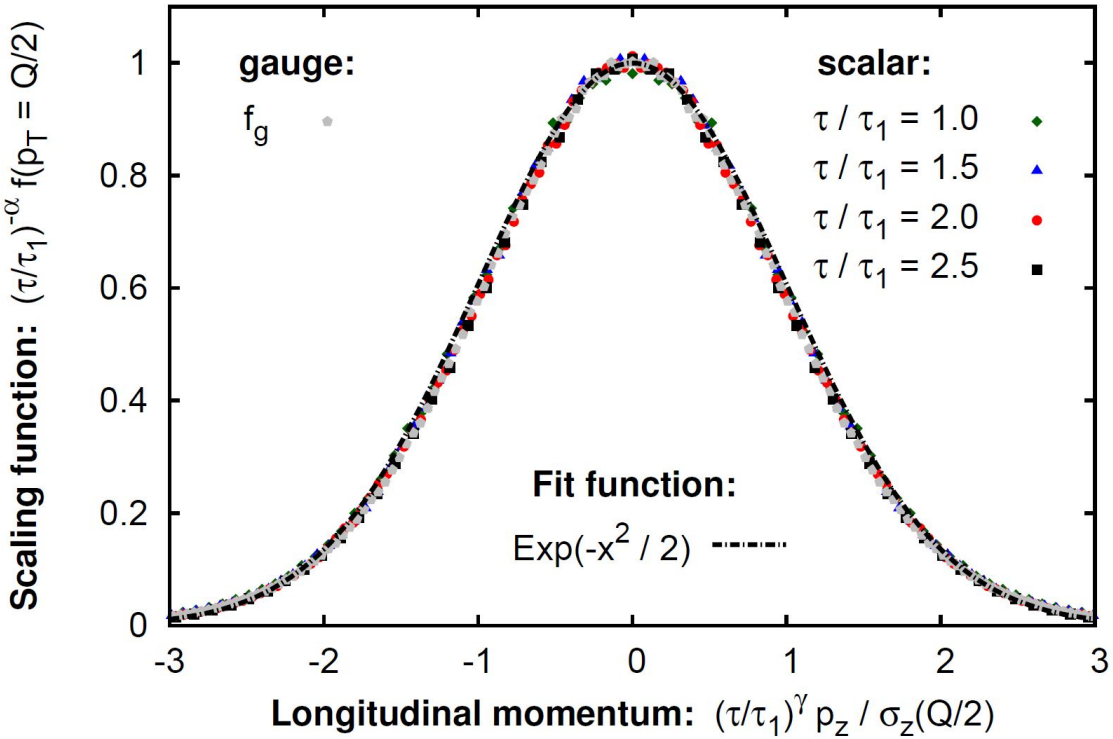
Dynamical exponents

$$\alpha = -2/3$$
$$\gamma = 1/3$$

and Gaussian shape

*same as for gauge theory!*

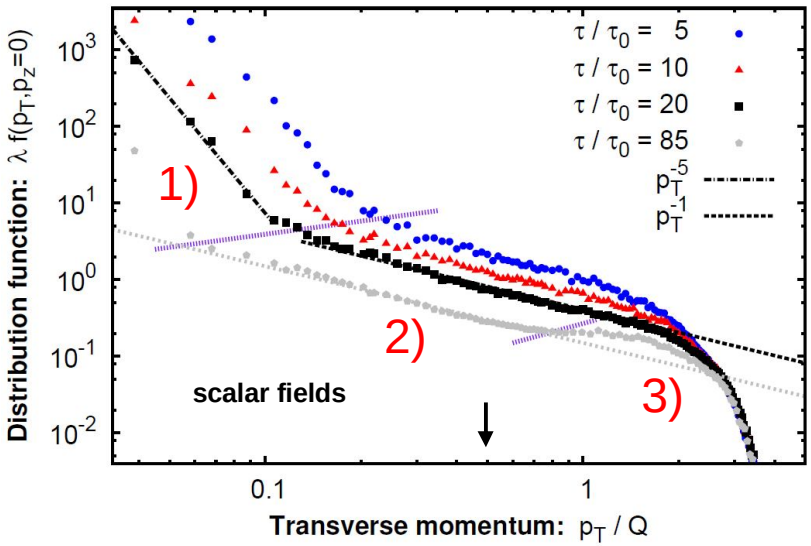
Manifestation of universality

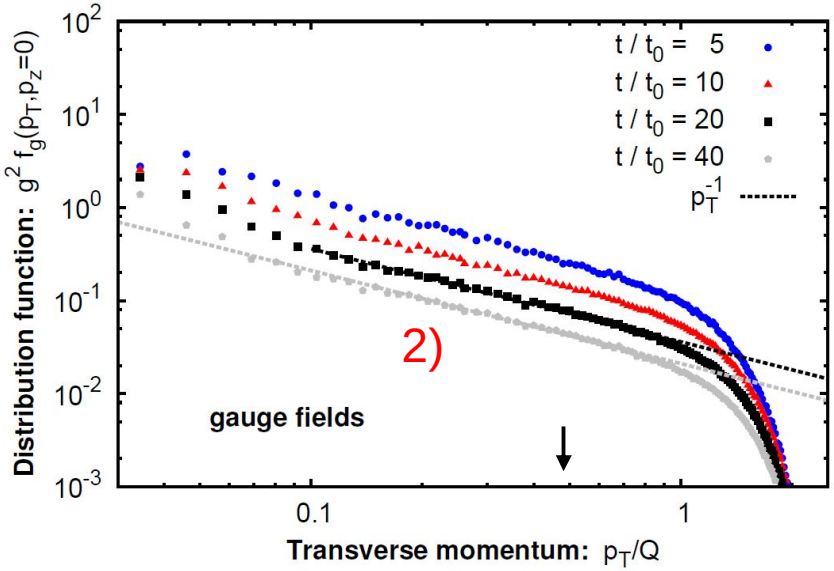



*Scattering process in gauge theory more general than small-angle scattering?*

# Longitudinally expanding scalar fields

Inertial range of the non-thermal fixed-point?




- Inertial range of shared non-thermal fixed-point 2) given by $f(p_T, p_z = 0) \sim 1/p_T$

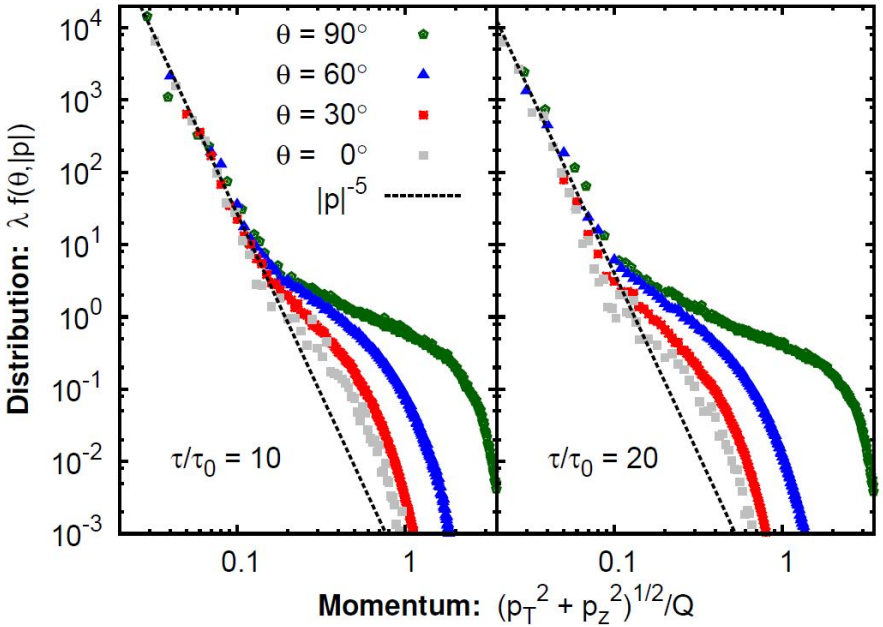
- *Scalars:* inertial range increases on logarithmic scale – robust property of dynamics

- Fixed-point approached for different initial amplitudes, anisotropies, initial times and for condensate driven initial conditions (Backup-slide), also for O(1) and O(2)

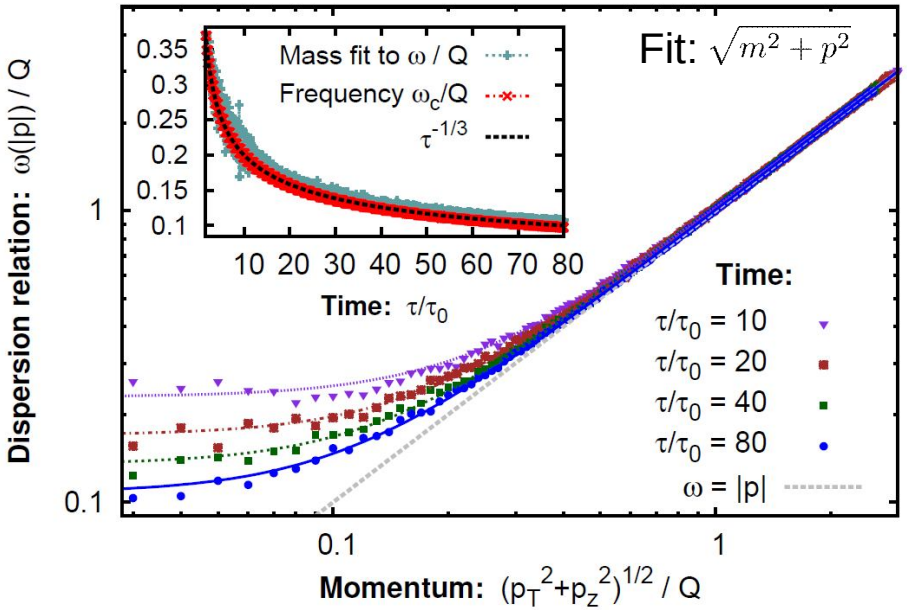
# Longitudinally expanding scalar fields

## Infrared fixed-point: Inertial range 1)

*Isotropic* , *non-relativistic* power law




Dynamically generated *mass*

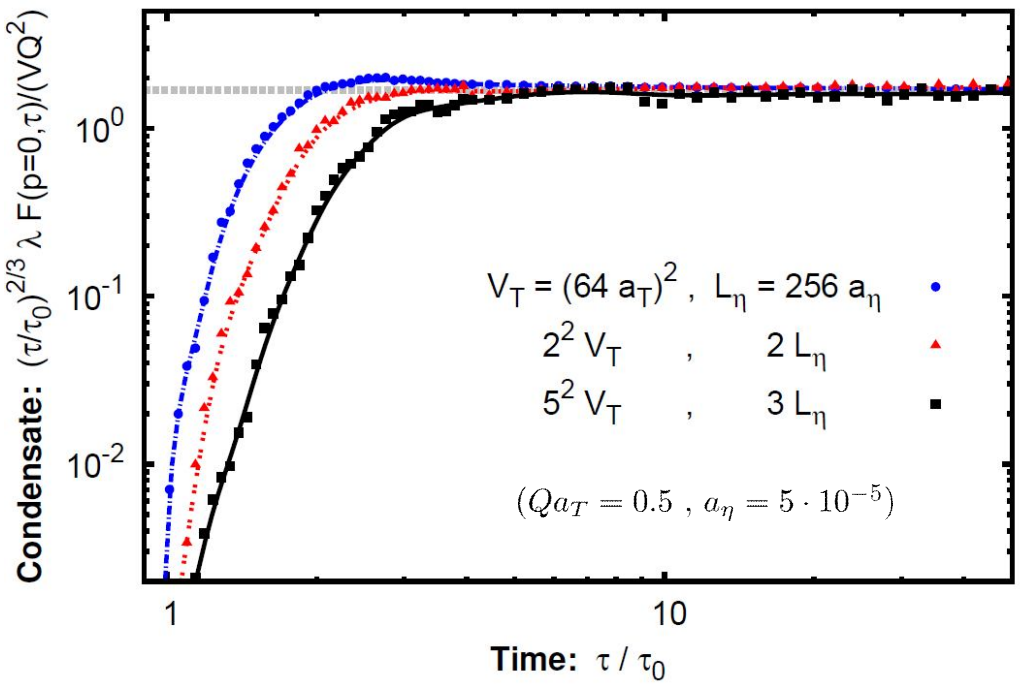



Similar to non-expanding scalar theory –
high scattering rates overcome expansion

$$\left(\omega(|p|) = \sqrt{\frac{\ddot{F}(|p|)}{F(|p|)}}\right)$$

# Longitudinally expanding scalar fields

## Bose-Einstein condensation




If global Bose condensate exists, then the scalar field zero mode should scale as

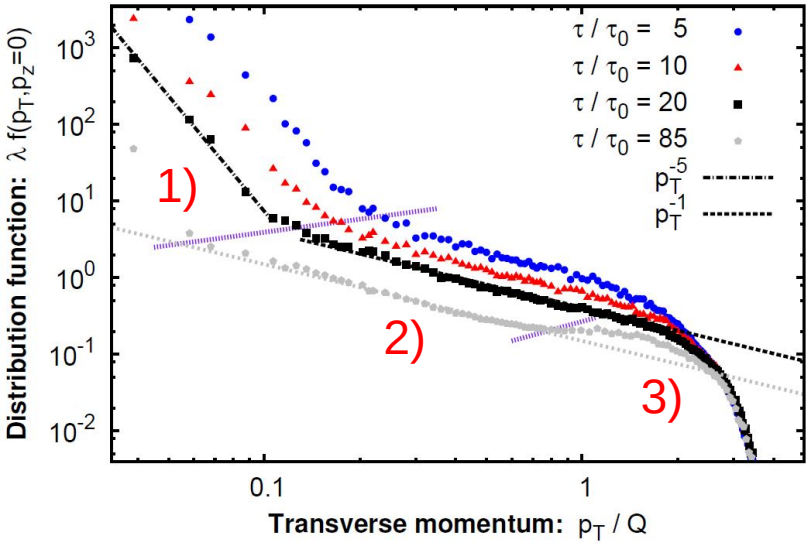
$$F(p=0) \sim \phi^2 \delta^{(3)}(\boldsymbol{p}) \sim \phi^2 V$$

J. Berges & D. Sexty, *PRL 108 (2012), 161601*

Bose-Einstein condensation far from equilibrium observed!

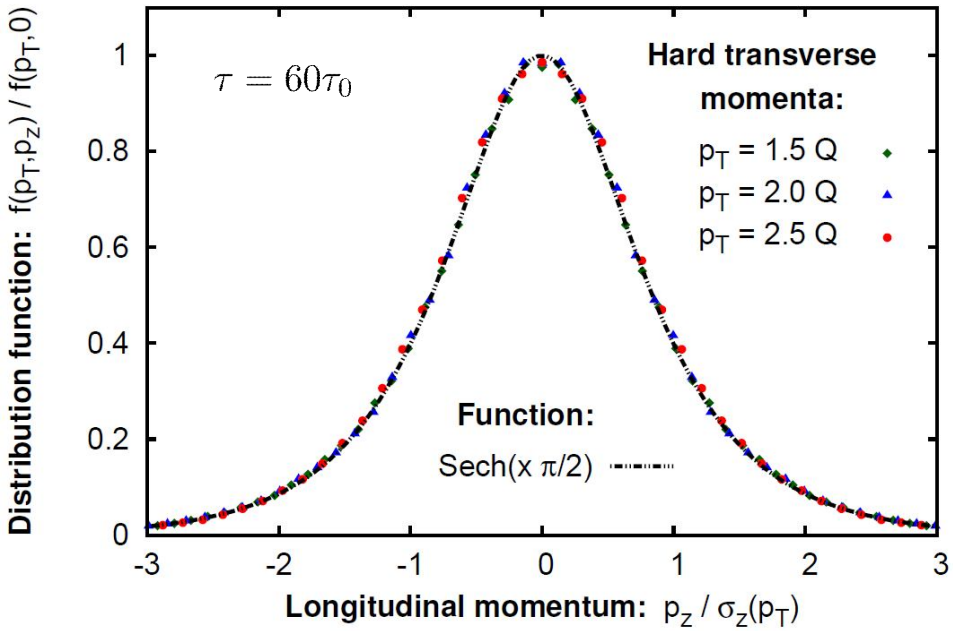
# Longitudinally expanding scalar fields

## Hard-momentum fixed-point: Inertial range 3)




At late times a non-thermal fixed-point emerges at *large momenta*.

Longitudinal distribution at hard $p_T$




It has a hyperbolic secant shape, which has a broader tail than the Gaussian function.

*Sign for large angle scatterings?*

# Longitudinally expanding scalar fields

## Hard-momentum fixed-point: Inertial range 3)

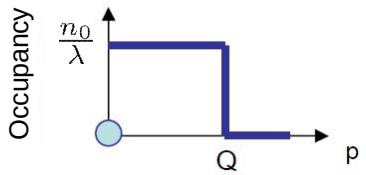
*Self-similar evolution at large $p_T$*

$$f(p_T, p_z, \tau) = \tau^{\alpha'} f'_S(p_T, \tau^{\gamma'} p_z)$$

### Longitudinal hard scale

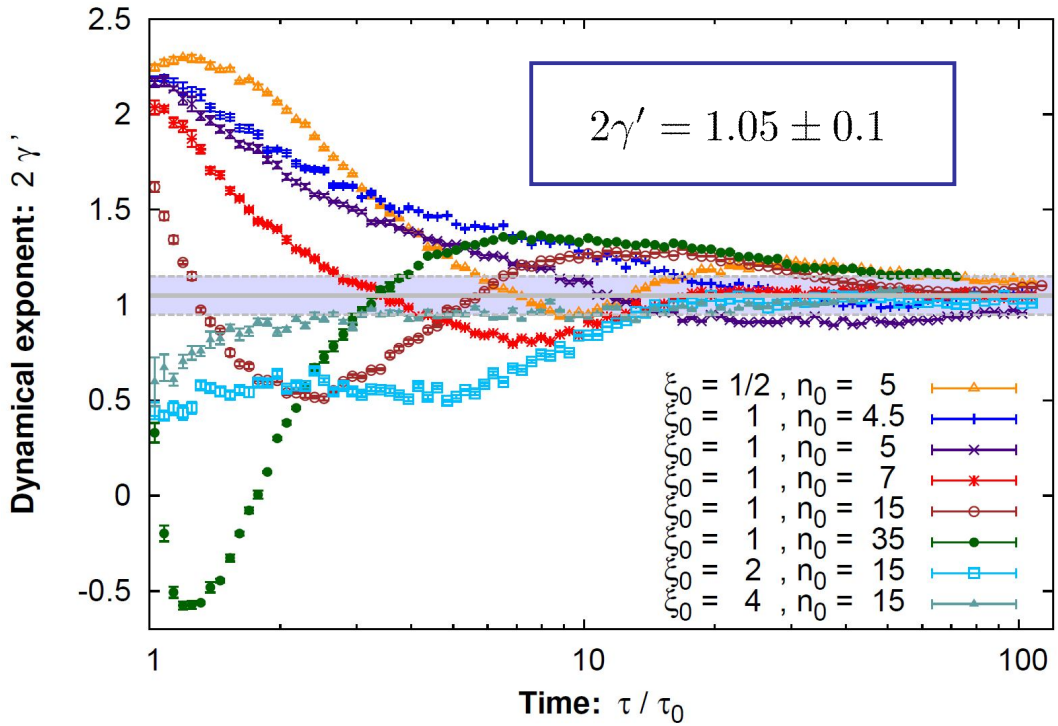
$$\Lambda_L^2(\tau) \approx \frac{\int d^2p_T \int dp_z \; p_z^2 \, \omega(\boldsymbol{p}) \; f(p_T, p_z, \tau)}{\int d^2p_T \int dp_z \; \omega(\boldsymbol{p}) \; f(p_T, p_z, t)} \sim \tau^{-2\gamma'}$$

Independent of initial conditions!



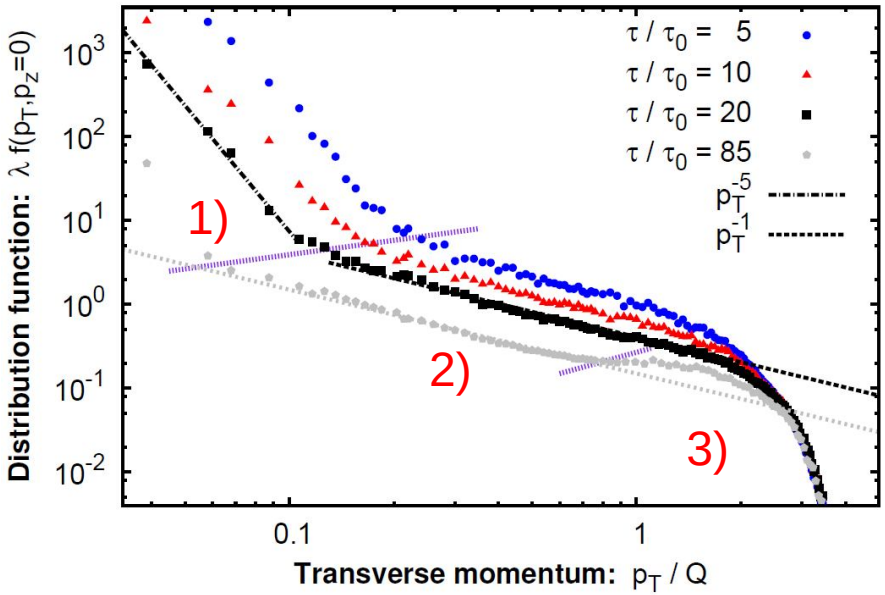

$$f(p_T, p_z, \tau_0) = \frac{n_0}{\lambda} \Theta\left(Q - \sqrt{p_T^2 + (\xi_0 p_z)^2}\right)$$

### Logarithmic slope of hard scale



# Longitudinally expanding scalar fields

## Summary: non-thermal fixed-points




| Inertial range | 1) | 2) | 3) |
|---|---|---|---|
| $\alpha$ | $3\beta$ | $-\frac{2}{3}$ | $-\frac{1}{2}$ |
| $\beta$ | $\gamma$ | $0$ | $0$ |
| $\gamma$ | | $\frac{1}{3}$ | $\frac{1}{2}$ |
| $f_S(p_T, p_z = 0)$ | $\lvert p\rvert^{-5}$ | $p_T^{-1}$ | $const$ |
| $f_S(p_T = const, p_z)$ | $\lvert p\rvert^{-5}$ | $e^{-p_z^2}$ | $\mathrm{sech}(p_z)$ |

1) Shares universal properties with non-relativistic scalars (fixed box)

2) Evidence: same universality class as non-Abelian gauge theory

# Conclusion:

- Highly correlated quantum systems may approach non-thermal fixed-points, classification to non-equilibrium universality classes is a crucial step

- Relativistic and non-relativistic scalars lie in the same IR universality class

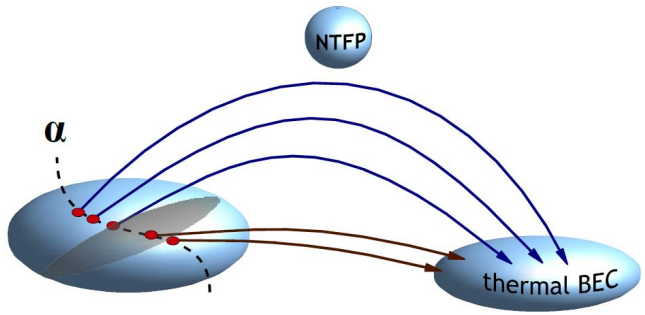
- Expanding scalar and gauge systems show anisotropic fixed-points with conserved energy and particle densities locally in $p_T$

- Evidence for the same universality class in characteristic $p_T$-range

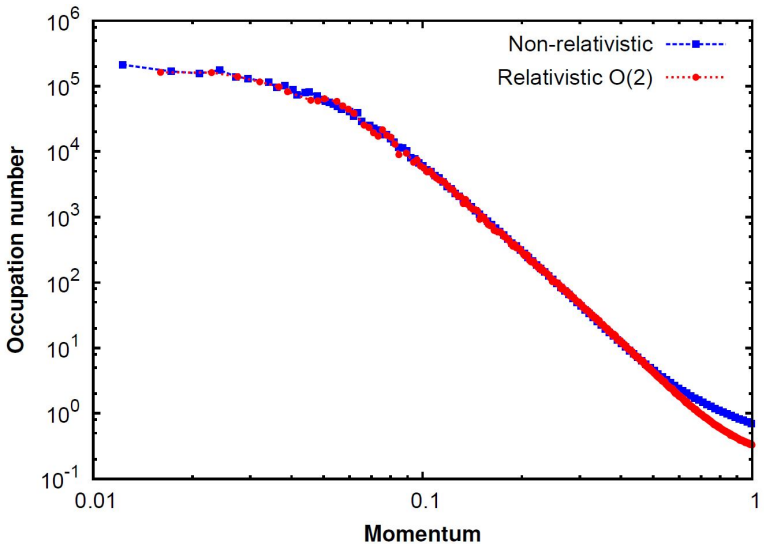
# Outlook:

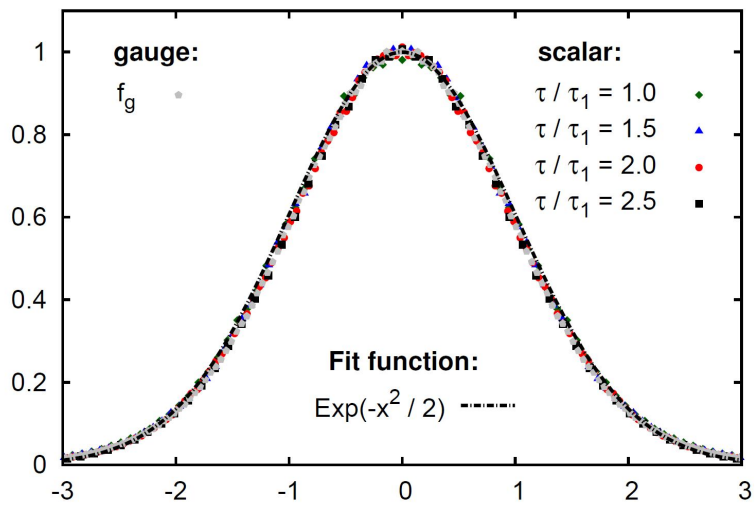
- Universality in expanding systems: *Break-down of naive kinetic theory?* Is there a *general principle* for scattering processes?

- Use *ultracold Bose gases to experimentally* access universal properties of systems under extreme conditions (*inflaton cosmology, heavy-ion collisions*)?




# Thank you for your attention!



# BACKUP SLIDES

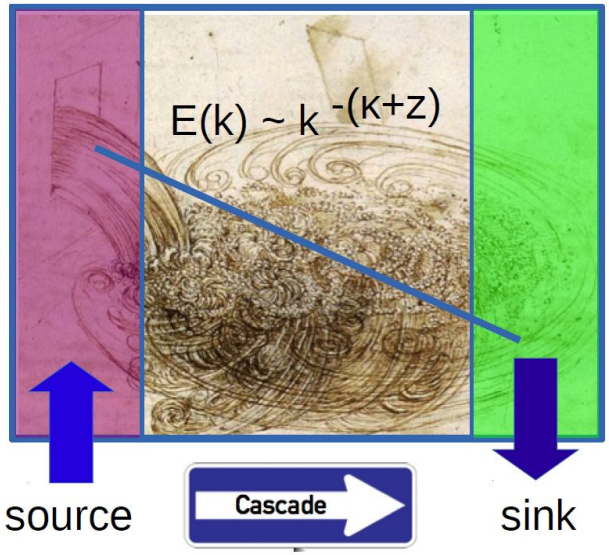
# Manifestation of turbulence

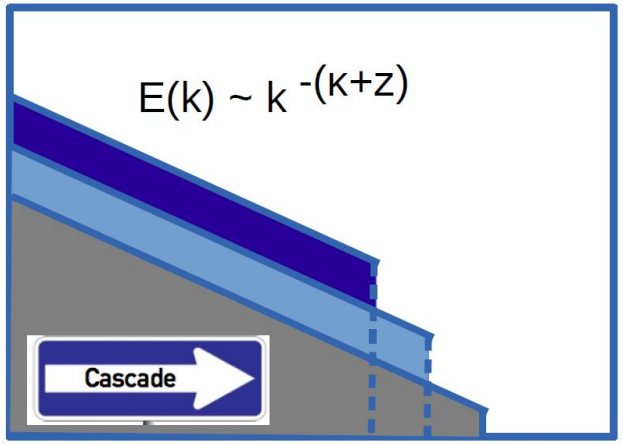
Figures by
S. Schlichting

***"Driven" Turbulence – Kolmogorov wave turbulence***

***vs.***

***"Free" Turbulence – Turbulent Thermalization***




closed system

- ***Stationary scaling solution*** associated to scale invariant energy flux

- ***quasi-stationary solution*** with universal ***non-thermal spectral exponents***

- ***Self-similar evolution*** with universal ***dynamical scaling exponents***

Uriel Frisch, *"Turbulence. The Legacy of A. N. Kolmogorov."*

Zakharov, V. E.; L'vov, V. S.; Falkovich, G, *"Kolmogorov spectra of turbulence 1. Wave turbulence."*

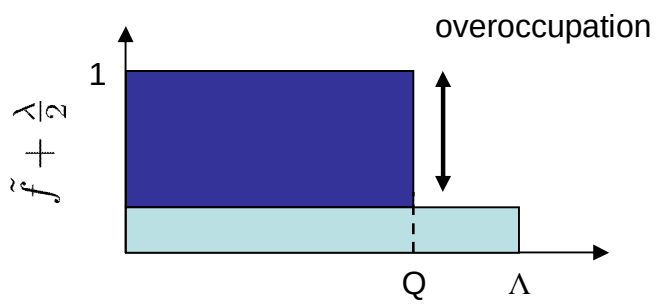
# Limitations of classical-statistical simulations

Rescaling: $\Box\,\varphi + \frac{\lambda}{6}\varphi^3 = 0 \xrightarrow{\varphi=\tilde{\varphi}/\sqrt{\lambda}} \Box\,\tilde{\varphi} + \frac{1}{6}\tilde{\varphi}^3 = 0$

Coupling drops out of classical dynamics

→ enters initial conditions, e.g. $\langle\tilde{\varphi}\tilde{\varphi}\rangle(t=0,p) = \frac{\tilde{f}(p)+\frac{\lambda}{2}}{\omega(p)}$

Coupling controls „quantum-$\frac{1}{2}$"




For finite cutoff $\Lambda \gg Q$

accurate low-energy description for sufficiently small coupling $\lambda$

If this classicality condition not met:

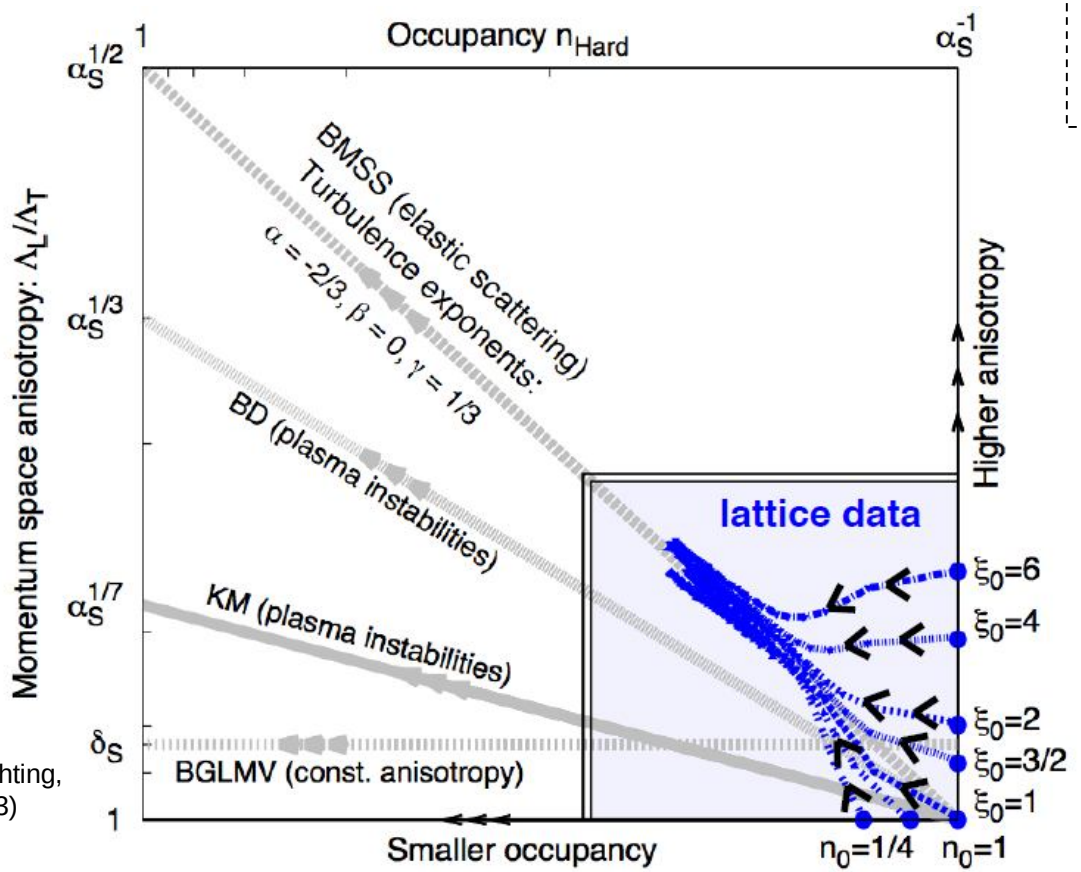
rapid approach to classical thermal equilibrium,

Evolution sensitive to cutoff, leads to negative occupancies

Berges, KB, Schlichting & Venugopalan (2013) ; Epelbaum, Gelis, Tanji & Wu (2014)

→ *Break-down of this numerical approach!*

# Turbulent thermalization of expanding gauge theory




Berges, KB, Schlichting, Venugopalan (2013)
arXiv: **1303.5650**

Reminder: Self-similar evolution

$$f(p_T, p_z, t) = t^{\alpha} f_S(t^{\beta} p_T, t^{\gamma} p_z)$$

Simulations approach nonthermal fixed point with universal scaling exponents

$$\begin{aligned} \alpha &= -2/3 \\ \beta &= 0 \\ \gamma &= 1/3 \end{aligned}$$

Nonthermal fixed point matches the BMSS scenario!

Thermalization scenarios

Baier, Mueller, Schiff, Son ( BMSS ), (2001)

Bodeker ( BD ), (2005)

Kurkela, Moore ( KM ), (2011)

Blaizot, Gelis, Liao, McLerran, Venugopalan ( BGLMV ), (2012)

# Low-momentum universality class of scalar theories

Non-expanding relativistic scalar field theory:

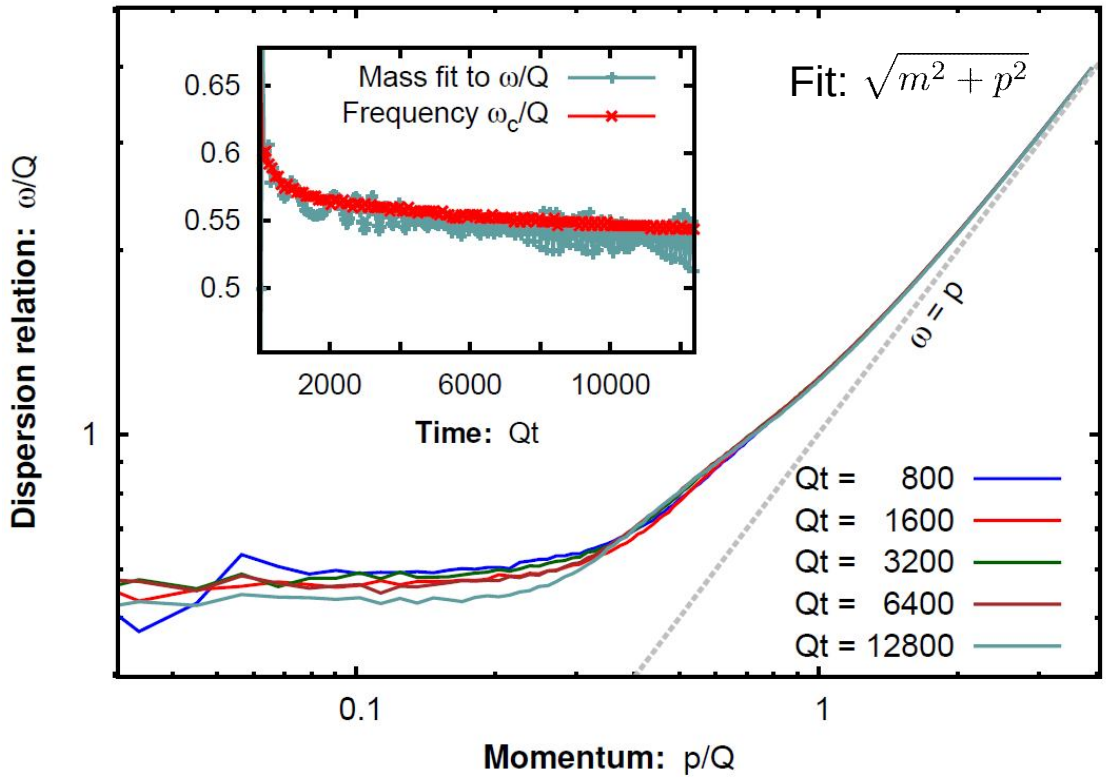
Dynamically generated mass

Dispersion relation

$$\omega(|p|) = \sqrt{\frac{\ddot{F}(|p|)}{F(|p|)}}$$

In inset shown:

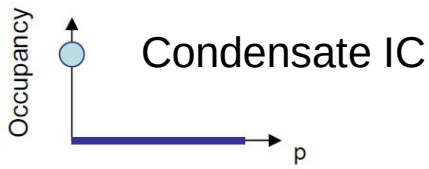
Oscillation frequency $\omega_c$ of the condensate $\phi$



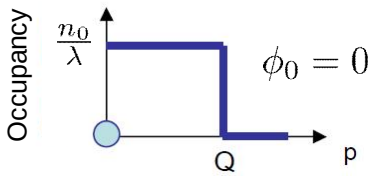
# Longitudinally expanding scalar fields

Non-thermal fixed-point 2): *Independence of initial conditions*

Initial conditions



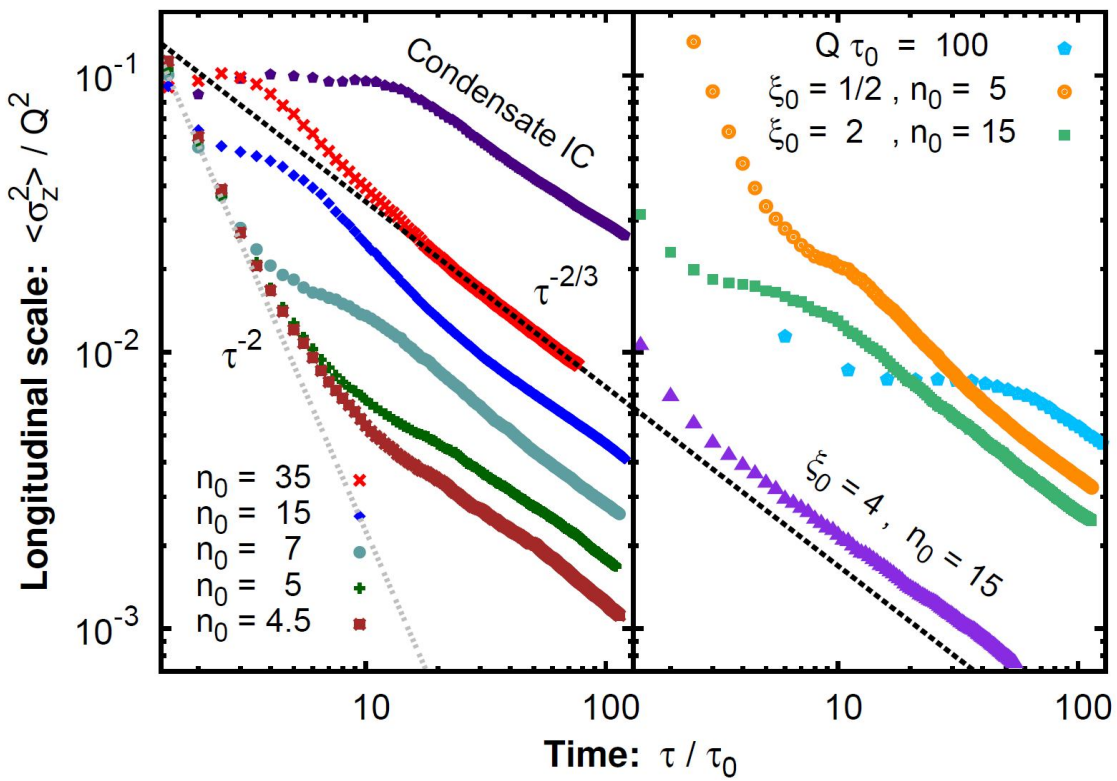

$\phi_0 \sim 1/\sqrt{\lambda}$ $\quad f = 0$




$$f(p_T, p_z, \tau_0) = \frac{n_0}{\lambda}\Theta\left(Q - \sqrt{p_T^2 + (\xi_0 p_z)^2}\right)$$

From comparison

$$2\gamma = 0.66 \pm 0.05$$

Longitudinal scale (averaged over region 2)

$$\langle\sigma_z^2\rangle(\tau) := \langle\int dp_z\, p_z^2 f\rangle / \langle\int dp_z f\rangle$$