Exploring the Phase Structure and the Dynamics of QCD

RIKEN Lunch Seminar
12/04/14 BNL

Nils Strodthoff, ITP Heidelberg
Outline

QCD Phase Structure
- QCD phase structure from functional approaches
- Quenched QCD in the vacuum
Outline

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- QCD phase structure from functional approaches
- Quenched QCD in the vacuum

Dynamics I Spectral functions
- Spectral functions from a Euclidean framework
- Mesonic spectral functions in simple models
QCD Phase Structure
- QCD phase structure from functional approaches
- Quenched QCD in the vacuum

Dynamics I Spectral functions
- Spectral functions from a Euclidean framework
- Mesonic spectral functions in simple models

Dynamics II Transport Coefficients
- Kubo formula with expansion in full propagators/vertices
- Transport coefficients in YM and QCD
QCD Phase Structure

The QCD phase diagram?

- Fukushima, Hatsuda Rept.Prog.Phys. 74 (2011) 014001
- adapted from GSI
The QCD phase diagram?

- Fukushima, Hatsuda Rept.Prog.Phys. 74 (2011) 014001

Phase structure at large chemical potentials largely unknown due to sign problem in lattice QCD...

- adapted from GSI
Continuum perspective
...using functional approaches

Functional relations between off-shell Green’s functions
Continuum perspective
...using functional approaches

Functional relations between off-shell Green’s functions

e.g. Dyson-Schwinger equation for quark propagator

\[
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Continuum perspective
...using functional approaches

Functional relations between off-shell Green’s functions

e.g. Dyson-Schwinger equation for quark propagator

✓ Easy access to mechanisms:
  • Chiral symmetry breaking
  • Confinement

✓ Complementary to the lattice
✓ No sign problem
✓ Effective models incorporated
Functional Approaches: finite T

PQM model, Nf=2+1, FRG

- Herbst, Mitter, Pawlowski, Schaefer, Stieble

Matter+Glue system, Nf=2, FRG

- Braun, Haas, Marhauser, Pawlowski
  PRL 106 (2011) 022002

Quark propagator, Nf=2+1, DSE

- Fischer, Luecker, Welzbacher
Functional Approaches: finite $T$ & $\mu$

**PQM model, Nf=2, FRG**


**Quark propagator, Nf=2+1, DSE**

Functional Approaches: finite T & μ

PQM model, Nf=2, FRG


But: so far all require additional phenomenological input

PQM-model: UV parameters, glue input (Polyakov-loop potential)
DSE calculation: vertex models e.g. for the quark-gluon vertex

Quark propagator, Nf=2+1, DSE

fQCD Collaboration

fQCD Collaboration (J. Braun, L. Fister, T. K. Herbst, M. Mitter, J. M. Pawlowski, F. Rennecke and N. Strodthoff)

- Mitter, Pawlowski, NSt arXiv:1411.7978
- Braun, Fister, Pawlowski, Rennecke arXiv:1412.1045
✓ Finite $\mu$ requires fluctuations to be **quantitatively** under control

✓ Mismatches in fluctuation scales lead to large systematic errors at finite $\mu$
  - Helmboldt, Pawlowski, NSt arXiv:1409.8414
Finite $\mu$ requires fluctuations to be quantitatively under control.

Mismatches in fluctuation scales lead to large systematic errors at finite $\mu$.

No phenomenological input (vertex models, running couplings...)

Input parameters only the fundamental parameters of QCD:

- $\alpha_s(20 \text{ GeV})$ strong running coupling
- $M_q(20 \text{ GeV}) \approx 1-2 \text{ MeV}$ current quark mass

at large perturbative momenta.
Finite $\mu$ requires fluctuations to be quantitatively under control.

Mismatches in fluctuation scales lead to large systematic errors at finite $\mu$.

No phenomenological input (vertex models, running couplings...)

Input parameters only the fundamental parameters of QCD

$\alpha_s (20 \text{ GeV})$ strong running coupling

$M_q(20 \text{ GeV}) \approx 1-2 \text{ MeV}$ current quark mass

at large perturbative momenta.

Quantitative FRG approach towards the investigation of the phase diagram and the hadron spectrum.
Functional RG for QCD

- Spirit of **Wilson RG**: Calculate full quantum effective action $\Gamma$ by integrating fluctuations with momentum $k$

\[ k \rightarrow 0 \quad \Gamma \quad \Gamma_k \quad S \quad k \rightarrow \Lambda_{UV} \]
Functional RG for QCD

- Spirit of **Wilson RG**: Calculate full quantum effective action $\Gamma$ by integrating fluctuations with momentum $k$

$$\Gamma_k \rightarrow 0 \Gamma \rightarrow \Lambda_{UV}$$

Functional Renormalization Group (FRG)

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}$$
- Free energy/Grand potential
- gluon
- ghost
- quark
- hadrons
**Functional RG for QCD**

- **Spirit of Wilson RG**: Calculate full quantum effective action $\Gamma$ by integrating fluctuations with momentum $k$

$$k \rightarrow 0 \quad \Gamma_k \quad \Gamma \quad k \rightarrow \Lambda_{UV}$$

**Functional Renormalization Group (FRG)**

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}$$

Gluon - Ghost - Quark + $\frac{1}{2}$

Free energy / Grand potential

**Dynamical hadronization**


Store resonant 4-Fermi structures in terms of effective mesonic interactions
Truncation

Vertex expansion

FRG Yang-Mills results
Truncation

Vertex expansion

FRG Yang-Mills results

mom. dep.
classical tensor structure

mom. dep.
classical tensor structure
Truncation

Vertex expansion

FRG Yang-Mills results

mom. dep.
classical tensor structure

full mom. dep.
Truncation

Vertex expansion

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full mom. dep.
all tensor structures
Truncation

Vertex expansion

FRG Yang-Mills results

mom. dep. classical tensor structure

mom. dep. classical tensor structure

full mom. dep. all tensor structures

STI-consistent dressing

-1

full mom. dep.
Truncation

Vertex expansion

FRG Yang-Mills results

-1
-1

full mom. dep.

full mom. dep.
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mom. dep.
classical tensor structure

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mom. dep.

STI-consistent dressing

Fierz-complete basis

at $p = 0$ and mom. dep.
Truncation

Vertex expansion

- FRG Yang-Mills results
- Full mom. dep.
- Fierz-complete basis at $p = 0$ and mom. dep.
- STI-consistent dressing
- Mom. dep.
- Mom. dep.
- Mom. dep. classical tensor structure
Truncation

Vertex expansion

- FRG Yang-Mills results
- mom. dep. classical tensor structure
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Vertex expansion

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mom. dep. classical tensor structure

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full mom. dep. all tensor structures

STI-consistent dressing

Fierz-complete basis at \( p = 0 \) and mom. dep.

mom. dep.

mom. dep.

full effective potential
Truncation

Vertex expansion

Computer-algebraic generation of equations using DoFun

Propagators (T=0)

Quenched gluon propagator (input)

\[ \Gamma^{\mu\nu}_{A^2}(p) = Z_A(p)p^2 \Pi^{\mu\nu}_T(p) \]

- Fischer, Maas, Pawlowski Annals Phys. 324, 2408 (2009)
- Fister, Pawlowski in prep.
Propagators (T=0)

Quenched gluon propagator (input)

\[ \Gamma_{A^2}^{\mu\nu}(p) = Z_A(p)p^2 \Pi_T^{\mu\nu}(p) \]

Quark propagator

\[ \Gamma_{\bar{q}q}(p) = Z_q(p)(i\not{p} + M_q(p)) \]

Very good agreement with (quenched) lattice results!

- Fischer, Maas, Pawlowski Annals Phys. 324, 2408 (2009)
- Fister, Pawlowski in prep.
- Mitter, Pawlowski, NSt arXiv:1411.7978
Chiral symmetry breaking

β-function:

\[ k \partial_k \hat{\lambda}_\psi = (d - 2) \hat{\lambda}_\psi - a \hat{\lambda}_\psi^2 - b \hat{\lambda}_\psi g^2 - cg^4 \]

Chiral symmetry breaking

\( \beta \)-function:

\[
k \partial_k \hat{\lambda}_\psi = (d - 2) \hat{\lambda}_\psi - a \hat{\lambda}_\psi^2 - b \hat{\lambda}_\psi g^2 - cg^4
\]

- reflects gluon mass gap
- area above the critical value decides

Effective model perspective

- Independence of initial scale and initial condition
Effective model perspective

- Independence of initial scale and initial condition
- only requirement: decoupling of gluons
- Low-energy models completely fixed by QCD flow
Unquenching: first qualitative results available

- Braun, Fister, Pawlowski, Rennecke arXiv:1412.1045
Outlook

Unquenching: first qualitative results available

Shopping list
✓ Quantitative results in the vacuum
☐ Full unquenching
☐ Quantitative investigations in the vacuum (YM vertices, 4-Fermi)
☐ Transition to low-energy effective models
☐ Finite temperature
☐ Finite Density (important: role of baryonic/diquark d.o.f.)

Braun, Fister, Pawlowski, Rennecke arXiv:1412.1045
Dynamics I
Spectral Functions

- Helmboldt, Pawlowski, NSt arXiv:1409.8414
Spectral Functions

Real-time observable from Euclidean framework

\[
\Gamma_R^{(2)}(\omega, \vec{p}) = -\lim_{\epsilon \to 0} \Gamma_E^{(2)}(-i(\omega + i\epsilon), \vec{p})
\]

\[
\rho(\omega, \vec{p}) = \frac{\text{Im} \, \Gamma_R^{(2)}(\omega, \vec{p})}{\text{Im} \, \Gamma_R^{(2)}(\omega, \vec{p})^2 + \text{Re} \, \Gamma_R^{(2)}(\omega, \vec{p})^2}
\]

requires analytical continuation from Euclidean to Minkowski signature numerically hard or even ill-posed problem
Spectral Functions

Real-time observable from Euclidean framework

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requires analytical continuation from Euclidean to Minkowski signature numerically hard or even ill-posed problem

Popular approaches (based on Euclidean data)
- Maximum Entropy Method (MEM)
- Padé Approximants
Spectral Functions

Real-time observable from Euclidean framework

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\]

requires analytical continuation from Euclidean to Minkowski signature numerically hard or even ill-posed problem

Popular approaches (based on Euclidean data)

- Maximum Entropy Method (MEM)
- Padé Approximants

Alternative: analytic continuation on the level of the functional equation

- Floerchinger JHEP 1205 (2012) 021
- Strauss, Fischer, Kellermann PRL **109** (2012) 252001
Analytical continuation

Analytical continuation

- Compute flow equation for Euclidean 2-point function
  perform analytically for 3d regulator function

\[ R = \bar{p}^2 r(\bar{p}^2) \]
• Compute flow equation for Euclidean 2-point function
  perform analytically for 3d regulator function
  \[ R = \bar{p}^2 r(\bar{p}^2) \]

• Perform analytical continuation in ext. momentum
  \[ p_0 \rightarrow -i(\omega + i\epsilon) \]
• Compute flow equation for Euclidean 2-point function perform analytically for 3d regulator function $R = \bar{p}^2 r(\bar{p}^2)$

$\Gamma^{(3)}_k$

$\Gamma^{(3)}_k$

$p + q$

• Perform analytical continuation in ext. momentum $p_0 \rightarrow -i(\omega + i\epsilon)$

• Ensure correct continuation $n_{B/F}(E + ip_0) \rightarrow n_{B/F}(E)$
Compute flow equation for Euclidean 2-point function perform analytically for 3d regulator function \( R = \vec{p}^2 r(\vec{p}^2) \)

Perform analytical continuation in ext. momentum
\[
p_0 \to -i(\omega + i\epsilon)
\]

Ensure correct continuation
\[
n_B/F(E + ip_0) \to n_B/F(E')
\]

For small but finite \( \epsilon \) compute real and imaginary part of
\[
-\Gamma^{(2)}_E(-i(\omega + i\epsilon), \vec{p})
\]
Analytical continuation

- Compute flow equation for Euclidean 2-point function perform analytically for 3d regulator function $R = \vec{p}^2 r(\vec{p}^2)$

- Perform analytical continuation in ext. momentum

- Ensure correct continuation

- For small but finite $\varepsilon$ compute real and imaginary part of $-\Gamma^{(2)}_E (-i(\omega + i\varepsilon), \vec{p})$

**Test cases:** simple bosonic/ Yukawa models
Mesonic Spectral Functions

O(N) model (T=0)

- Kamikado, NSt, von Smekal, Wambach
Mesonic Spectral Functions

O(N) model (T=0)

Quark-meson (Yukawa) model

QM Model at $T>0$

$\rho_\sigma$

$\rho_\pi$

$\Lambda_{UV}^{-2}$

$T=10$ MeV

1: $\sigma^* \rightarrow \sigma\sigma$
2: $\sigma^* \rightarrow \pi\pi$
3: $\sigma^* \rightarrow \tilde{\psi}\tilde{\psi}$
4: $\pi^* \rightarrow \sigma\pi$
5: $\pi^*\pi \rightarrow \sigma$
6: $\pi^* \rightarrow \tilde{\psi}\tilde{\psi}$

QM Model at T > 0

QM Model at $T > 0$

QM Model at $T>0$

Outlook
Outlook

Generalization towards a fully numerical procedure

First step: Euclidean momenta (via an iterative procedure)

- Helmboldt, Pawlowski, NSt arXiv:1409.8414

<table>
<thead>
<tr>
<th>step</th>
<th>( m_{\text{cur}} ) [MeV]</th>
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<td>412.8</td>
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<td>1</td>
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**Second step:** Minkowski external momenta

- Pawlowski, NSt in prep.
Outlook

Generalization towards a fully numerical procedure

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- Helmboldt, Pawlowski, NSt arXiv:1409.8414

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**Second step:** Minkowski external momenta
- Pawlowski, NSt in prep.

**Shopping list**
- Continuation procedure set-up
- Tested in simple models
- Generalization towards a fully numerical procedure
- Quark & gluon spectral functions
- Vector meson spectral functions
- Charmonium spectral functions
Dynamics II
Transport Coefficients

- Christiansen, Haas, Pawlowski, arXiv:1411.7986
Transport Coefficients

- Evolution of the hot plasma well-described by hydrodynamics
- Extract viscosity from $v_2$
- Transport coefficients as important microscopic input
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Transport Coefficients

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![Graph showing transport coefficients](image)


**Kubo formula**

$$\eta = \lim_{\omega \to 0} \frac{1}{20} \frac{\rho_{\pi\pi}(\omega, \vec{0})}{\omega}$$

Require

$$\rho_{\pi\pi}(\omega, \vec{p}) = \int_x e^{-i\omega x_0 + ip\vec{x}} \langle [\pi_{ij}(x), \pi_{ij}(0)] \rangle$$
DSE-like expansion formula

\[ \langle \pi_{i,j} [\hat{A}] \pi_{i,j} [\hat{A}] \rangle = \pi_{i,j} [G_{A\phi_{k}} \frac{\delta}{\delta \phi_{k}} + A] \pi_{i,j} [G_{A\phi_{k}} \frac{\delta}{\delta \phi_{k}} + A] \]
Computing EM Correlators

DSE-like expansion formula

\[ \langle \pi_{i,j} [\hat{A}] \pi_{i,j} [\hat{A}] \rangle = \pi_{i,j} \left[ G A \phi_k \frac{\delta}{\delta \phi_k} + A \right] \pi_{i,j} \left[ G A \phi_k \frac{\delta}{\delta \phi_k} + A \right] \]

Finite number of diagrams involving full propagators/vertices

All diagrams to 2-loop order
Computing EM Correlators

DSE-like expansion formula

\[ \langle \pi_{ij}[\hat{A}]\pi_{ij}[\hat{A}] \rangle = \pi_{ij} [G A \phi_k \frac{\delta}{\delta \phi_k} + A] \pi_{ij} [G A \phi_k \frac{\delta}{\delta \phi_k} + A] \]

Finite number of diagrams involving full propagators/vertices

Input: **gluon spectral function** from Euclidean FRG data using MEM

$\eta/s$ in Yang-Mills Theory

$T_{\text{min}} = 1.26 \, T_c$
Value = 0.14

Christiansen, Haas, Pawlowski, NSt arXiv:1411.7986
η/s in Yang-Mills Theory

Direct sum: 
\[ \eta_s(T) = \frac{a}{\alpha_s^{\gamma}} + \frac{b}{(T/T_c)^\delta} \]

High T: consistent with HTL-resummed pert. theory supporting quasiparticle picture

Small T: algebraic decay glueball resonance gas

T_{min} = 1.26 T_c
Value = 0.14

Christiansen, Haas, Pawlowski, NSt arXiv:1411.7986
More 2-Loop

- Consistent with 1-loop around $T_c$
- Dominant contribution from Maki-Thompson and Eight at large $T$
From YM to QCD in three simple steps

1. Replace $\alpha_s$; impose equality at $T_c$
   $$\alpha_s^{N_f=0}|_{T_c} = \alpha_s^{N_f=3}|_{T_c}$$

2. Genuine quark contributions to $\eta$ and $s$

3. Replace GRG by HRG
   ➤ Demir, Bass PRL 102 (2009) 172302

$\eta/s$ in QCD
From YM to QCD in three simple steps

1. Replace $\alpha_s$; impose equality at $T_c$
   $$\alpha_s^{N_f=0}|_{T_c} = \alpha_s^{N_f=3}|_{T_c}$$

2. Genuine quark contributions to $\eta$ and $s$

3. Replace GRG by HRG

Demir, Bass PRL 102 (2009) 172302

$T_{\text{min}} = 1.3 \, T_c$
Value: 0.17
Outlook

Quark contributions

Require quark and gluon spectral functions in QCD
Outlook

Quark contributions

Require quark and gluon spectral functions in QCD

Shopping List

☑ Formalism set-up
☑ Quantitative results for \( \eta/s \) in YM
☐ Bulk viscosity
☐ Relaxation times
☐ Application to non-relativistic systems e.g. ultracold atoms
Summary

- QCD phase structure
towards a quantitative continuum approach to QCD
  ✓ Quantitative grip on fluctuation physics in the vacuum
  ❑ finite temperature and density
Summary

• **QCD phase structure**
  towards a quantitative continuum approach to QCD
  ✓ Quantitative grip on fluctuation physics in the vacuum
  ❑ finite temperature and density

• **Spectral Functions**
  new approach to analytical continuation problem
  ✓ tested in simple models (O(N), QM model)
  ❑ quark & gluon spectral functions, vector mesons, charmonium
Summary

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• **Transport Coefficients**
  from loop expansion involving full propagators and vertices
  ✓ Global quantitative prediction for $\eta/s$ in YM theory
  ❑ Full QCD, bulk viscosity, relaxation times
Summary

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Thank you for your attention!
More Matter system?

Quark-Gluon vertex

- Take into account all 8 tensor structures of the trans. projected vertex
- 5,7 most important (non-classical) tensor structures in the symmetric phase
- **but keep in mind gauge-invariance**

\[
\begin{align*}
[T_{\overline{q}Aq}^{(1)}]^{\mu}(p,q) &= \gamma^{\mu}, \\
[T_{\overline{q}Aq}^{(2)}]^{\mu}(p,q) &= -i(p-q)^{\mu}, \\
[T_{\overline{q}Aq}^{(3)}]^{\mu}(p,q) &= -i(\bar{q} - q)\gamma^{\mu}, \\
[T_{\overline{q}Aq}^{(4)}]^{\mu}(p,q) &= i(\bar{q} + q)\gamma^{\mu}, \\
[T_{\overline{q}Aq}^{(5)}]^{\mu}(p,q) &= (\bar{q} + q)(p-q)^{\mu}, \\
[T_{\overline{q}Aq}^{(6)}]^{\mu}(p,q) &= -(\bar{q} - q)(p-q)^{\mu}, \\
[T_{\overline{q}Aq}^{(7)}]^{\mu}(p,q) &= \frac{1}{2}[\bar{q}, q]\gamma^{\mu}, \\
[T_{\overline{q}Aq}^{(8)}]^{\mu}(p,q) &= -\frac{1}{2}[\bar{q}, q](p-q)^{\mu},
\end{align*}
\]
STI-consistent expansion

Setting up a sensible truncation scheme:
- Use an expansion in terms of gauge-invariant operators
- Construct from combinations of covariant derivatives
STI-consistent expansion

Setting up a sensible truncation scheme:

• **Use an expansion in terms of gauge-invariant operators**
• **Construct from combinations of covariant derivatives**

• e.g. for the dominant (non-classical) structure in the chirally symmetric regime: gauge invariant operator

\[ i\sqrt{4\pi\alpha_s} \bar{q} \gamma_5 \gamma_\mu \epsilon_{\mu\nu\rho\sigma} \{ F_{\nu\rho}, D_\sigma \} q \]

gives rise to contribution proportional to

\[ \frac{1}{2} T_{\bar{q}Aq}^{(5)} + T_{\bar{q}Aq}^{(7)} \]
STI-consistent expansion

Setting up a sensible truncation scheme:
- Use an expansion in terms of gauge-invariant operators
- Construct from combinations of covariant derivatives

- e.g. for the dominant (non-classical) structure in the chirally symmetric regime: gauge invariant operator

\[ i \sqrt{4\pi \alpha_s} \bar{q} \gamma_5 \gamma_\mu \epsilon_{\mu \nu \rho \sigma} \{ F_{\nu \rho}, D_\sigma \} q \]

gives rise to contribution proportional to

\[ \frac{1}{2} \mathcal{T}_{\bar{q}Aq}^{(5)} + \mathcal{T}_{\bar{q}Aq}^{(7)} \]

- Associated non-classical vertices are quantitatively important
(a) Renormalisation group scale dependence of dimensionless four-fermi interactions, see App. B 2 c and bosonised $\sigma-\pi$ channel. Grey: respects chiral symmetry, blue: breaks $U(1)_A$, red: breaks $SU(2)_A$, magenta: breaks $U(2)_A$. 
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- Bosonizing the $\sigma$-$\pi$ channel only is sufficient to remove divergence
- In the vacuum: other channels not quantitatively relevant
Euclidean Iteration

Why momentum dependence?

Quantitative precision
Why momentum dependence?

Quantitative precision

- QCD perspective on low-energy effective models:
  - **UV parameters fixed** by QCD flows
    - Talks by L. Fister, M. Mitter, J. Pawlowski, F. Rennecke
  - Increase in predictive power
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• **Benchmark** of popular **truncation schemes** (LPA and LPA’)

• Momentum dependence crucial for critical physics
Euclidean Iteration I

Momentum dependence of 2-point functions in an iterative procedure
Example: mesonic propagators in a quark meson model

\[ \partial_t U_k = \frac{1}{2} \]

\[ \partial_t \Delta \Gamma_k^{(2)}(p^2) = \begin{bmatrix} p - q & q & q \\ p + q - \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} \rightarrow 0 \end{bmatrix} \]
Euclidean Iteration I

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momentum-independent vertices from eff. potential
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Iteration procedure

\[ \Gamma_{cl,k}^{(2)} \rightarrow U_k \rightarrow \Gamma_k^{(2)} \rightarrow U_k \rightarrow \Gamma_k^{(2)} \rightarrow \ldots \]
• **Numerically inexpensive upgrade** for existing Euclidean calculations

• Here: **Quark-meson model at finite T; fixed ren. Yukawa coupling**

• **4d exponential** regulator function
Euclidean Iteration II

- **Numerically inexpensive upgrade** for existing Euclidean calculations

- Here: **Quark-meson model at finite T; fixed ren. Yukawa coupling**

- **4d exponential** regulator function

- Convergence properties:

<table>
<thead>
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<th>step</th>
<th>$m_{\text{cur}}$ [MeV]</th>
<th>$m_{\text{pol}}$ [MeV]</th>
<th>$\sigma_{\text{min}}$ [MeV]</th>
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<td>16.8</td>
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<td>2</td>
<td>136.4</td>
<td>$135 \pm 2$</td>
<td>91.8</td>
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<tr>
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<td>135.1</td>
<td>$134 \pm 2$</td>
<td>93.1</td>
</tr>
<tr>
<td>4</td>
<td>134.9</td>
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<td>93.2</td>
</tr>
<tr>
<td>5</td>
<td>134.9</td>
<td>$133 \pm 2$</td>
<td>93.2</td>
</tr>
</tbody>
</table>
Mass Definitions

Renormalized 2-point function:

\[ \bar{\Gamma}^{(2)}(p_0, \vec{p}^2) = \Gamma^{(2)}(p_0, \vec{p}^2) / \bar{Z} \]
Mass Definitions

Renormalized 2-point function:

\[ \tilde{\Gamma}^{(2)}(p_0, \vec{p}^2) = \Gamma^{(2)}(p_0, \vec{p}^2) / \tilde{Z} \]

Pole mass:

\[ \tilde{\Gamma}^{(2)}(im_{pol}, 0) = 0 \]

Temporal screening:

\[ T \sum_{p_0} \Gamma^{(2)}(p_0, 0)^{-1} e^{ip_0 t} \sim e^{-m_{pol} |t|} \]
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No physical observable; dependent on renormalization procedure, parameterization of the propagator.
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Onset mass: Silver Blaze property links mass to critical chemical potential; coincides with pole mass
Physics Results

From converged iteration

\[ m_{\text{cur}} \]
\[ m_{\text{pol}} \]
\[ m_{\text{scr}} \]
**Physics Results**

From converged iteration:

\[ m_{\text{cur}} \]
\[ m_{\text{pol}} \]
\[ m_{\text{scr}} \]

\[ T=0: \quad m_{\text{pol}} = m_{\text{scr}} \quad \text{by O(4) invariance} \]

\[ m_{\text{pol}} \approx m_{\text{cur}} : \quad m_{\text{cur}}^2 = \frac{Z\parallel(p_0=im_{\text{pol}},p^2=0)}{Z} m_{\text{pol}}^2 \]
Physics Results

From converged iteration

$T > 0$:

$m_{\text{pol}} = m_{\text{scr}}$ by O(4) invariance

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$T = 0$:

$m_{\text{pol}} = m_{\text{scr}}$ by O(4) invariance

$\frac{m_{\text{pol}}^2}{m_{\text{scr}}^2} = \frac{Z_{\perp}(p_0=0, \vec{p}^2=-m_{\text{scr}}^2)}{Z_{\parallel}(im_{\text{pol}}, \vec{p}^2=0)}$
LPA: Mismatches of Fluctuation Scales

More than an academic exercise...

\[ m_{\text{cur}} \approx m_{\text{pole}} = m_{\text{ons}} \]

\[
\left[ \frac{\mu_c}{T_c} \right]_{\text{full}} / \left[ \frac{\mu_c}{T_c} \right]_{\text{LPA}} \approx \left[ \frac{m_{\text{cur}}}{m_{\text{ons}}} \right]_{\text{LPA}} \approx 1.4
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- quantum fluctuations
- density fluctuations

Rough estimate:

\[ \left[ \frac{\mu_c}{T_c} \right]_{\text{full}} / \left[ \frac{\mu_c}{T_c} \right]_{\text{LPA}} \approx \left[ \frac{m_{\text{cur}}}{m_{\text{ons}}} \right]_{\text{LPA}} \approx 1.4 \]

- mismatch of fluctuation scales
  => large systematic errors at finite \( \mu \) (curvature, CEP)
- resolved by including momentum dependence
Comparison: Fixed UV

QCD perspective

- LPA with these initial conditions => no $\chi_{SB}$
- Full calculation and LPA' in quantitative agreement
• LPA’ includes only a **scale-dependent Z**
• Very good approximation to the full calculation (**deviation < 3 %**)
• Upgrade: calculate momentum dependence on LPA’ solution (1 step)
Comparison: Fixed IR

Model perspective

- Reasonably good agreement at $\mu=0$ (in terms of relative scales)
- But in LPA still **large systematic error at finite $\mu$**
Spectral Functions

- Tripolt, NSt, von Smekal, Wambach; Phys.Rev. D89 (2014) 034010
QM Model at $\mu > 0$
QM Model at $\mu > 0$

$\mu = 200$ MeV

$\Lambda_{\text{UV}}^2$

$\rho_{\pi}$

$\rho_{\sigma}$

$\omega$ [MeV]

1: $\sigma^* \rightarrow \sigma \sigma$
2: $\sigma^* \rightarrow \pi \pi$
3: $\sigma^* \rightarrow \bar{\psi} \psi$
4: $\pi^* \rightarrow \sigma \pi$
5: $\pi^* \pi \rightarrow \sigma$
6: $\pi^* \rightarrow \bar{\psi} \psi$
QM Model at $\mu > 0$

$\mu = 292$ MeV

1: $\sigma^* \rightarrow \sigma \sigma$
2: $\sigma^* \rightarrow \pi \pi$
3: $\sigma^* \rightarrow \bar{\psi} \psi$
4: $\pi^* \rightarrow \sigma \pi$
5: $\pi^* \pi \rightarrow \sigma$
6: $\pi^* \rightarrow \bar{\psi} \psi$
QM Model at $\mu > 0$

\[ \mu = 292.97 \text{ MeV} \]

\[ \mu = 292.8 \text{ MeV} \]

1: $\sigma^* \rightarrow \sigma \sigma$
2: $\sigma^* \rightarrow \pi \pi$
3: $\sigma^* \rightarrow \bar{\psi} \psi$
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Going beyond...

So far: 3d regulator function \[ R = \vec{p}^2 r(\vec{p}^2) \]
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- Either regulators which allow to perform Matsubara sums analytically
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  • Require: analytical regulator function for complex momenta
  • for free: finite chemical potential
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- Or **fully numerical procedure**- perform Matsubara sum numerically
  - Require: analytical regulator function for complex momenta
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**Shopping list:**

- Proper regulator for complex external momenta
- Suitable for numerical applications
- Analytical functions
- Analytical structure of regularized propagator: as few poles as possible
  - Require pole procedures to obtain the correct real-time result
4d Spectral Functions

Preliminary

Pawlowski, NSt [in prep.]