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Dysonian dynamics of the Ginibre ensemble

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Outline

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1. Dysons Brownian motion and the complex Burgers equation
2. The Non-Hermitian version
3. Examples

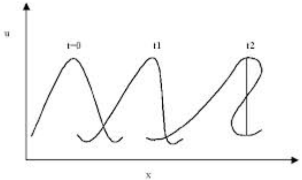
Diffusing Hermitian matrices

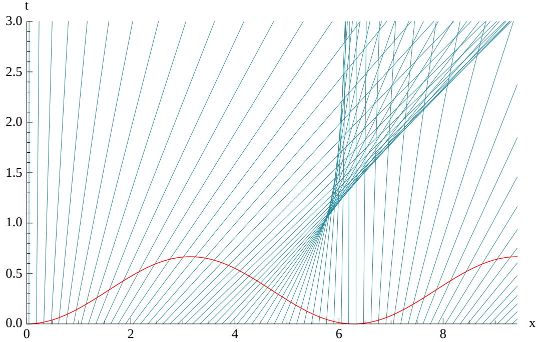
- $H_{ij} \to H_{ij} + \delta H_{ij}$ with $< \delta H_{ij} = 0 >$ and $< (\delta H_{ij})^2 >= (1 + \delta_{ij})\delta t$
- Eigenvalues x_i, as mutually repulsing particles performing brownian motion (Dyson) $< \delta x_i > \equiv E(x_i)\delta t = \sum_{i \neq j} \left(\frac{1}{x_j - x_i} \right) \delta t$ and $< (\delta x_i)^2 >= \delta t$
- SFP eq: $\partial_t P(\{x_j\}, t) = \frac{1}{2} \sum_i \partial_{ii}^2 P(\{x_j\}, t) - \sum_i \partial_i (E(x_i) P(\{x_j\}, t))$
- Resulting SFP equation for the resolvent
- in the limit $N = \infty$ and $\tau = Nt$ reads $\partial_\tau G(z, \tau) + G(z, \tau)\partial_z G(z, \tau) = 0$ where $G(z, \tau) = \frac{1}{N} \left\langle \mathrm{tr} \frac{1}{z - H(\tau)} \right\rangle$ is the resolvent
- Non-linear, inviscid complex Burgers (Hopf, Voiculescu) equation

Two words about the real viscous Burgers equation

- $\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = \mu \partial_{xx} u(x,t)$
  $u(x,t)$ is the velocity field at time t and position x of the fluid with viscosity μ.
- One-dimensional toy model for turbulence? No. Turned out to be exactly integrable $u = -\frac{1}{2\mu}\partial_x \ln h$.

  - $\mu = 0$ caustics and shocks!
    $u = u_0(x - tu)$
  - method of characteristics





Complex inviscid Burgers Equation

- Complex Burgers equation $\partial_\tau G + G\partial_z G = 0$
- Complex characteristics, trivial initial conditions
  $G(z,\tau) = G_0(\xi[z,\tau)])$ $\qquad G_0(z) = G(\tau = 0, z) = \frac{1}{z}$
  $\xi = z - G_0(\xi)\tau \quad (\xi = x - vt)$, so solution reads
  $G(z,\tau) = G_0(z - \tau G(z,\tau))$
- Shock wave when $\frac{d\xi}{dz} = \infty$
- Equivalently, $dz/d\xi = 0$, then $\xi_c = \pm\sqrt{\tau}$, so
  $z_c = \xi_c + G_0(\xi_c)\tau = \pm 2\sqrt{\tau}$
- Since explicit solution easily reads $G(z,\tau) = \frac{1}{2\pi\tau}(z - \sqrt{z^2 - 4\tau})$, i.e. $\rho(x,\tau) = \frac{1}{2\pi\tau}\sqrt{4\tau - x^2}$, we see that shock waves appear at the edges of the spectrum ($x = \pm 2\sqrt{\tau}$).

Where is the viscosity?

- Let us define $D_N(z,\tau) \equiv \langle \det(z - H(\tau)) \rangle$
- Opening the determinant with the help of auxilliary Grassmann variables and perfoming the averaging one gets easily $D_N(z,\tau) = \int \exp\left(\sum_i \bar{\eta}_i z \eta_i - \frac{\tau}{N} \sum_{i<j} \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j\right) \prod_{l,r} d\bar{\eta}_l d\eta_r$
- Differentiating and using the properties of the Grassmann variables one gets that D_N obeys complex equation $\partial_\tau D_N(z,\tau) = -\frac{1}{2N} \partial_{zz} D(z,\tau)$.

Where is the viscosity? - cont.

- $\partial_\tau D_N(z,\tau) = -\frac{1}{2N}\partial_{zz} D(z,\tau)$.
- Then complex Cole Hopf transformation $f_N(z,\tau) = \frac{1}{N}\partial_z \ln D_N(z,\tau)$ leads to exact for any N, viscid complex Burgers equation
  $\partial_\tau f_N + f_N \partial_z f_N = -\mu \partial_{zz} f_N \qquad \mu = \frac{1}{2N}$
- Positive viscosity "smoothens" the shocks, negative is "roughening" them, triggering violent oscillations
- Note than $G(z,\tau) = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z-H(\tau)}\right\rangle = \partial_z \left\langle \frac{1}{N}\mathrm{Tr}\ln(z-H(\tau))\right\rangle = \partial_z \left\langle \frac{1}{N}\ln\det(z-H(\tau))\right\rangle$ so f_N and G coincide only when $N=\infty$ (cumulant expansion).
- $\left\langle \frac{1}{N}\ln\det(z-H(\tau))\right\rangle \overset{N=\infty}{=} \frac{1}{N}\ln\left\langle\det(z-H(\tau))\right\rangle$,
- This approach allows to study macro and microscopic features of the spectral statistics. [J.-P. Blaizot, M. A. Nowak, PW]

Outline

- X, $N \times N$, random, non-Hermitian matrix (here, complex entries)
- Electrostatic potential:

$$V(z,\bar{z}) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\left\langle \mathrm{Tr}\log\left[(z-X)(\bar{z}-X^\dagger)+\epsilon^2\right]\right\rangle,$$

- Electric field

$$G\left(z,\bar{z}\right) \equiv \partial_z V(z,\bar{z}) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\left\langle \mathrm{Tr}\frac{\bar{z}-X^\dagger}{(z-X)(\bar{z}-X^\dagger)+\epsilon^2}\right\rangle,$$

- Gauss law

$$\rho(z) = \frac{1}{\pi}\partial_{\bar{z}} G = \frac{1}{\pi}\partial_{z\bar{z}} V.$$

- Spectral density: $\rho(z) \equiv \frac{1}{N}\left\langle\sum_i \delta^{(2)}\left(z-z_i\right)\right\rangle$.
- complex Dirac delta function represented as

$$\pi\delta^{(2)}\left(z-z_i\right) = \lim_{\epsilon\to 0}\frac{\epsilon^2}{\left(\epsilon^2+|z-z_i|^2\right)^2}.$$

[L. Brown;1986],[Sommers,Crisanti,Sompolinsky,Stein;1988]

- Duplication trick

$$\mathcal{G}(z,\bar{z}) = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix} = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\left\langle \mathrm{bTr}\frac{1}{\mathrm{Q}-\mathcal{X}}\right\rangle,$$

with the block-trace defined as $\mathrm{bTr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}A & \mathrm{Tr}B \\ \mathrm{Tr}C & \mathrm{Tr}D \end{pmatrix}$ and notation $Q-\mathcal{X} = \begin{pmatrix} z-X & i\epsilon \\ i\epsilon & \bar{z}-X^\dagger \end{pmatrix}$.

- $G(z,\bar{z}) = \mathcal{G}_{11}$
- condition for the boundary of the spectrum $\mathcal{G}_{1\bar{1}}\mathcal{G}_{\bar{1}1} = 0$

[Fyodorov,Khoruzhenko,Sommers;1996]
[Janik,Nowak,Papp,Zahed;1996],[Feinberg,Zee;1997]
[Jarosz,Nowak;2004] (FRV)

- Eigenvector correlations:

$$O(z) \equiv \frac{1}{N} \left\langle \sum_{\alpha} O_{\alpha\alpha} \delta^2(z - z_\alpha) \right\rangle ,$$

with $O_{\alpha\beta} = \langle L_\alpha | L_\beta \rangle \langle R_\alpha | R_\beta \rangle$ where $|L_\alpha\rangle$ ($|R_\alpha\rangle$) are the left (right) eigenvectors of matrix X.

[Savin,Sokolov; 1997],[Chalker,Mehlig; 1998]

- in the large N limit:

$$O(z) = \frac{N}{\pi} \mathcal{G}_{1\bar{1}} \mathcal{G}_{\bar{1}1}$$

[Janik,Noerenberg,Nowak,Papp,Zahed; 1998]

- How to gain acces to the complex eigenvalues and the correlation of the eigenvectors in the diffussing matrix framwork?

Left ($\langle L_\alpha|$) and right ($|R_\alpha\rangle$) eigenvectors:
$XR = R\Lambda$ and $L^\dagger X = \Lambda L^\dagger$.

We want to study a determinant of the form $\det(Q - \mathcal{X}) =$

$$\det \begin{pmatrix} z - X & i\epsilon \\ i\epsilon & \bar{z} - X^\dagger \end{pmatrix} =$$

$$\det\left[S^{-1}(Q - \mathcal{X})S\right] =$$

$$\det \begin{pmatrix} z - \Lambda & i\epsilon L^\dagger L \\ i\epsilon R^\dagger R & \bar{z} - \Lambda^\dagger \end{pmatrix}$$

with $S = \mathrm{diag}(R, L)$!

- Define (w a complex variable replacing $i\epsilon$)

$$D(z,\bar{z},w,\bar{w}) \equiv D \equiv \left\langle \det\left((z-X)(\bar{z}-X^\dagger)+|w|^2\right)\right\rangle$$

$$= \left\langle \det \begin{pmatrix} z-X & -\bar{w} \\ w & \bar{z}-X^\dagger \end{pmatrix} \right\rangle = \int \mathcal{D}[X] P(X,\tau) \det \begin{pmatrix} z-X & -\bar{w} \\ w & \bar{z}-X^\dagger \end{pmatrix}$$

with the measure $\mathcal{D}[X] = \prod_{i,j} dx_{ij} dy_{ij}$.

- $r \equiv |w|$. Define $v \equiv \frac{1}{2N}\partial_r \ln D$ and $g \equiv \frac{1}{N}\partial_z \ln D$.
- For $N \to \infty$ and $r \to 0$:

$$g \to \mathcal{G}_{11} = G(z) \quad \text{and} \quad v^2 \to \mathcal{G}_{1\bar{1}}\mathcal{G}_{\bar{1}1} \sim O(z)$$

- Diffusion of the elements (X_0 initial condition) $X_{ij} = x_{ij} + iy_{ij}$:

$$dx_{ij} = \frac{1}{\sqrt{2N}} dB^x_{ij}, \quad dy_{ij} = \frac{1}{\sqrt{2N}} dB^y_{ij},$$

- The joint probability density function:

$$\partial_\tau P(X,\tau) = \frac{1}{4N} \sum_{i,j} (\partial^2_{x_{ij}} + \partial^2_{y_{ij}}) P(X,\tau).$$

- Express the determinant in terms of an integral over auxiliary Grassmann variables
$\det \begin{pmatrix} z - X & -\bar{w} \\ w & \bar{z} - X^\dagger \end{pmatrix} =$
$\int \mathcal{D}[\eta,\xi] \exp \left[\begin{pmatrix} \bar{\eta} & \bar{\xi} \end{pmatrix} \begin{pmatrix} z - X & -\bar{w} \\ w & \bar{z} - X^\dagger \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \right]$
- and use the PDE satisfied by $P(X,\tau)$, to show that:

$$\partial_\tau D = \frac{1}{N} \partial_{w\bar{w}} D$$

for any initial condition, for finite N!

- v satisfies (!):

$$\partial_\tau v = v\partial_r v + \frac{1}{N}\left(\Delta_r - \frac{1}{4r^2}\right) v,$$

a Burgers-like equation, where $\Delta_r = \frac{1}{4}(\partial_{rr} + \frac{1}{r}\partial_r)$ is the radial part of the two-dimensional Laplacian. The $1/N$ factor is a viscosity-like parameter.

- v and g are related: $\partial_z v = \frac{1}{2}\partial_r g$; and therefore

$$g = 2\int dr\partial_z v.$$

- g satisfies (!):

$$\partial_\tau g = v\partial_r g + \frac{1}{N}\Delta_r g.$$

- In the inviscid limit ($N \to \infty$):

$$\partial_\tau v = v \partial_r v,$$

- Can be solved by the method of characteristics. The curves along which the solution is constant are given by

$$r = \xi - v_0(\xi)\tau,$$

and labeled with ξ. v_0 plays the role of velocity of the front-wave.

- We have

$$v = v_0(r + \tau v).$$

- For g:

$$\partial_\tau g = 2v \partial_z v.$$

Outline

$X_0 = 0$

- For the initial condition $X_0 = 0$ - the Ginibre ensemble
- correspons to $v_0(r) = r/(z\bar{z} + r^2)$,
  we obtain a cubic algebraic equation for v
- As $r \to 0$

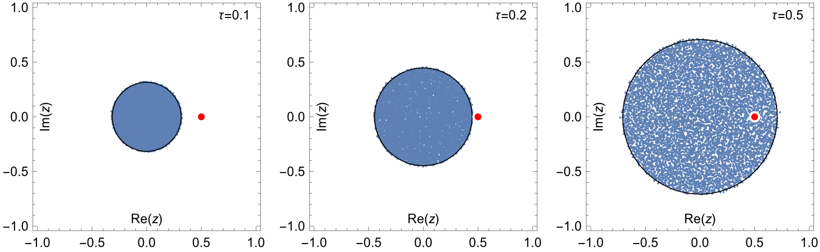
$$v^2 = (\tau - |z|^2)/\tau^2 \quad \text{and} \quad v = 0.$$

matched on the border of the spectrum.

- $g_0(r) = \bar{z}/(|z|^2 + r^2)$, in particular $g_0(r = 0) = 1/z$. For $v = 0$ we have $\partial_\tau g = 0$ so g is constant in time, and therefore it is equal to $g = 1/z$.
- Inside the circle:

$$g = \bar{z}/\tau.$$

$X_0 = 0$



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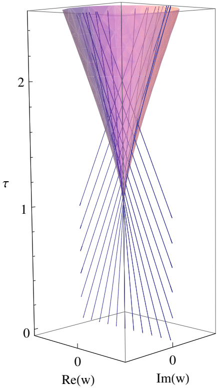
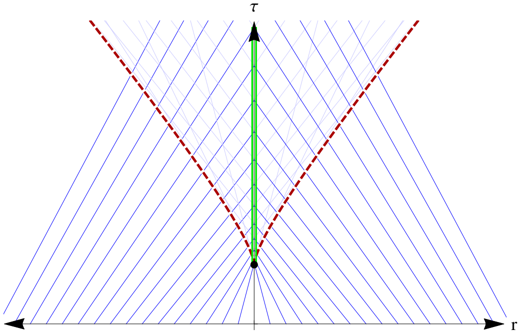
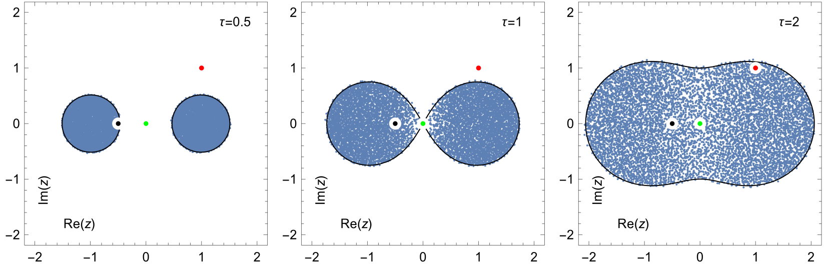
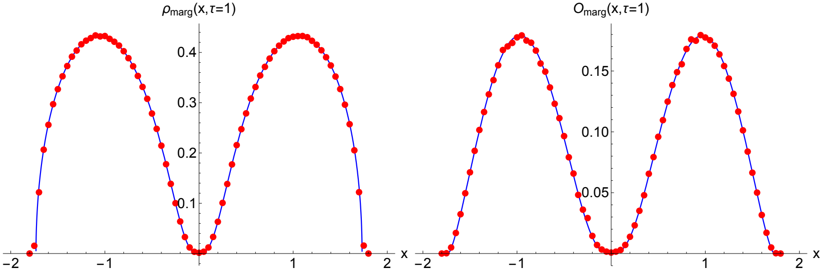




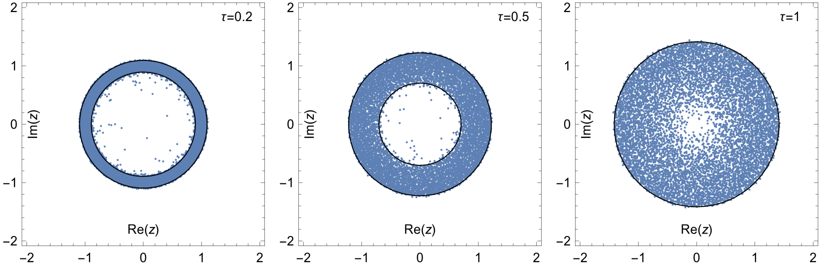
$X_0 = \mathrm{diag}(-1,\ldots,-1,1,\ldots,1) \to$ Spiric



$X_0 = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1) \to$ Spiric



$(X_0)_{ij} = \delta_{i\,j-1} \rightarrow$ non-normal



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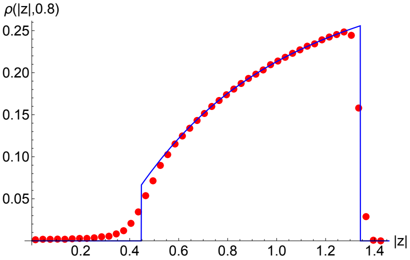

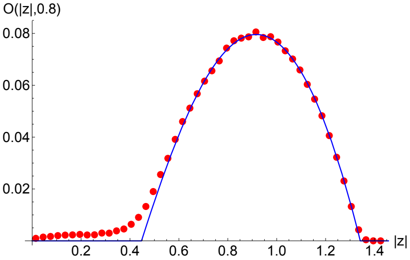

Conclusions

- To study the eigenvalue evolution, one needs to take into account the eigenvectors. Their dynamics is intertwined.
- One can gain access to the evolution of the eigenvectors by introducing an auxiliary variable.
- Correlators of eigenvectors for non-Hermitian matrices are important and deserve more attention!

What we don't know how to formulate yet:

- the Langevin equations for the eigenvalues (and eigenvector correlators),
- the associated SFP equation.

Thank you!

- Dysonian dynamics of the Ginibre ensemble, **Phys. Rev. Lett. 113 (2014) 104102, arXiv:1403.7738**.
- Unveiling the significance of eigenvectors in diffusing non-hermitian matrices by identifying the underlying Burgers dynamics, **arXiv:1503.06846** .

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