One Flavor QCD and the Dirac Spectrum at $\theta = 0$

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BNL -- June 2015

Acknowledgments

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Relevant Papers

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J.J.M. Verbaarschot and T. Wettig, The Spectrum of the Dirac Operator for QCD with on flavor at fixed θ angle, Phys. Rev. D90 (2014) 070 [arXiv:1410.0883[hep-lat]].

P.H. Damgaard, Topology and the Dirac Operator Spectrum in Finite Volume Gauge Theory, Nucl. Phys. B556 327 (1999).

H. Leutwyler and A. Smilga, Spectrum of Dirac Operator and Role of Winding Number in QCD, Phys. Rev. D 46 (1992) 5607.

M. Creutz, One Flavor QCD, Ann. Phys. 322 (2007) 1518 [arXiv[hep-th/0609187]].

- J.J.M. Verbaarschot and T. Wettig, The Chiral Condensate of One-Flavor QCD and the Dirac Spectrum at $\theta = 0$, PoS LATTICE2014 (2014) 072 [arXiv:1412.5483 [hep-lat]].

Contents

I. Chiral Symmetry Breaking in One Flavor QCD

Chiral Condensate

Dirac Spectrum of One-Flavor QCD

Sign Problem

One Flavor QCD

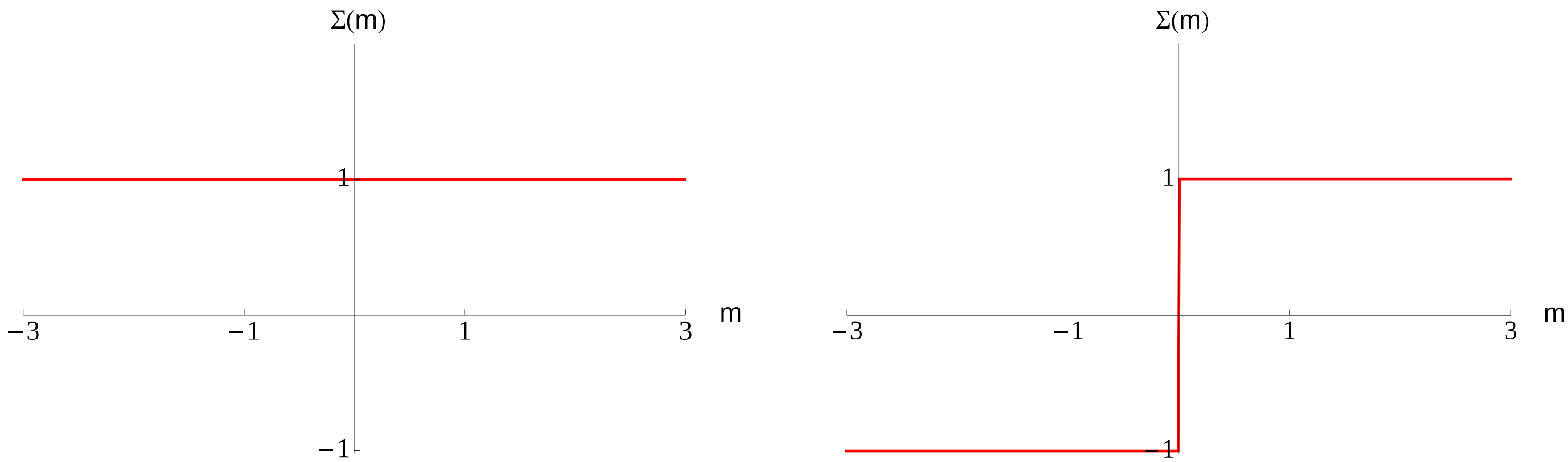
- Chiral symmetry is broken by the anomaly.
- There is no spontaneous symmetry breaking and no Goldstone bosons.
- The mass dependence of the one flavor QCD partition function is given by

$$Z = e^{mV\Sigma\cos\theta + O(m^2V)}.$$

- For $N_f = 2$ with spontaneous symmetry breaking, the mean field estimate of the partition function is given by (for $\theta = 0$)

$$Z = e^{|m|V\Sigma + O(m^2V)}.$$

Chiral Condensate



Behavior of the chiral condensate for $N_f = 1$ (left) and $N_f \geq 2$ (right).

$$\Sigma(m) = -\langle \bar{q} q \rangle = \frac{d}{dm} \log Z(m)$$

The goal of this talk is to explain this behavior in terms of the Dirac spectrum.

Dirac Spectrum

The spectral density of the Dirac operator is given by

$$\rho(x,m,\theta) = \frac{\sum_\nu e^{i\nu\theta} Z_\nu(m)\rho_\nu(x,m)}{\sum_\nu e^{i\nu\theta} Z_\nu(m)}.$$

The partition function at fixed topology is given by

$$Z_\nu(m) = m^\nu \langle \prod_k (\lambda_k^2 + m^2)\rangle.$$

For $m < 0$ the partition function is not positive definite and $\rho(x, m < 0, \theta)$ may become negative.

Banks-Casher

$$
\begin{aligned}
-\langle \bar{q}q \rangle &= \left\langle \frac{1}{V} \sum_k \frac{1}{i\lambda_k + m} \right\rangle \\
&= \left\langle \frac{1}{V} \sum_k \frac{m - i\lambda_k}{\lambda_k^2 + m^2} \right\rangle \\
&= \left\langle \frac{1}{V} \int d\lambda \rho(\lambda, m) \frac{m}{\lambda^2 + m^2} \right\rangle \\
&\underset{m \to 0}{=} \frac{\pi}{V} \rho(0, m) \mathrm{sign}(m)
\end{aligned}
$$

To obtain a continuous chiral condensate we need that $\rho(0, m = 0_+) = -\rho(0, m = 0_-)$.

Could it be that $\rho(0) = 0$?

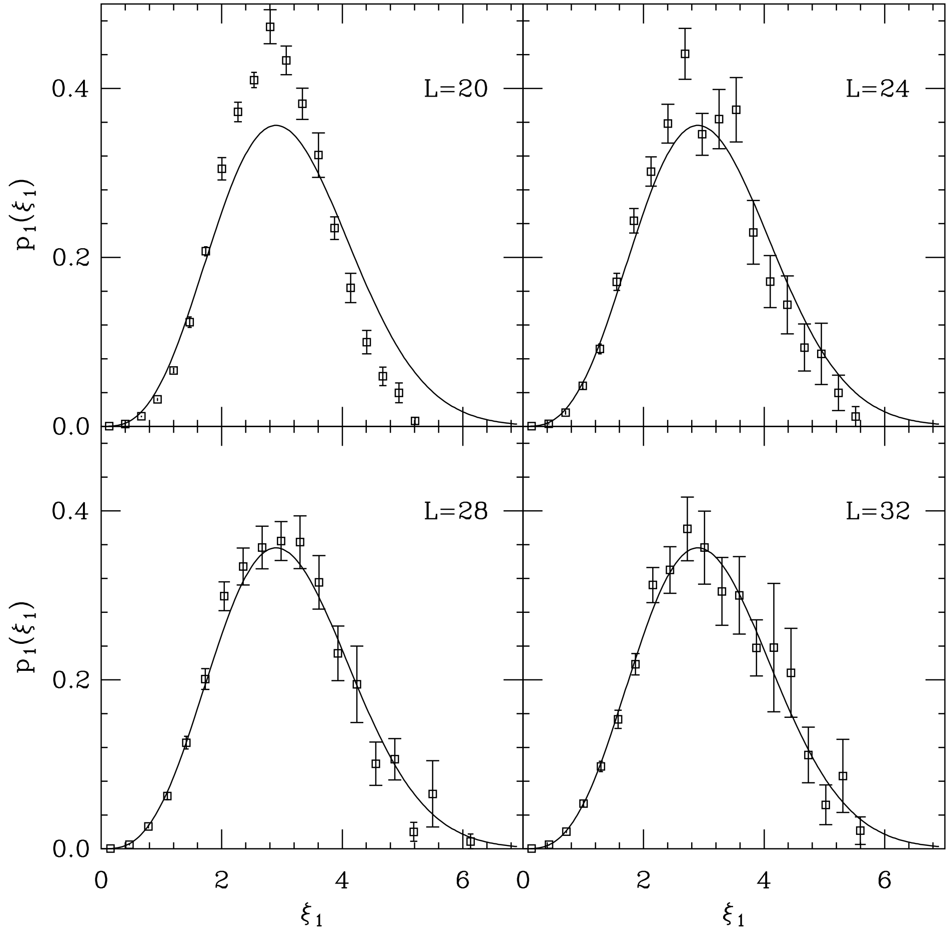
We can also interpret the Banks-Casher relation as

$$\rho(m=0, \lambda=0) = \frac{V}{\pi}(\langle \bar{q}q \rangle_{m=0_-} - \langle \bar{q}q \rangle_{m=0_+}).$$

We could conclude that because of the absence of a discontinuity in the chiral condensate we have $\rho(0) = 0$. Creutz-2006

We will see that the solution to this puzzle is much more subtle than this.

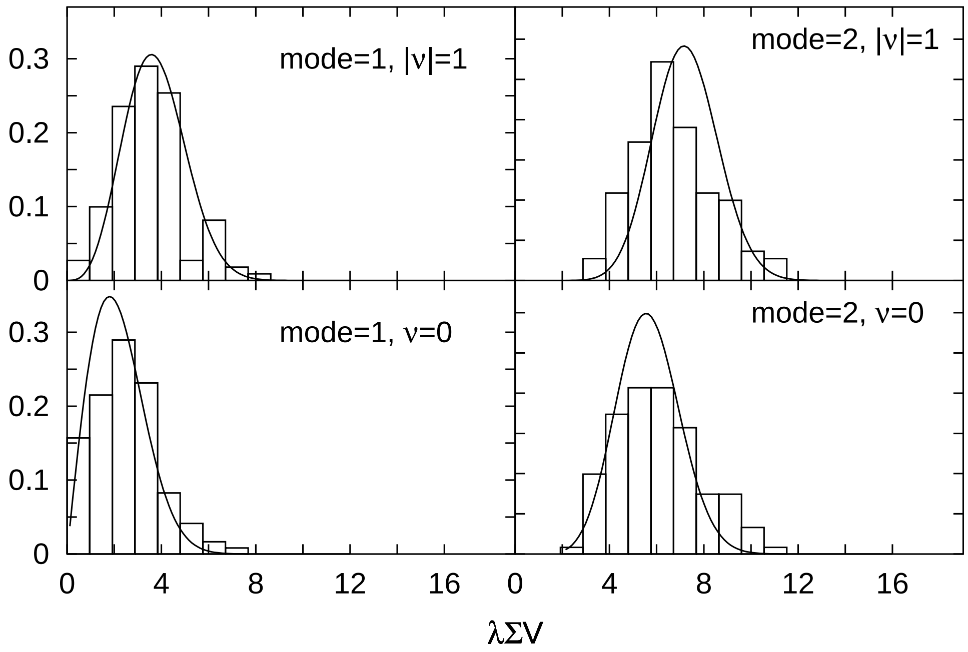
Lattice Results for the Schwinger Model



Distribution of the lowest eigenvalue of the Dirac operator of the Schwinger model compared to chiral random matrix theory (solid curve).

Damgaard-Heller-Narayanan-Svetitsky-2005

Lattice results for one-flavor QCD



Distribution of the lowest two Dirac eigenvalues for QCD with one flavor compared to the result from chiral random matrix theory (solid curve).

Degrand-Hoffmann-Schäfer-Liu-2006

What Can We Learn from Lattice Dirac Spectra?

- The smallest Dirac eigenvalues of one-flavor QCD behave in exactly the same was as the Dirac spectrum of QCD and QCD-like theories with spontaneously broken chiral symmetry.
- The Dirac spectrum is determined by the $U(2|1)$ -theory which has Goldstone particles as well as a massive η' .
- Although the Dirac spectrum at fixed topology has all signatures of spontaneous chiral symmetry breaking, it should synthesize a chiral condensate that is due to explicit chiral symmetry breaking.

The Microscopic Domain of QCD

We will do our calculations in the microscopic domain of QCD, and the explicit results we quoted before were already in this domain.

In this domain, also know as the ϵ -domain, the quark mass and the Dirac eigenvalues scale as

$$m \sim \frac{1}{V}, \qquad \lambda \sim \frac{1}{V}.$$

Correction terms will enter when $m, \lambda \approx 1/\Lambda_{\rm QCD}\sqrt{V}$.

In this domain, the spectral density can be evaluated analytically.

Spectral Density at Fixed ν for $N_f = 1$

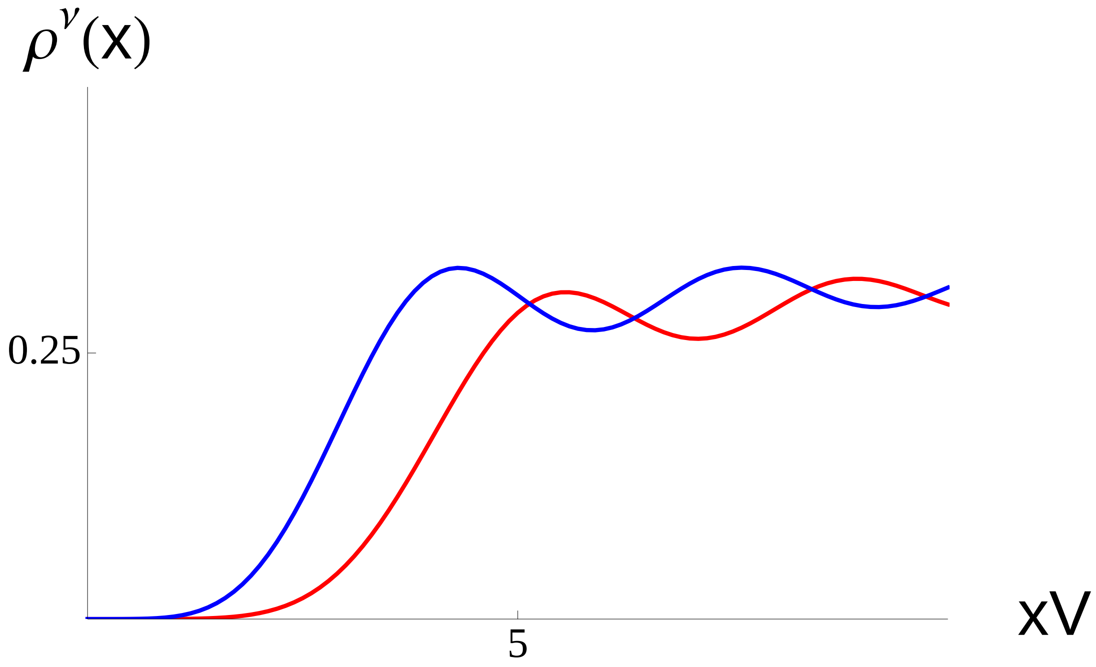
The one-flavor spectral density in the ϵ -domain is given by

$$\begin{aligned}
\rho_\nu(\lambda, m) \quad &= \quad \frac{\hat{x}}{2}(J_\nu^2(\hat{x}) - J_{\nu+1}(\hat{x})J_{\nu-1}(\hat{x})) + |\nu|\delta(\hat{x}) \\
&\quad - \frac{\hat{x}}{\hat{m}^2 + \hat{x}^2}\left[\hat{x}J_\nu(\hat{x})J_{\nu+1}(\hat{x}) - \hat{m}\frac{I_{\nu+1}(\hat{m})}{I_\nu(\hat{m})}J_\nu^2(\hat{x})\right].
\end{aligned}$$

Damgaard-Osborn-Toublan-JV-1999

$$\hat{x} \equiv \lambda\Sigma V, \qquad \hat{m} \equiv m\Sigma V$$

Microscopic Spectral Density at fixed ν



The one microscopic spectral density for $\nu = 2$ and $mV = 1$ (red) compared to the quenched result for $\nu = 2$ (blue).

II. Role of Topology

Decomposition in Topological Sectors

Is the Chiral Condensate Due to Zero Modes

Chiral Condensate at $\theta = 0$

Sign Problem

Topological Decomposition

$$Z(m,\theta) = e^{mV\Sigma\cos\theta} = \sum_\nu e^{i\nu\theta} Z_\nu(m).$$

The partition function at fixed ν is given by

$$Z_\nu(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{mV\Sigma\cos\theta} = I_\nu(mV\Sigma).$$

Asymptotic behavior of Bessel functions

$$I_\nu(\hat{m}) \sim \begin{cases} \frac{1}{\sqrt{2\pi\hat{m}}} e^{|\hat{m}|-\nu^2/2|\hat{m}|}, & \hat{m} > 0, \\ \frac{(-1)^\nu}{\sqrt{2\pi|\hat{m}|}} e^{|\hat{m}|-\nu^2/2|\hat{m}|}, & \hat{m} < 0 \end{cases}.$$

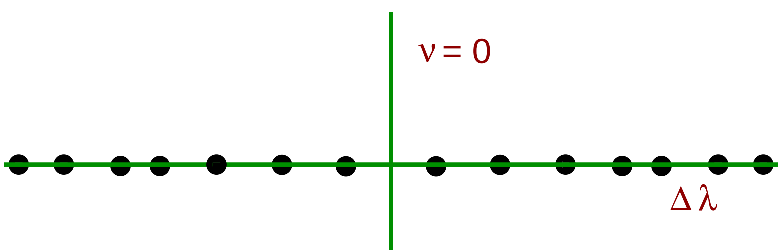
Notice that there are exponential cancellations to recover the partition function at $\theta = 0$ for $m < 0$.

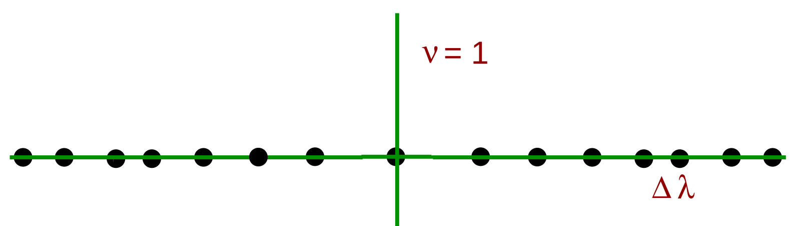
The topological susceptibility is equal to $\chi = mV\Sigma$.

Can the Chiral Condensate be Due to the Zero Modes?

If we take the chiral limit before the thermodynamic limit then

$$
\begin{aligned}
-\langle \bar{q} q \rangle \quad &= \quad \frac{1}{VZ(m)} \left\langle \sum_\nu \sum_k \frac{1}{i\lambda_k + m} m^{|\nu|} \prod_k (i\lambda_k + m) \right\rangle \\
&\underset{m \to 0}{=} \quad \frac{1}{V} \frac{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=1} \right\rangle}{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=0} \right\rangle} + \frac{1}{V} \frac{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=-1} \right\rangle}{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=0} \right\rangle} \approx \frac{1}{V \Delta \lambda_0}
\end{aligned}
$$





The nonzero eigenvalues shift on average by $\nu \Delta \lambda / 2$.

Even in the chiral limit, the value of the chiral condensate is due to the nonzero modes.

Condensate Due to the Anomaly in the Gapped Phase

At high temperature the gap in the Dirac spectrum, $\Delta\lambda_0$ remains finite in the thermodynamical limit.

If the $U_A(1)$ symmetry remains broken, the condensate in the chiral limit is given by

$$\Sigma \sim \frac{1}{V\Delta\lambda_0}$$

and the condensate is at least $1/V$ suppressed and vanishes in the thermodynamical limit.

What Happens if we Reverse the Thermodynamics and Chiral Limits?

If the thermodynamic limit is taken before the chiral limit we have that $mV\Sigma \gg 1$

$$
\begin{aligned}
-\langle \bar{q}q \rangle &= \frac{1}{V} \frac{\sum_\nu \frac{|\nu|}{m} e^{-\nu^2/2|m|V\Sigma}}{\int d\nu e^{-\nu^2/2|m|V\Sigma}} \\
&= \frac{1}{Vm} \frac{\sum_\nu |\nu| e^{-\nu^2/2|m|V\Sigma}}{\int d\nu e^{-\nu^2/2|m|V\Sigma}} \\
&= \mathrm{sign}(m) \frac{2\Sigma}{\sqrt{\pi 2|m|V\Sigma}}.
\end{aligned}
$$

To get a constant chiral condensate we need the contribution of the nonzero modes.

This analysis is incorrect for $m < 0$ because of the exponential cancellations.

Chiral condensate at $\theta = 0$

$$\Sigma(m, \theta = 0) = \frac{\sum_{\nu=-\infty}^{\infty} Z_\nu(m) \Sigma^\nu(m)}{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)}.$$

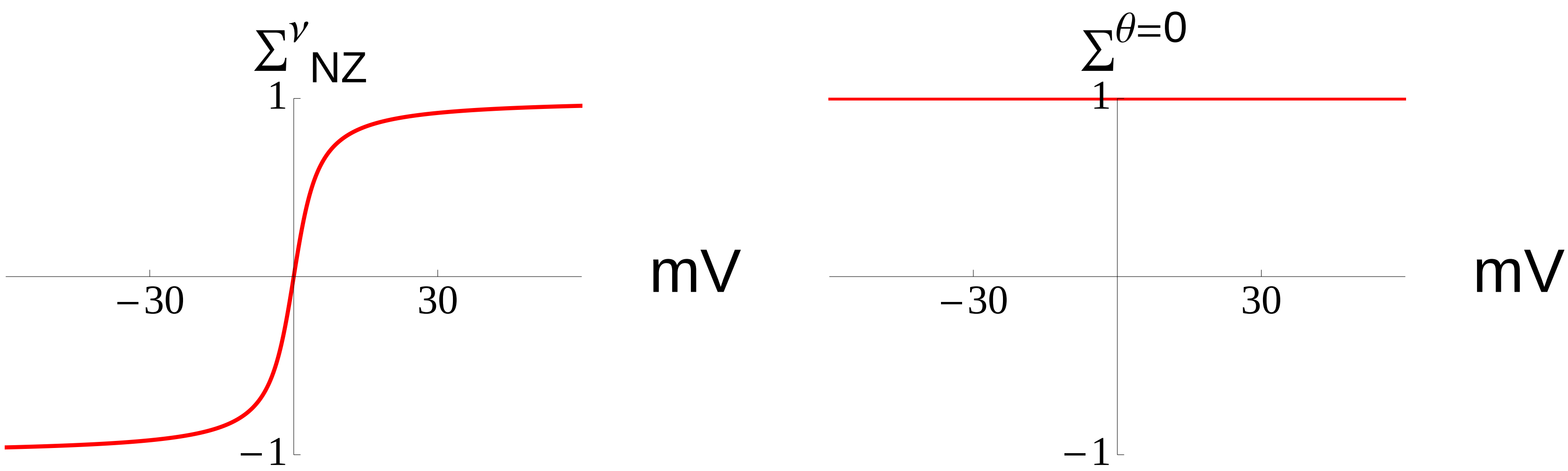
This condensate follows from the spectral density at $\theta = 0$

$$\rho(\lambda, m, \theta = 0) = \frac{\sum_{\nu=-\infty}^{\infty} Z_\nu(m) \rho^\nu(\lambda, m)}{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)}.$$

Can be evaluated numerically in the ϵ -domain of QCD.

Damgaard-1999, Kanazawa-Wettig-2011

Chiral Condensate for $N_f = 1$



Mass dependence of the chiral condensate due to the nonzero modes for $\nu = 2$ (left) and the mass dependence of the chiral condensate at $\theta = 0$. Note that $\Sigma^{\nu}_{NZ}(m) = \Sigma^{\nu}(m) - \frac{|\nu|}{mV}$.

For $m < 0$, the negative values of $\Sigma^{\nu}(m)$ should average to a positive number. This is possible because the weight $Z_{\nu}(m)$ is not positive definite.

Sign Problem for QCD with $N_f = 1$

Because

$$\det(D+m) = m^\nu \prod_k (\lambda_k^2 + m^2).$$

QCD at $\theta = 0$ has a severe sign problem for $m < 0$.

Magnitude of the Sign Problem for QCD with $N_f = 1$

Partition function at $\theta = 0$

$$Z_{QCD}(m) = \sum_{\nu=-\infty}^{\infty} I_\nu(mV\Sigma) = e^{mV\Sigma}.$$

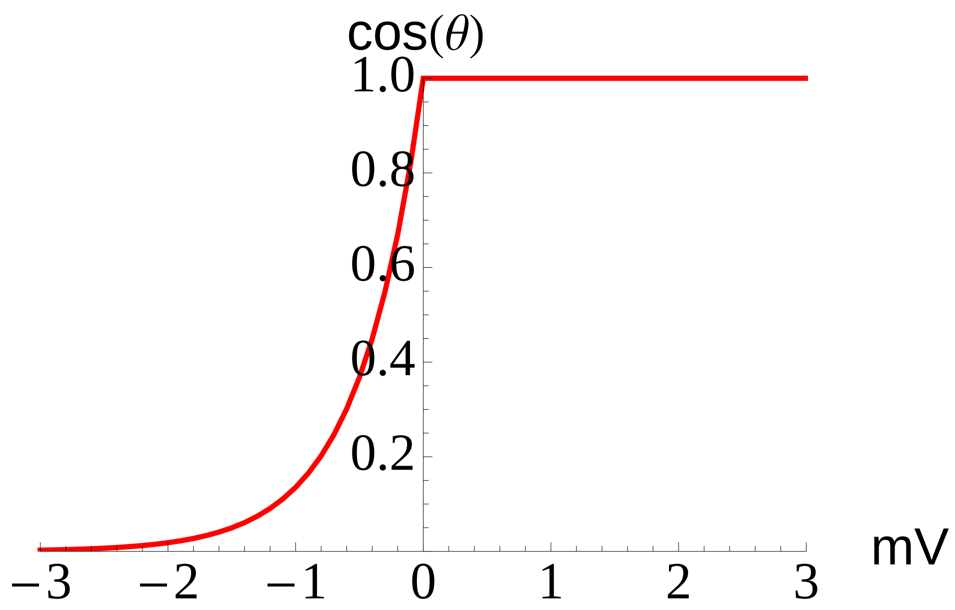
Phase quenched partition function

$$Z_{|QCD|}(m) = \sum_{\nu=-\infty}^{\infty} |I_\nu(mV\Sigma)| = e^{|m|V\Sigma}.$$

Average sign

$$\cos\theta = \frac{Z_{QCD}(m)}{Z_{|QCD|}(m)} = e^{(m-|m|)V\Sigma}.$$

Sign Problem for QCD with $N_f = 1$



$$\cos\theta = \frac{Z_{QCD}(m)}{Z_{|QCD|}(m)}.$$

Silver Blaze Problem

- The spectrum of the Dirac operator for one flavor QCD at fixed topological charge is as if chiral symmetry is broken spontaneously.
- In particular, in the thermodynamic limit, the chiral condensate has a discontinuity when the mass crosses the line of eigenvalues.
- General arguments show that the chiral condensate for $\theta = 0$ does not have a discontinuity.
- What is the solution of the “Silver Blaze Problem”?
- One flavor QCD has a sign problem for $m < 0$.
- This problem first arose in QCD at nonzero chemical potential, and the original motivation to study one flavor QCD was to improve our understanding of the relation between the chiral condensate and the Dirac spectrum for QCD at nonzero chemical potential.

III. One dimensional QCD at Nonzero Chemical Potential

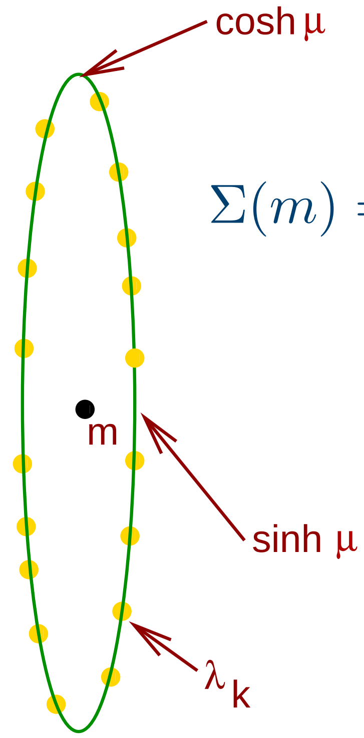
Chiral Condensate

Sign Problem

Spectral Density

OSV Mechanism

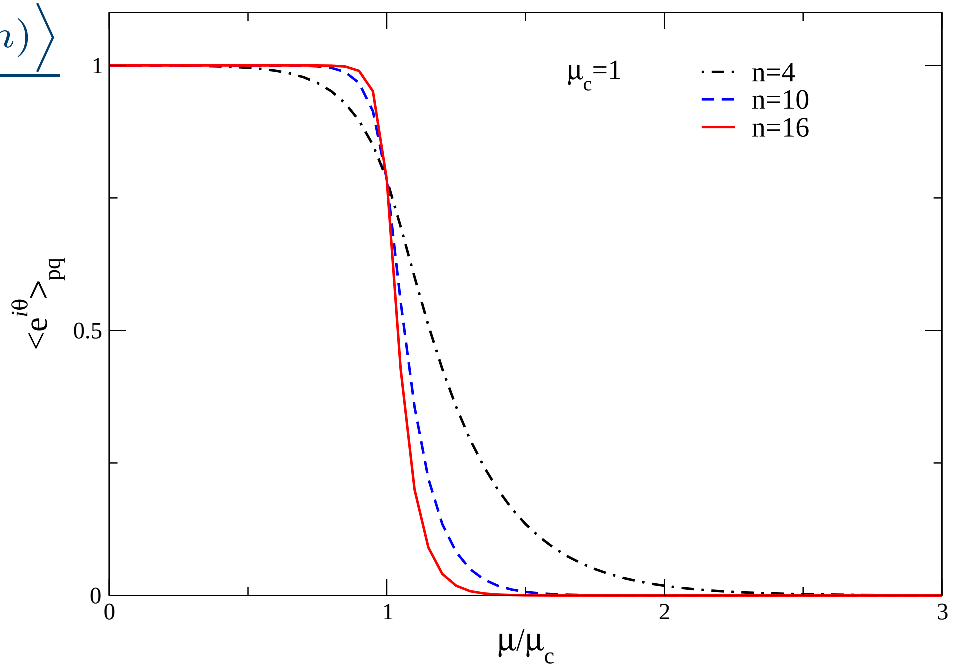
Chiral Condensate $U(1)$ QCD in 1d



Dirac spectrum of 1d QCD

$$\Sigma(m) = \frac{\left\langle \sum_k \frac{1}{\lambda_k + m} \prod_k (\lambda_k + m) \right\rangle}{\left\langle \prod_k (\lambda_k + m) \right\rangle}$$

determinant with
a complex phase

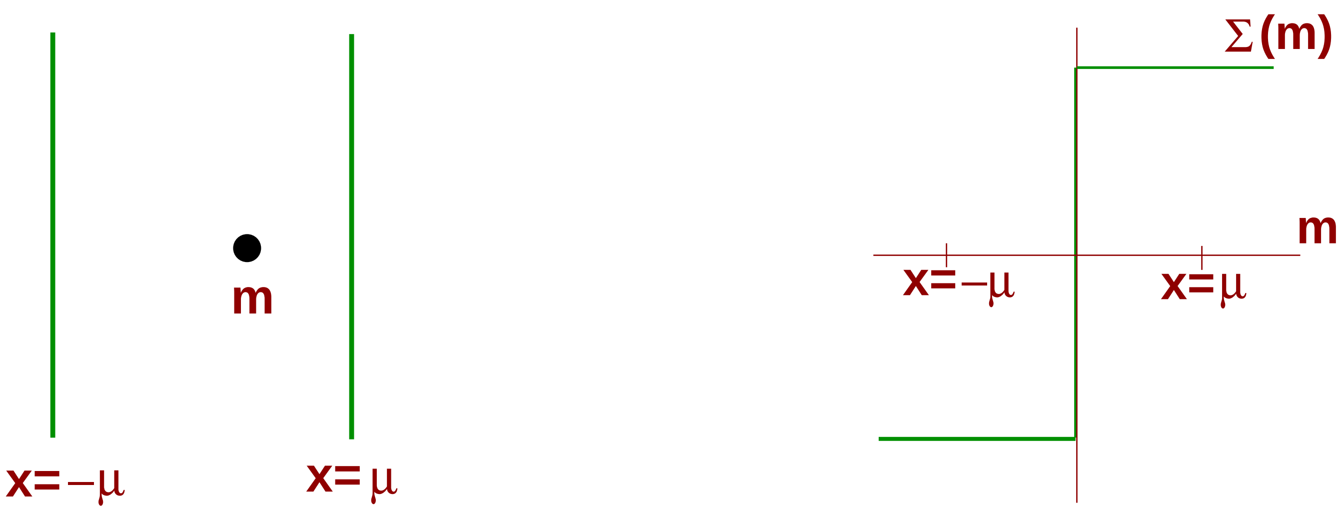


Ravagli-JV-2007, Aarts-Splittorff-2010

Eigenvalues are equally spaced on an ellipse with a random overall phase.

The dependence of the partition function on the chemical potential can be eliminated by shifting the integration variable $\theta_k \to \theta_k + i\mu$.

Chiral Condensate for $U(1)$ QCD in 1d



The chiral condensate is continuous across the ellipse where the eigenvalues are located.

In the limit of a dense spectrum, $\Sigma(m)$ is discontinuous across the imaginary axis despite the fact that there are no eigenvalues for $\mu \neq 0$.

Since we have a $U(1)$ theory there are no baryons and the partition function is independent of μ

Spectral Density for 1d QCD

For large V and small μ the eigenvalues of the Dirac operator are located on two parallel lines $x \pm \mu$ resulting in the spectral density and the chiral condensate

$$\begin{aligned} \Sigma(m) &= \int \frac{dxdy}{2\pi} \frac{1}{m-x-iy} \underbrace{\delta(|x|-\mu)\left[1 - \frac{(e^{V(x+iy)} + e^{-V(x+iy)})}{e^{Vm}+e^{-Vm}}\right]}_{\rho(x,y) \quad \text{for} \quad N_f = 1} \\ &= \tanh(Vm). \end{aligned}$$

In the thermodynamic limit $(V \to \infty)$ this results in a discontinuity across $m = 0$, but not at $m \pm \mu$.

Osborn-Splittorff-JV-2005, Ravagli-JV-2008

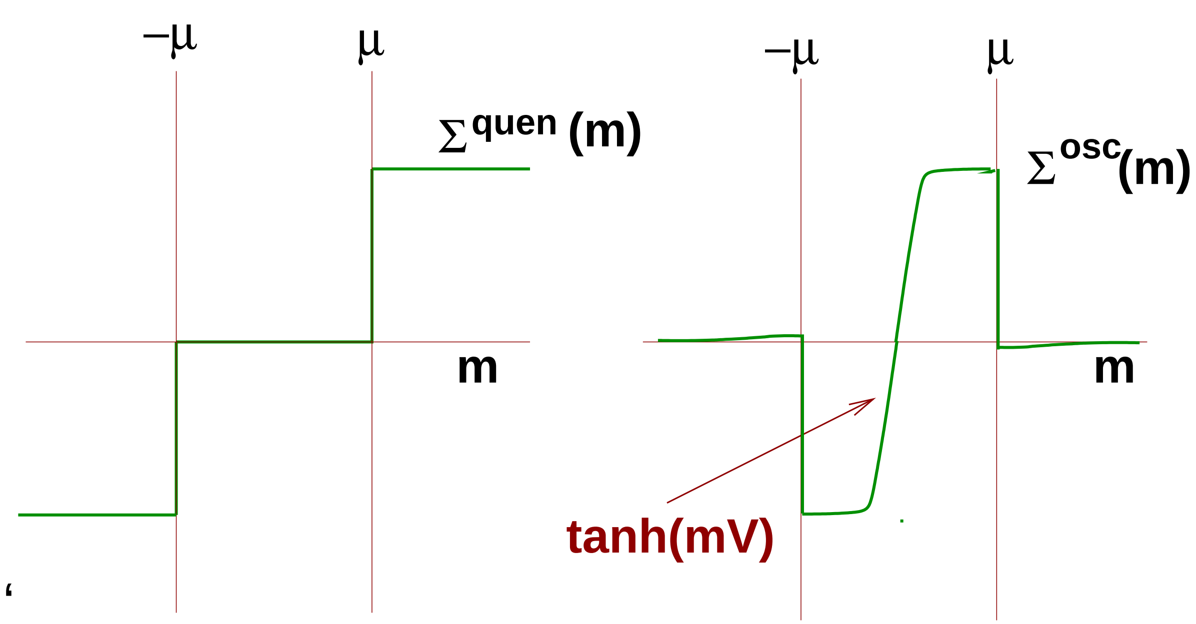
Chiral Condensate in 1d

The first term ($\sim \delta(|x| - \mu)$) gives the quenched contribution

$$\Sigma^{\rm quenched}(m) = {\rm sign}(m - \mu) + {\rm sign}(-m + \mu).$$

This follows from electrostatic arguments with eigenvalues as charges.
The second term is evaluated as

$$\Sigma^{\rm osc}(m) = \tanh(mV) - {\rm sign}(m - \mu) - {\rm sign}(-m + \mu).$$



The chiral condensate becomes discontinuous in the continuum limit.
Ravagli-JV-2007

Alternative to the Banks-Casher Formula

- This mechanism makes it possible to obtain a chiral condensate that does not change when the mass crosses a line or area of eigenvalues.
- For a positive definite eigenvalue density this is not possible according to the Banks-Casher formula.
- When the eigenvalue density is not positive definite (due to the fermion determinant), the OSV mechanism replaces the Banks-Casher formula.
- This mechanism, where the chiral condensate results from an oscillating spectral density with an amplitude that diverges exponentially with the volume and a period proportional to the inverse volume, was discovered for a chiral random matrix theory at nonzero chemical potential. Osborn-Splittorff-JV-2005

Let us see how it can work for QCD with one flavor.

IV. Role of Dynamical Quarks

Decomposition into Quenched and Dynamical Part

How To Obtain a Constant Chiral Condensate

Decomposition of the Spectral Density

- To confirm if the OSV mechanism holds we have to calculate the spectral density of the Dirac operator for $N_f = 1$ QCD.
- Actually this can be done analytically in the ϵ domain of QCD. The result can be expressed as a simple one dimensional integral.

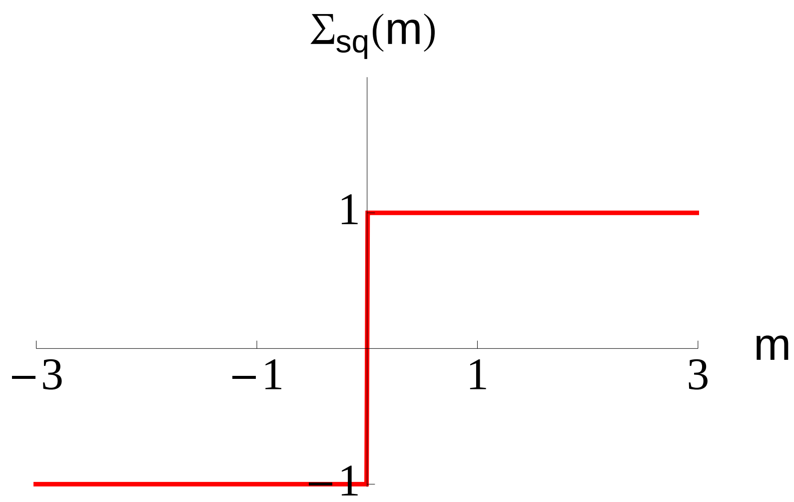
We decompose

$$\rho(\lambda, m) = \begin{cases} \rho_{\mathsf{zm}}(\lambda, m) + \rho_{\mathsf{nz}}(\lambda, m) \, , \\ \rho_{\mathsf{sq}}(\lambda, m) + \rho_{\mathsf{osc}}(\lambda, m) + \rho_{\,\delta\,\mathsf{zm}}(\lambda, m) \, , \end{cases}$$

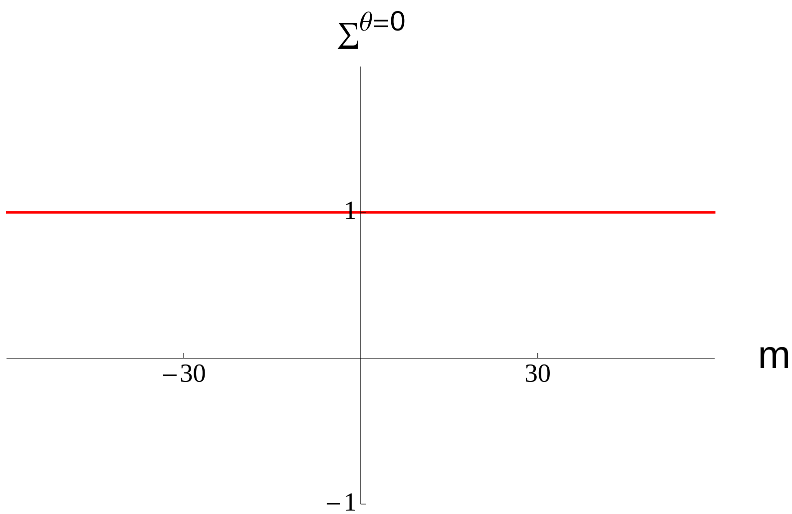
where

$$\begin{aligned} \rho_{\mathsf{sq}}(\lambda, m) &= \rho(\lambda, |m|) \, , \\ \rho_{\mathsf{osc}}(\lambda, m) &= \rho_{\mathsf{nz}}(\lambda, m) - \rho_{\mathsf{nz}}(\lambda, |m|) \, , \\ \rho_{\,\delta\,\mathsf{zm}}(\lambda, m) &= \rho_{\mathsf{zm}}(\lambda, m) - \rho_{\mathsf{zm}}(\lambda, |m|) \, . \end{aligned}$$

How to get a Constant Chiral Condensate?



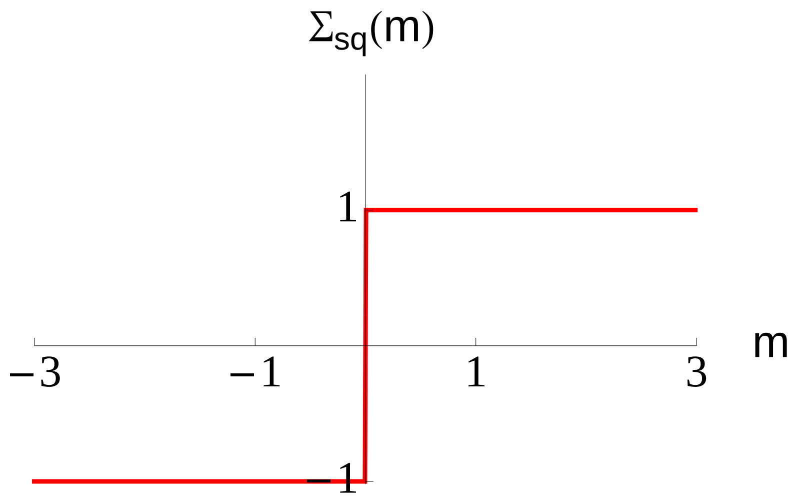
Behavior of the chiral condensate due to a line of eigenvalues for the quenched theory at $\theta = 0$.

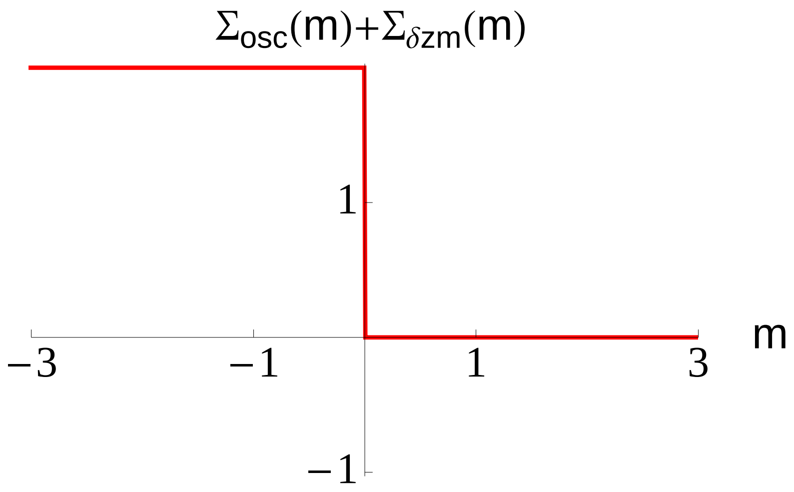


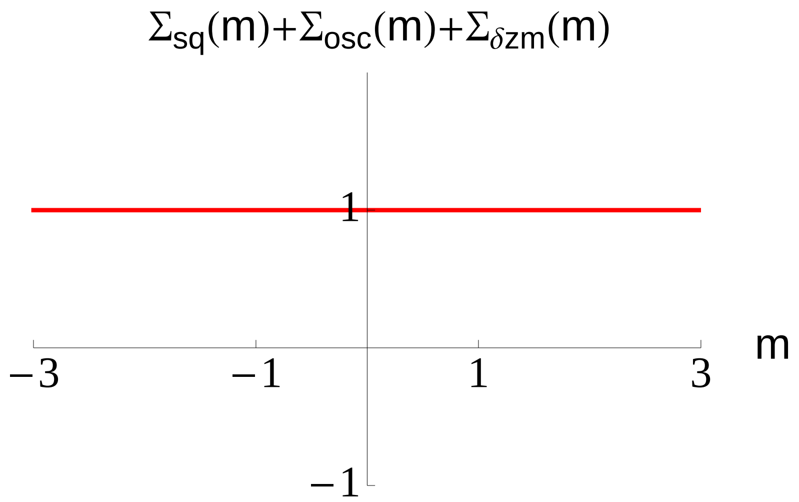
Behavior of the chiral condensate due to a line of eigenvalues for the one flavor theory at $\theta = 0$.

This implies that the not positive definite measure should give a correction to the spectral density that results in a mass dependence of the chiral condensate given by $\Sigma_{\rm osc}(m) + \Sigma_{\delta zm} = 2\theta(-m)$

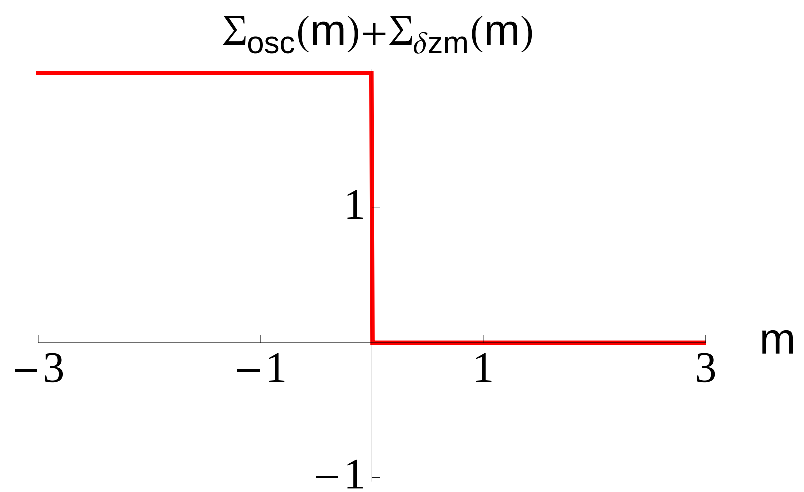
OSV Mechanism in Pictures







How can this be Generated by a Spectral Density?



$$2\theta(-m) = \int d\lambda \frac{\rho_{\rm osc}(\lambda, m)}{i\lambda - m}.$$

What is $\rho_{\rm osc}(\lambda, m)$?

Hint,

$$\theta(m) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\tau \frac{e^{im\tau + m\epsilon}}{\tau - i\epsilon}.$$

Solution satisfying $\rho(\lambda) = \rho(-\lambda)$

$$\rho_{\rm osc}(\lambda, m) = \frac{1}{\pi}(e^{iV\lambda - Vm} + e^{-iV\lambda - Vm})$$

$$\int_{-\infty}^{\infty} d\lambda \frac{1}{i\lambda - m}\frac{1}{\pi}(e^{V(i\lambda - m)} + e^{V(i\lambda - m)}) = 2\theta(-m) - 2\theta(m)e^{-2Vm}$$

satisfying $\rho(\lambda) = \rho(-\lambda)$ is given by Therefore, the chiral condensate due to the spectral density

$$\rho(\lambda, m) = \frac{1}{\pi}(1 - e^{V(i\lambda - m)} - e^{-V(i\lambda + m)}).$$

does not have a discontinuity across $m = 0$.

What is the Most General Class of solutions?

At least in the thermodynamical limit, the solution for the spectral density is not unique. Another solution that gives $2\theta(-m)$ in the thermodynamic limit is given by

$$\rho(x,m) = -\frac{4}{\pi}\frac{x^2}{x^2+m^2}\int_0^1 \frac{tdt}{\sqrt{1-t^2}} e^{-2mVt^2} J_1(2xVt)$$

However, we have the stronger requirement that also in the microscopic domain the contribution to the condensate is given by $2\theta(-m)$.

V. One Flavor QCD

Spectral Density

Zero Modes

Solution of Silver Blaze Problem

Technical Detail

We have to calculate

$$\rho(\hat{x}, \hat{m}, \theta = 0) = \frac{1}{Z(\hat{m}, \theta = 0)} \sum_\nu I_\nu(\hat{m}) \rho_\nu(\hat{x}, \hat{m}),$$

where

$$\begin{aligned} \rho_\nu(\hat{x}, \hat{m}) \quad &= \quad \frac{\hat{x}}{2}(J_\nu^2(\hat{x}) - J_{\nu+1}(\hat{x}) J_{\nu-1}(\hat{x})) + |\nu| \delta(\hat{x}) \\ &\quad + - \frac{\hat{x}}{\hat{m}^2 + \hat{x}^2} \left[\hat{x} J_\nu(\hat{x}) J_{\nu+1}(\hat{x}) - \hat{m} \frac{I_{\nu+1}(\hat{m})}{I_\nu(\hat{m})} J_\nu^2(\hat{x}) \right], \end{aligned}$$

and

$$Z(\hat{m}, \theta = 0) = e^{\hat{m}}.$$

Technical Detail

To evaluate the microscopic spectral density at fixed θ -angle we need sums of the form

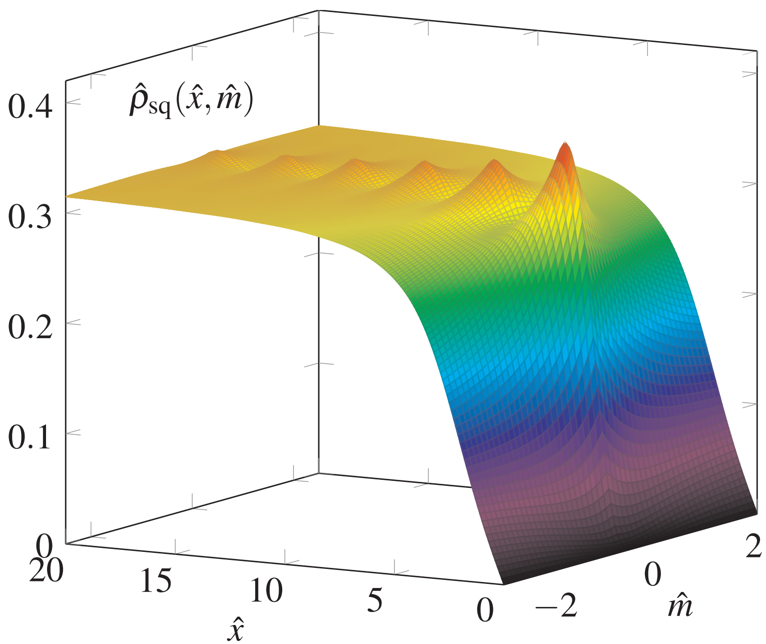
$$S_{a,b,c}(x,m,\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} I_{\nu+a}(m) J_{\nu+b}(x) J_{\nu+c}(x)$$

They can be reduced to one-dimensional integrals. Examples are

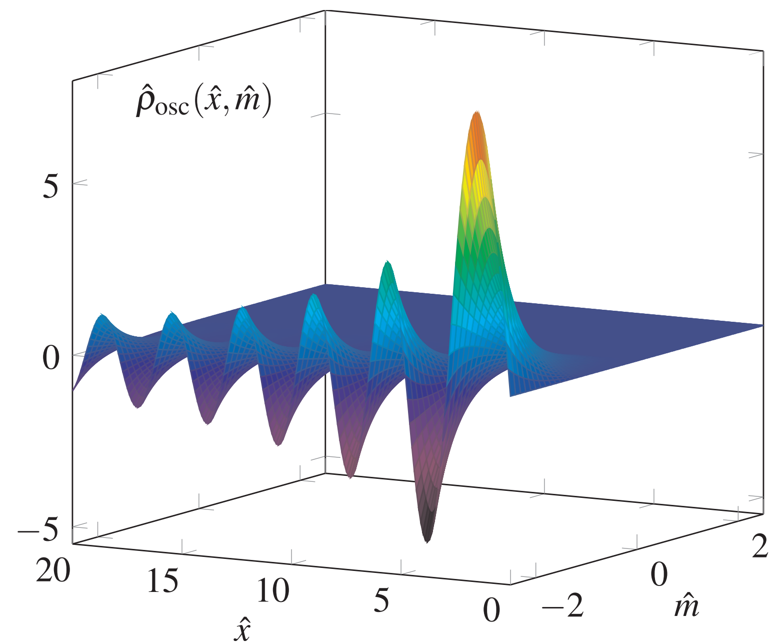
$$\begin{aligned} \sum_\nu I_\nu(m) J_\nu^2(x) &= \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^2}}\, e^{m-2mt^2} J_0(2xt)\,, \\ \sum_\nu I_\nu(m) J_{\nu+1}(x) J_{\nu-1}(x) &= -\frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^2}}\, e^{m-2mt^2} J_2(2xt)\,. \end{aligned}$$

JV-Wettig-2014

The Dirac Spectrum for $N_f = 1$ at $\theta = 0$



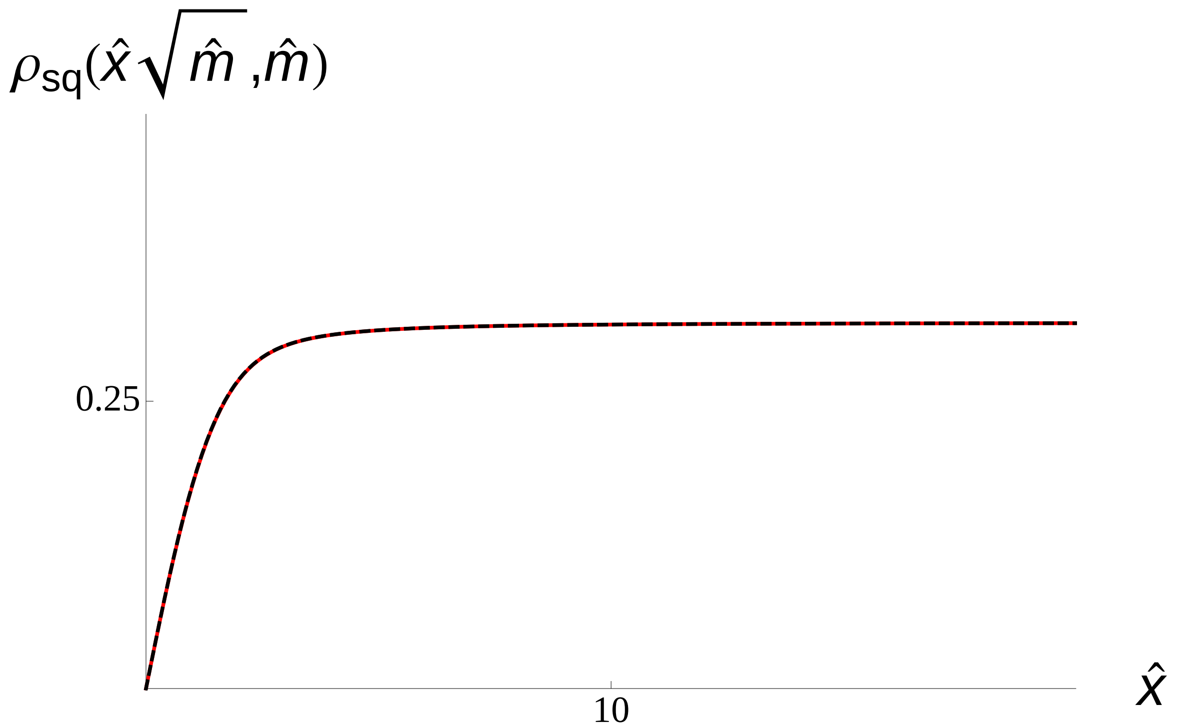
The sign quenched part of the spectral density at $\theta = 0$, $\hat{\rho}_{\rm sq}(\hat{x},\hat{m})$.



The oscillating part of the spectral density at $\theta = 0$, $\hat{\rho}_{\rm osc}(\hat{x},\hat{m})$.

$$\hat{\rho}_{\rm nz}(x,m) = \frac{1}{\pi}\int_0^1 \frac{e^{-2mVt^2}dt}{t\sqrt{1-t^2}} J_1(2xVt) - \frac{2}{\pi}\frac{x}{x^2+m^2}\int_0^1 \frac{e^{-2mVt^2}dt}{\sqrt{1-t^2}} \times \left[xtJ_1(2xVt) + m(1-2t^2)J_0(2xVt)\right] .$$

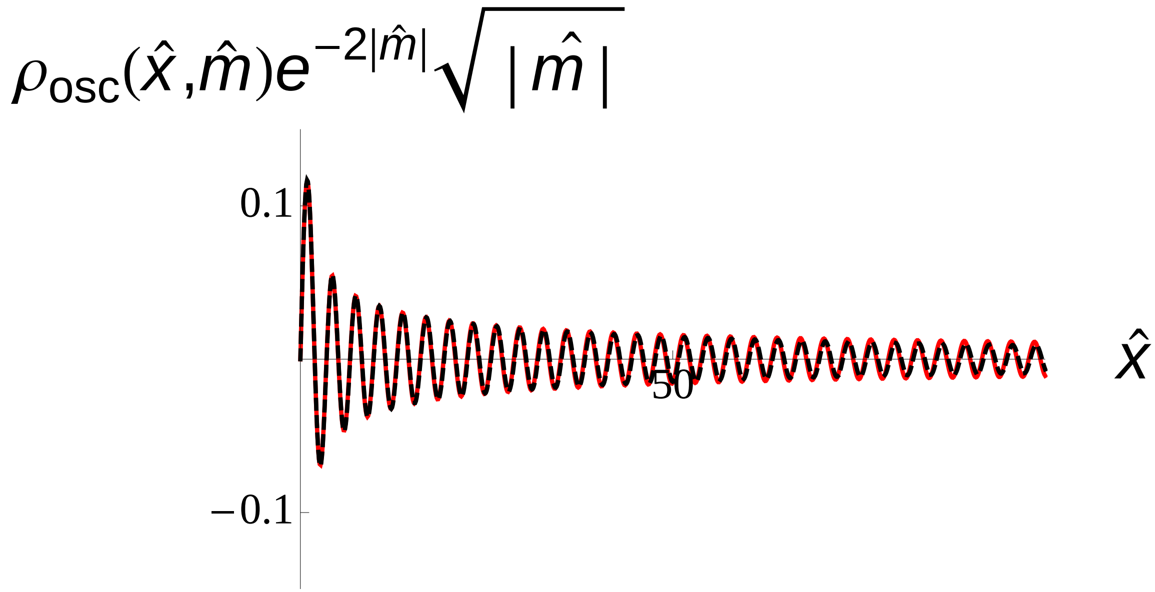
Asymptotic Scaling for $m > 0$



Spectral density for $mV\Sigma \gg 1$.

$$\rho_{\text{sq}}(x\sqrt{m}, m > 0) \sim \frac{xV}{\sqrt{2\pi mV}} e^{-Vx^2/4m}[I_0(Vx^2/4m) + I_1(Vx^2/4m)]$$

Asymptotic Scaling for $m < 0$



Spectral density for $mV\Sigma \ll -1$.

$$\rho_{\rm osc}(x, m < 0) \sim \frac{e^{2|m|V}}{\sqrt{8\pi|m|V}} J_1(2xV).$$

Wettig-JV-2014

Spectral Density due to Zero Modes

Contribution from zero modes   Leutwyler-Smilga-1992

$$\rho_{\rm zm}(x,m) \quad = \quad e^{-mV}\sum_\nu |\nu| I_\nu(mV)\delta(x) = e^{-mV}(I_0(mV)+I_1(mV))\delta(x).$$

Spectral density becomes exponentially large for $m<0$.

Chiral Condensate

The chiral condensate can be obtained by integration over the spectral density

$$\Sigma(m) = \frac{1}{V}\int_{-\infty}^{\infty} \frac{2m\rho(\lambda, m)}{\lambda^2 + m^2}.$$

Role of the Zero Modes

For $m < 0$ the contribution from the zero modes diverges in the thermodynamic limit as

$$\Sigma_{\delta \mathrm{zm}}(m) \underset{\substack{V\to\infty \\ m<0}}{\sim} \frac{e^{2|mV|}}{\sqrt{8\pi|mV|^3}}.$$

This contribution cancels against a similar contribution from the nonzero modes. Kanazawa-Wettig-2012

Indeed we do find the asymptotic behavior JV-Wettig-2014

$$\Sigma_{\mathrm{osc}}(m) \underset{\substack{V\to\infty \\ m<0}}{\sim} -\frac{e^{2|mV|}}{\sqrt{8\pi|mV|^3}}.$$

Cancellation to All Orders

Actually, this cancellation takes place to all orders in $1/mV\Sigma$.

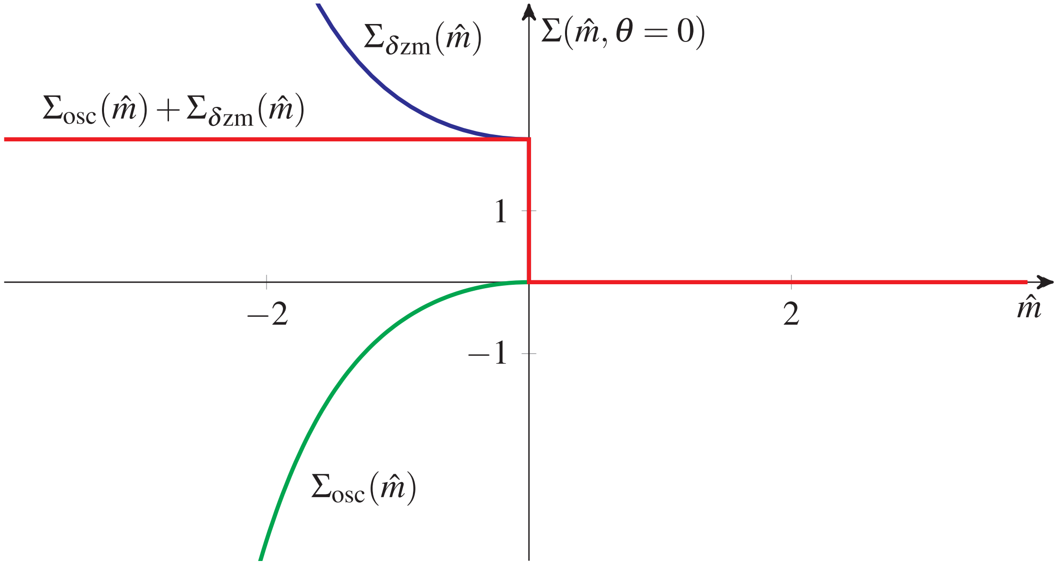
Using the exact analytical expressions, we have shown the identity

$$\Sigma_{\rm osc}(m) + \Sigma_{\delta {\rm zm}}(m) = 2\theta(-\hat{m}).$$

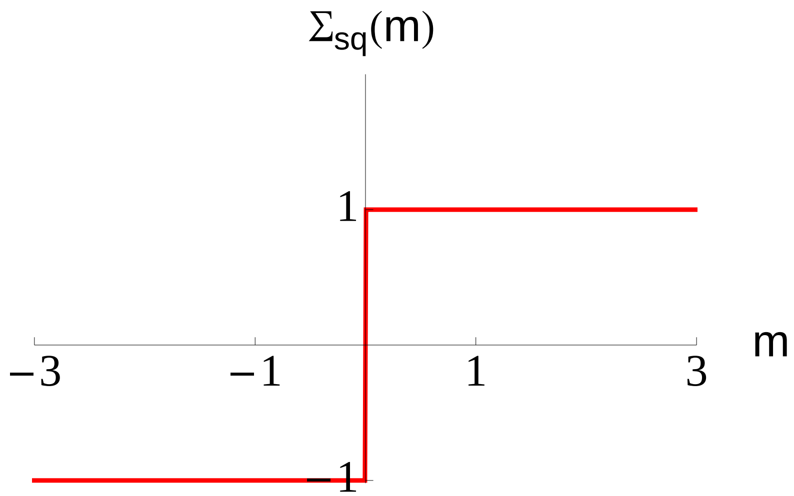
JV-Wettig-2014

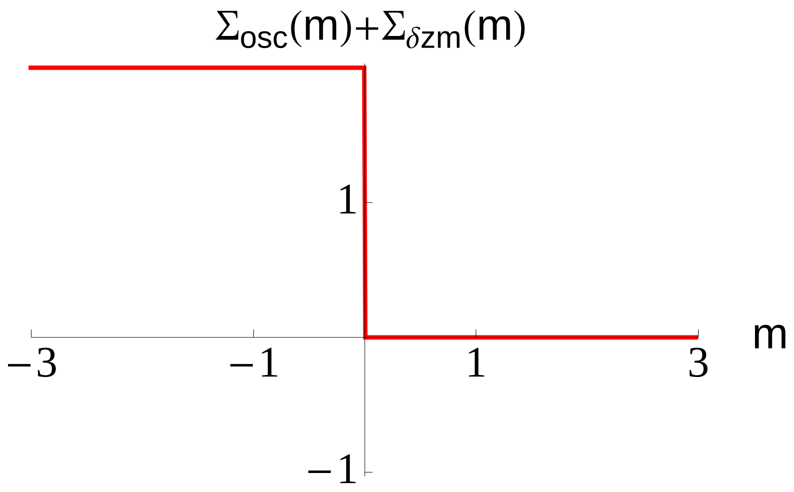
This is valid for $m\Lambda_{\rm QCD} \ll 1/\sqrt{V}$

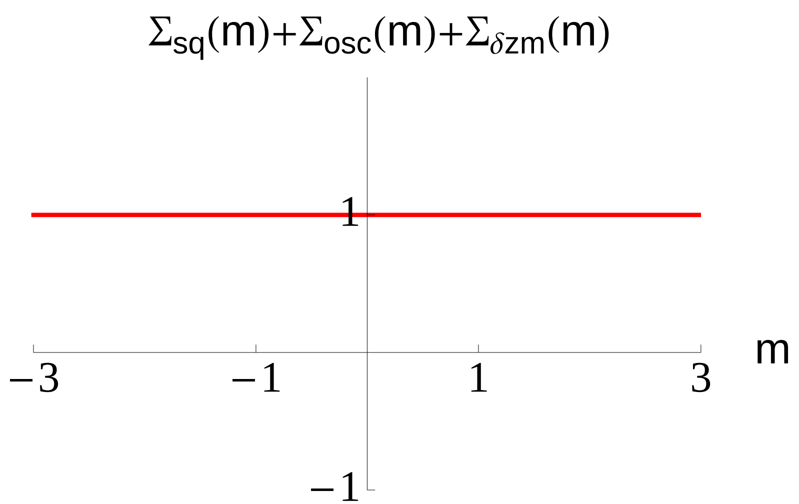
Total Contribution due to the Sign of the Mass



Solution of the Silver Blaze Problem







V. Conclusions

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- From QCD at nonzero chemical potential we have learnt that the solution of the Silver Blaze problem requires an oscillating spectral density with period $\sim 1/V$ and an amplitude that grows exponentially with the volume.
- In the ϵ domain of QCD we have obtained simple exact analytical expressions for the eigenvalue density of the Dirac operator at $\theta = 0$ and $\theta = \pi$. Indeed, an oscillating contribution to the spectral density results in a constant chiral condensate.

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- Rooting fails at a fundamental level.