Analytic solution of the Boltzmann equation in an expanding universe

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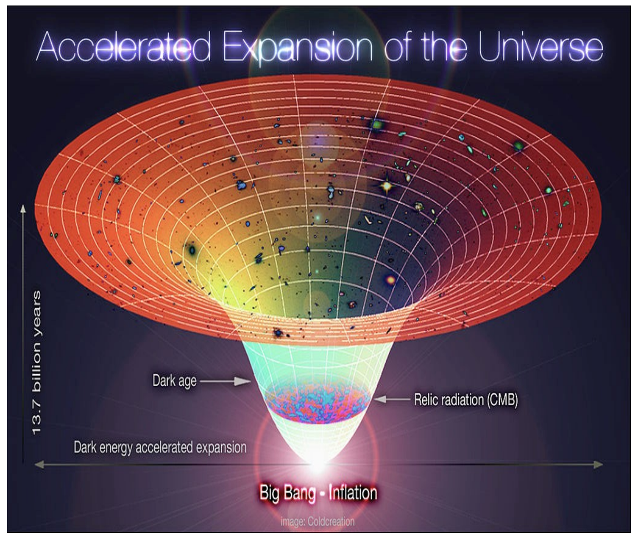
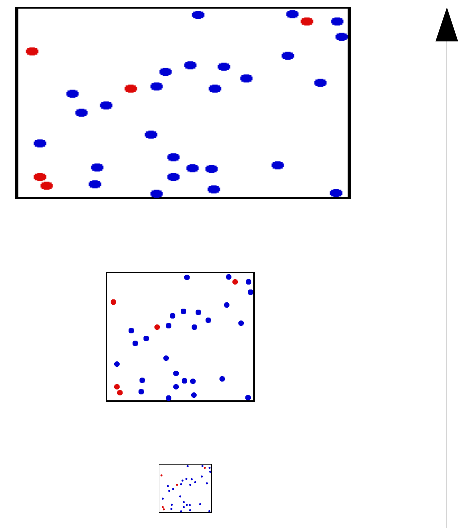
In this talk I will show you that the non-equilibrium dynamics of a gas with

- Ultrarelativistic particles (i.e., massless)

- In an expanding universe



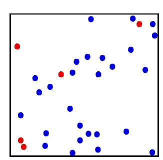
Expanding box



time

is equivalent to that of a non-relativistic gas of Maxwellian molecules

(repulsive $\sim 1/r^4$ interactions)



Fixed box

OUTLINE

- Boltzmann equation in Friedmann-Robertson-Walker spacetime

- General solution using scalar moments

- First analytical solution of the relativistic Boltzmann equation

- Conclusions and outlook

Relativistic Boltzmann Equation

- Dilute gases display complex non-equilibrium dynamics.

- The Boltzmann equation has been instrumental in physics and mathematics (e.g., 2010 Fields Medal).

Relativistic Boltzmann equation

$$\underbrace{k^\mu \partial_\mu f_k(x,k)}_{\text{Space-time variation}} = \underbrace{\mathcal{C}[f(x,k)]}_{\text{Collision term}}$$

- It describes how the particle distribution function $f_k(x,k)$ varies in time and space due to the effects of collisions (and external fields).

The relativistic Boltzmann equation has been applied in:

- Cosmology.

- Neutrino transport in supernovae.

And also in heavy ion collisions:

- Non-equilibrium processes involving quarks and gluons at sufficiently large temperatures.

- Numerical models (BAMPS, MPC, ZPC, AMPT, URQMD).

- Calculation of transport coefficients (e.g., shear viscosity of a hadron gas).

- Determine the regime of validity of relativistic hydrodynamics in rapidly expanding systems.

Example: Exact solution of the RTA Boltzmann equation for Gubser flow

PRL 113 (2014) 20, 202301

Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = \mathcal{C}[f](x,p)$$

Relaxation time approximation (RTA)

$$\mathcal{C}[f](x,p) = \frac{p\cdot u(x)}{\tau_{\rm rel}(x)}\left(f(x,p) - f_{\rm eq}(x,p)\right)$$

The long wavelength, long time limit should be

Viscous hydrodynamics

This is the "microscopic theory"

Effective theory

What is the domain of applicability of relativistic viscous hydrodynamics for rapidly evolving systems (such as the QGP)?

What we did: Phys.Rev.Lett. 113 (2014) 20, 202301

Key idea: Emergent Weyl symmetry.

RTA Boltzmann is Weyl invariant for massless particles.

Weyl transformation changes the curvature of spacetime: $g_{\mu\nu}(x) \to e^{-2\Omega(x)} g_{\mu\nu}(x)$

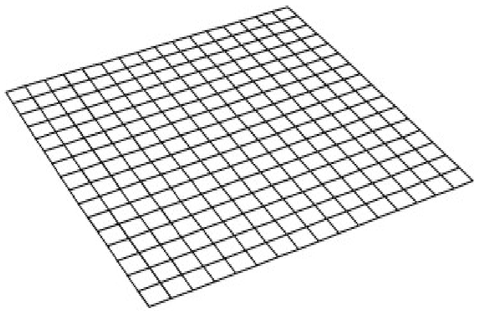
Complicated Gubser flow pattern in flat spacetime ⟷ Trivial static flow in curved Spacetime dS3 x R

$$\hat{u}_\mu = (-1, 0, 0, 0)$$

Gubser, PRD82 (2010) 085027.
Gubser and Yarom, NPB846 (2011) 469-511.

$$\boxed{g_{\mu\nu}(x) \to e^{-2\Omega(x)} g_{\mu\nu}(x)}$$

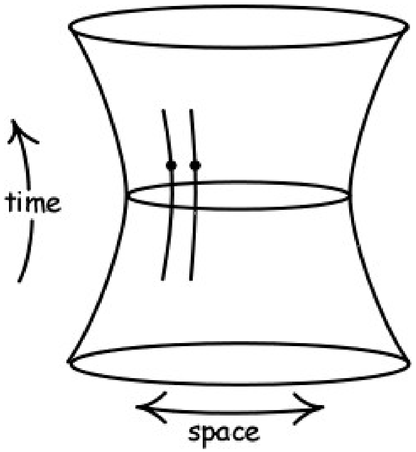
Flat Minkowski space



$$ds^2 = -d\tau^2 + dr^2 + r^2 d\phi^2 + \tau^2 d\xi^2$$

Complicated kinetic dynamics

dS_3 x R



3d de Sitter space

$$d\hat{s}^2 = -d\rho^2 + \cosh^2\rho\, d\theta^2 + \cosh^2\rho\, \sin^2\theta\, d\phi^2 + d\xi^2$$

Trivial flow

<u>What does the flow look like in flat space?</u>

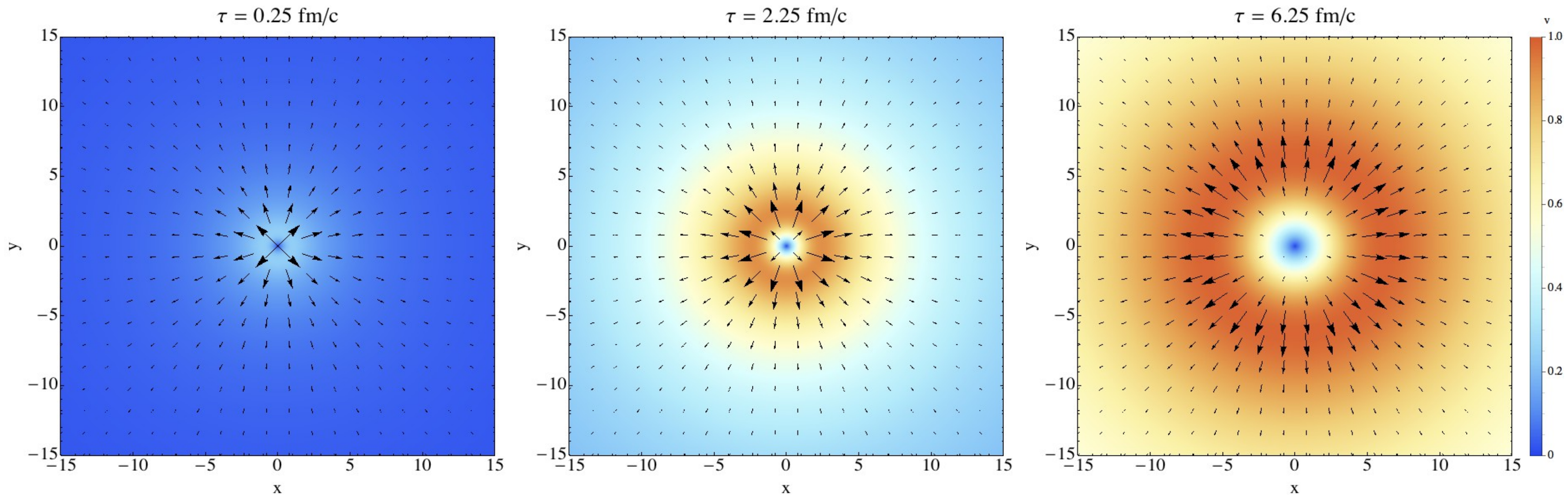
$$\sinh\rho = -\frac{1-\tilde{\tau}^2+\tilde{r}^2}{2\tilde{\tau}}, \qquad \tan\theta = \frac{2\tilde{r}}{1+\tilde{\tau}^2-\tilde{r}^2}.$$

Transverse flow velocity in flat spacetime – Gubser flow

Symmetry under $$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

$$u_r = \sinh\left[\tanh^{-1}\left(\frac{2q\tau r}{1+q^2\tau^2+r^2r^2}\right)\right]$$

Minkowski space



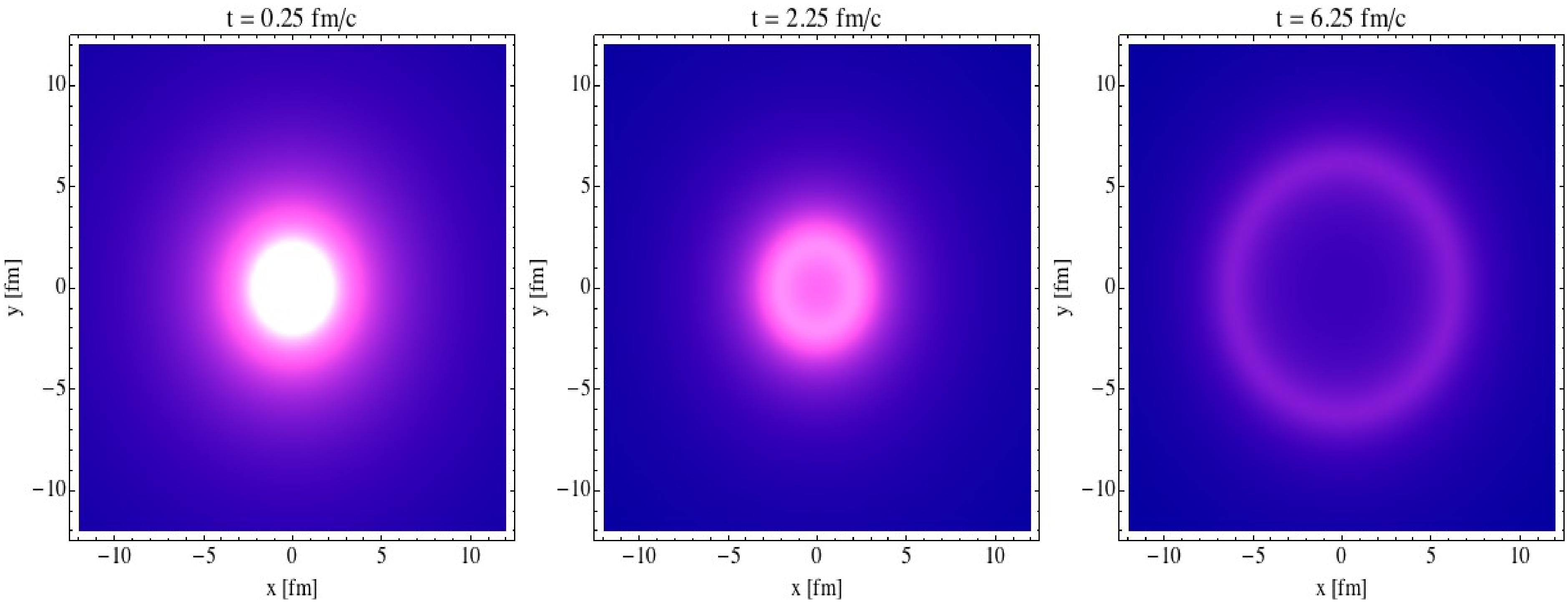
In the curved spacetime description the flow is static !!!

Gubser, PRD82 (2010) 085027.
Gubser and Yarom, NPB846 (2011) 469-511.

In curved spacetime dS3 x R, the RTA Boltzmann equation becomes very simple

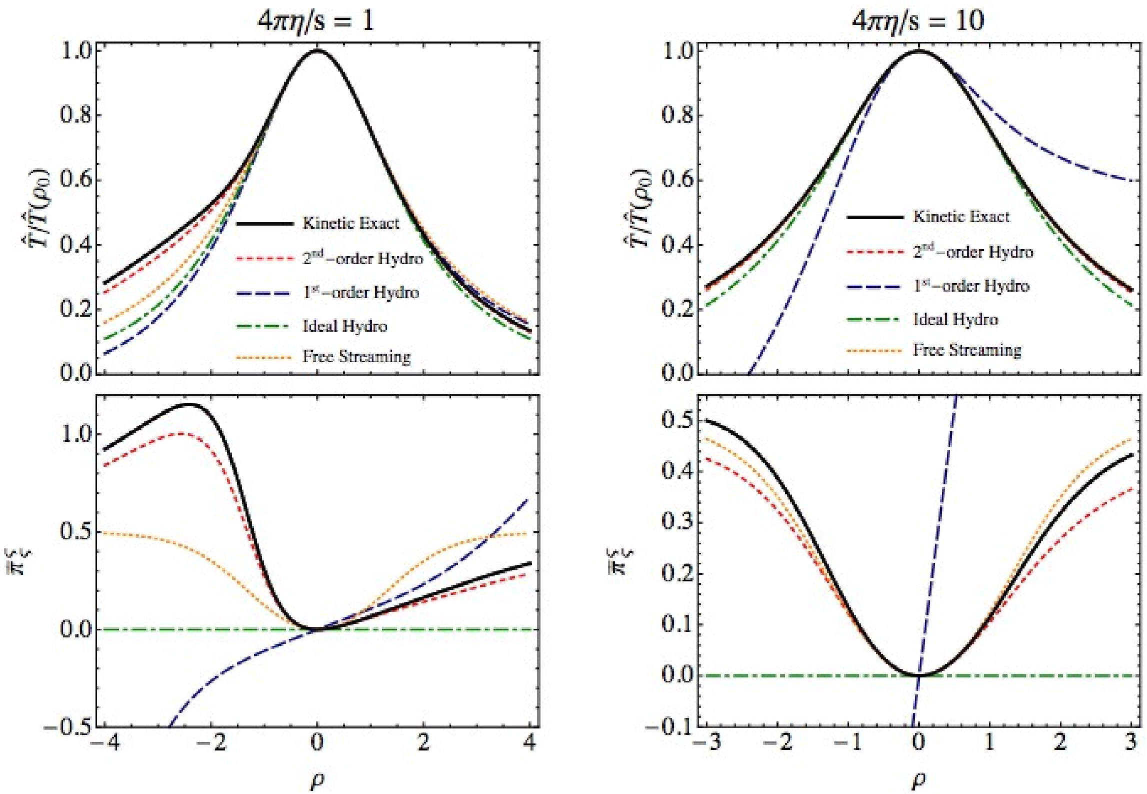
$$\boxed{\frac{\partial}{\partial\rho} f(\rho;\hat{p}_\Omega^2,\hat{p}_\varsigma) = -\frac{\hat{T}(\rho)}{c}\left[f(\rho;\hat{p}_\Omega^2,\hat{p}_\varsigma) - f_{\rm eq}\left(\frac{\hat{p}^\rho}{\hat{T}(\rho)}\right)\right]}$$

Using this we can compute the **energy density** and etc and Weyl it back to flat space



Checking the domain of applicability of different hydrodynamics theories

Hydro solution based on Marrochio et al., Phys.Rev. C91 (2015) 1, 014903

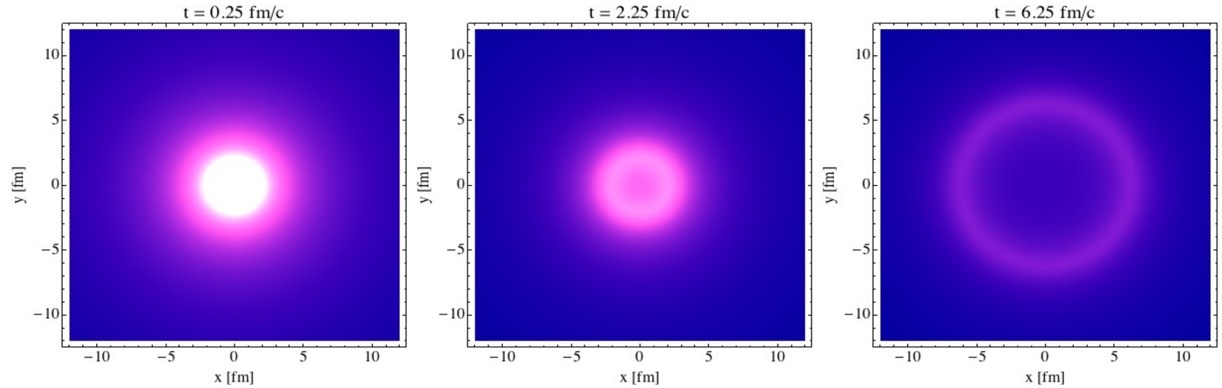


~~1st order (NS)~~

Hydro breaks down
at very early times
and very large radius

Domain of applicability of 2nd order hydro is surprisingly large !!!

- Viscous hydrodynamics can match the underlying kinetic theory dynamics pretty well even in this system that rapidly expands both in the longitudinal and transverse directions.



- However, the large domain of applicability of hydrodynamics found in this particular example may be an artifact of the assumptions regarding the collision term (RTA).

- A more realistic treatment of the collision term is needed …

What is the simplest expanding kinetic system that can be studied taking into account the inherent nonlinearities due to collisions?

A: Massless gas with a constant cross section in an expanding universe.

Boltzmann Equation in Friedmann-Robertson-Walker spacetime

We consider an isotropic and homogeneous expanding FRW spacetime

(zero spatial curvature)

metric

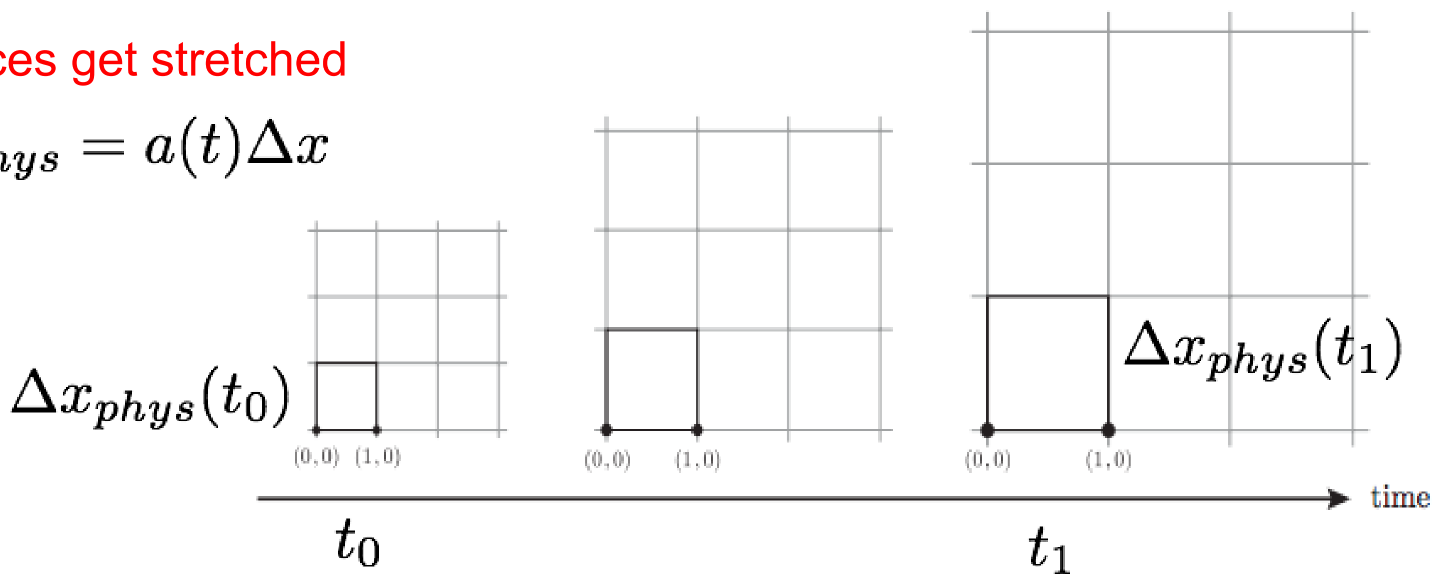
$$\boxed{ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)}$$

Cosmological scale factor $a(t) \sim t^{1/2}$

Hubble parameter $H = \dot{a}/a > 0$

Distances get stretched

$$\Delta x_{phys} = a(t)\Delta x$$



Due to the expansion, a locally static fluid $u^\mu = (1,0,0,0)$ possesses a nonzero

Expansion rate: $\theta = \nabla_\mu u^\mu = 3H(t)$

We consider massless particles and elastic 2 → 2 scattering.

Symmetries imply that energy-momentum and particle currents obey the eqs.

$$\partial_t n + 3nH(t) = 0, \qquad \partial_t \varepsilon + 4\varepsilon H(t) = 0.$$

Particle density Energy density

Solution: $n(t) = n(t_0)/a^3(t)$ and $\varepsilon(t) = \varepsilon(t_0)/a^4(t)$

However, this does not imply that the system is in equilibrium.

$$f_k(t) \neq f_{eq}$$

The Boltzmann equation in curved spacetime

See Debbasch, van Leuween, 2009

$$\boxed{k^\mu \partial_\mu f_{\boldsymbol{k}} + \Gamma^\lambda_{\mu i} k_\lambda k^\mu \frac{\partial f_{\boldsymbol{k}}}{\partial k_i} = C[f]}$$

$\Gamma^0_{ij} = a(t)\dot{a}(t)\,\delta_{ij}$ and $\Gamma^i_{0j} = \delta^i_j\, H(t)$

No external forces

On-shell particle distribution $f_{\boldsymbol{k}}(t) = f(t, \boldsymbol{k})$

(covariant momenta)

$k^0 = k/a(t)$ with $k = |\boldsymbol{k}|$

In this particular case the equation becomes $\boxed{k^0 \partial_t f_k = C[f]}$

Given that a(t) is known, a solution of the Boltzmann in fact is a solution of the coupled Einstein-Boltzmann equations.

Now we are ready to specify the collision term.

We assume classical statistics:

$$\boxed{C[f] = \frac{1}{2}\int_{k'pp'} W_{\boldsymbol{kk'}\to\boldsymbol{pp'}}\left(f_p f_{p'} - f_k f_{k'}\right)}$$

$\int_k \equiv \int d^3k / \left[(2\pi)^3 \sqrt{-g}\, k^0\right]$

Here we only consider a **constant cross section** σ

$$\boxed{W_{\boldsymbol{kk'}\to\boldsymbol{pp'}} = (2\pi)^5 \sqrt{-g}\,\sigma\, s\, \delta^4(k+k'-p-p')}$$

$s = (k^\mu + k'^\mu)(k_\mu + k'_\mu)$

And the Boltzmann equation becomes

$$\boxed{k^0 \partial_t f_k = \frac{(2\pi)^5}{2}\sqrt{-g}\,\sigma \int_{k'pp'} s\, \delta^4(k+k'-p-p')(f_p f_{p'} - f_k f_{k'})}$$

Our Boltzmann equation:

$$\boxed{k^0 \partial_t f_k = \frac{(2\pi)^5}{2}\sqrt{-g}\,\sigma \int_{k'pp'} s\,\delta^4(k+k'-p-p')(f_p f_{p'} - f_k f_{k'})}$$

This equation includes general relativistic effects + full nonlinear collision dynamics

<u>We want to find the general solution for the distribution function</u>

Given an initial condition: $f(t_0, k)$ and $n(t_0)$, $\varepsilon(t_0)$

How does one solve this type of nonlinear integro-differential equation?

The moments method

- Originally introduced by Grad (1949) and used by Israel and Stewart (1979) in the relativistic regime.

- Used more recently in Denicol et al. PRD 85, 114047 (2012).

The idea is simple.

Instead of solving for the distribution function itself directly, one uses the Boltzmann eq. to find equations of motion for the moments of the distribution function.

Ex: The particle density $n(t) = \int_k (u.k) f_k(t)$ is a scalar moment

with equation $\partial_t n + 3n\, H(t) = 0$

Ex: The energy density $\varepsilon(t) = \int_k (u.k)^2 f_k(t)$ is a scalar moment

with equation $\partial_t \varepsilon + 4\varepsilon\, H(t) = 0$

Clearly, due to the symmetries, only scalar moments can be nonzero.

Thus, if we can find the time dependence of the scalar moments

$$(m \in \mathbb{N}_0) \quad \boxed{\rho_m(t) = \int_{\boldsymbol{k}} (u \cdot k)^{m+1} f_k(t)} \qquad \rho_0 = n,\ \rho_1 = \varepsilon$$

via solving its exact equations of motion, one should be able to recover $f_k(t)$

This is how it is done.

$$\int_k (u.k)^m \text{ x } \boxed{k^0 \partial_t f_k = \frac{(2\pi)^5}{2}\sqrt{-g}\,\sigma \int_{k'pp'} s\,\delta^4(k+k'-p-p')(f_p f_{p'} - f_k f_{k'})}$$

and this gives

$$\boxed{\partial_t \rho_m(t) + (3+m)H(t)\rho_m(t) = \mathcal{C}^{(m)}_{\text{gain}}(t) - \mathcal{C}^{(m)}_{\text{loss}}(t)}$$

where

$$\mathcal{C}^{(m)}_{\text{gain}}(t) = \frac{(2\pi)^5}{2}\sqrt{-g}\,\sigma \int_{kk'pp'} s\,(u{\cdot}p)^m \delta^4(k+k'-p-p') f_k f_{k'},$$

$$\mathcal{C}^{(m)}_{\text{loss}}(t) = \frac{(2\pi)^5}{2}\sqrt{-g}\,\sigma \int_{kk'pp'} s\,(u{\cdot}k)^m \delta^4(k+k'-p-p') f_k f_{k'}.$$

It is easy to show that $$\boxed{\mathcal{C}^{(m)}_{\text{loss}}(t) = \sigma \rho_m(t) n(t)}$$

However, the gain term is much more complicated. One first writes

$$\mathcal{C}^{(m)}_{\text{gain}}(t) = \frac{\sigma}{2} \int_{kk'} s\, f_k f_{k'}\, \mathcal{P}_m$$

with

$$\mathcal{P}_m \equiv (2\pi)^5 \sqrt{-g} \int_{pp'} (u{\cdot}p)^m\, \delta^4(k{+}k'{-}p{-}p') \;=\; \frac{m!}{[2a(t)]^m} \sum_{j\ \text{odd}}^{m+1} \frac{(k{+}k')^{m+1-j}}{(m{+}1{-}j)!}\, \frac{|\boldsymbol{k}{+}\boldsymbol{k}'|^{j-1}}{j!}$$

which finally gives

$$\boxed{\mathcal{C}^{(m)}_{\text{gain}}(t) = 2\,(m{+}2)\, m!\, \sigma \sum_{j=0}^{m} \frac{\rho_j(t)}{(j{+}2)!}\, \frac{\rho_{m-j}(t)}{(m{-}j{+}2)!}}$$

Now we can write the exact equations for the moments.

It is convenient to define the scale time $\hat{t} = t/\ell_0$ where $\ell_0 = 1/(\sigma\, n(t_0))$

(constant mean free path)

And the normalized moments $M_m(\hat{t}) = \dfrac{\rho_m(\hat{t})}{\rho_m^{eq}(\hat{t})}$ which obey the **exact** set of eqs:

arXiv:1507.07834 [hep-ph]

$$\boxed{a^3(\hat{t}\,)\,\frac{\partial}{\partial \hat{t}} M_m(\hat{t}) + M_m(\hat{t}) = \frac{1}{m+1}\sum_{j=0}^{m} M_j(\hat{t}) M_{m-j}(\hat{t})}$$

GR effect

Simple recursive nonlinearity

Conservation laws require $M_0 = M_1 = 1$

Defining the time variable $\tau = \int_{\hat{t}_0}^{\hat{t}} dt'/a^3(t')$ to account for the expansion of the universe, one finds

$$\boxed{\partial_\tau M_m(\tau) + M_m(\tau) = \frac{1}{m+1}\sum_{j=0}^{m} M_j(\tau) M_{m-j}(\tau).}$$

- This equation is identical to the famous Bobylev-Krook-Wu equation found for the non-relativistic Boltzmann equation with Maxwellian molecules nearly 40 years ago.

- The underlying symmetry of the non-relativistic Boltzmann (Galilean invariance) is very different than our equation that was derived within GR.

- This shows that these very different physical systems are identical from a dynamical systems perspective and they evolve towards equilibrium in an universal manner.

Equilibrium corresponds to $\lim_{\tau \to \infty} M_m(\tau) = 1$

The only issue is that for a radiation dominated universe $\lim_{\hat{t} \to \infty} \tau(\hat{t}) = \text{finite}$

There is never enough time for the system to achieve equilibrium, regardless of the non-equilibrium initial conditions.

This is due to the fact that here $\mathrm{lim}_{\hat{t} \to \infty} \ell(\hat{t})\theta(\hat{t}) \to \infty$

A radiation dominated universe expands too quickly for this gas to “forget” its initial data via collisions.

This is a feature of the constant cross section assumption.

Does this system exhibit exact self-similar behavior?

scaling exponents

Assuming exact self-similar Ansatz $f_k(\hat{t}) = a^{\gamma}(\hat{t})\, f_S(a^{\beta}(\hat{t})\, u{\cdot}k)$

Fixed point distribution

Conservation laws impose that $\gamma = 0 \text{ and } \beta = 1$

This Ansatz leads $\rho_m^S(\hat{t}) = \frac{1}{2\pi^2}\frac{c_m^S}{a^{m+3}(\hat{t})}$ with $c_m^S = \int_0^\infty d\xi\, \xi^{m+2} f_S(\xi)$

Thus, $M_m^S = \rho_m^S/\rho_m^{eq}$ are time independent.

$M_0^S = M_1^S = 1$ + time independence = There is only 1 fixed point (equilibrium)

System never truly equilibrates but exact self-similar behavior is not observed.

Ok, so in general one needs to solve the (simple) dynamical equations for $M_m(\hat{t})$

But how do we go from the set $\{M_m(\hat{t})\} \rightarrow f_k(\hat{t})$???

This can be done by introducing the series expansion using the orthogonal basis

$$f_k(\hat{t}) = 2\pi^2 e^{-u.k/T} \sum_{n=0}^{\infty} \frac{\psi_n(\hat{t})}{(n+1)(n+2)} L_n^{(2)}\left(\frac{u.k}{T}\right)$$

Associated Laguerre

with $$\frac{\psi_n(\hat{t})}{(n+1)(n+2)} = \frac{\rho_0}{2T^3} \sum_{m=0}^{n} (-1)^m \binom{n}{m} M_m(\hat{t})$$

Therefore, the general solution of the problem can always be found.

Full Analytical Solution

Using the moments equations in this form $\tau = \int_{\hat{t}_0}^{\hat{t}} dt'/a^3(t')$

$$\partial_\tau M_m(\tau) + M_m(\tau) = \frac{1}{m+1}\sum_{j=0}^{m} M_j(\tau) M_{m-j}(\tau).$$

One can show that

$$M_m(\tau) = \mathcal{K}(\tau)^{m-1}\left[m - (m-1)\mathcal{K}(\tau)\right] \quad (m \geq 0)$$

is an analytical solution of the moments equations

$$\mathcal{K}(\tau) = 1 - \frac{e^{-\tau/6}}{4}$$

Full Analytical Solution

Since $M_m(\tau)$ are known, one can show that

$$\frac{\psi_n(\tau)}{(n+1)(n+2)} = \frac{\rho_0}{2T^3}(1-n)(1-\mathcal{K}(\tau))^n$$

which can be used to find the 1st analytical solution of the relativistic Boltzmann equation

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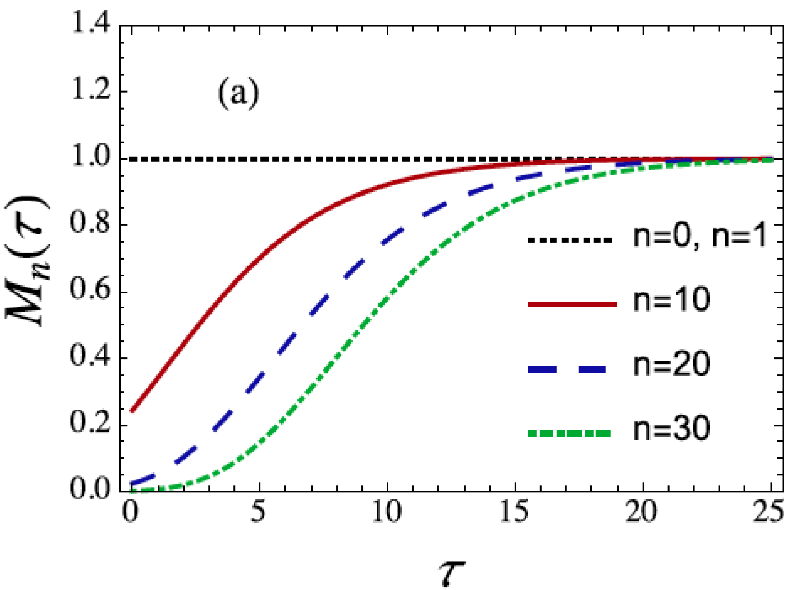
λ = fugacity

$$f_k(\tau) = \lambda \exp\left(-\frac{u\cdot k}{\mathcal{K}(\tau)T(\tau)}\right) \times \left[\frac{4\mathcal{K}(\tau)-3}{\mathcal{K}^4(\tau)} + \frac{u\cdot k}{T(\tau)}\left(\frac{1-\mathcal{K}(\tau)}{\mathcal{K}^5(\tau)}\right)\right]$$

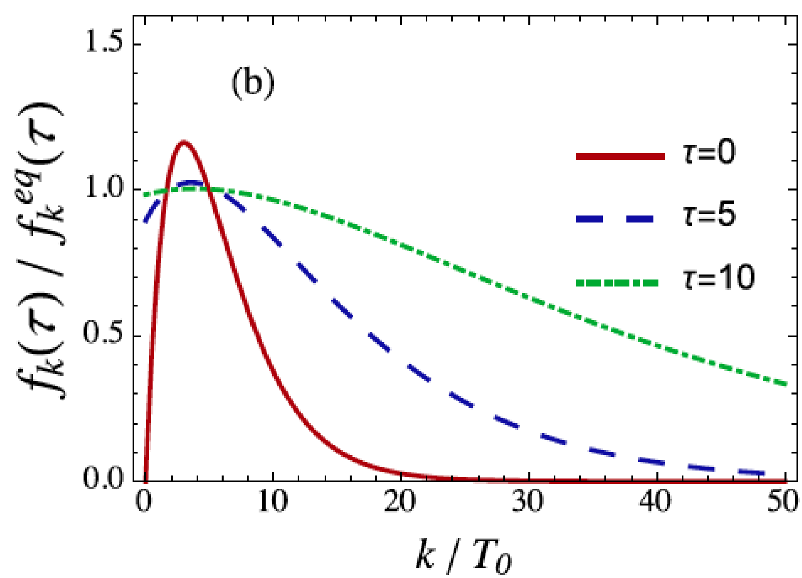
Initial condition $f_k(0) = \frac{256}{243}(k/T_0)\,\lambda\,\exp[-4k/(3T_0)] > 0$

Full Analytical Solution

Time evolution



Momentum dependence



Note that for radiation dominated universe $\tau(\hat{t}) = 2\hat{t_0}\left(1 - \sqrt{\frac{\hat{t}_0}{\hat{t}}}\right)$

So higher order moments will certainly not erase the info about initial conditions.

The approach to equilibrium here depends on the occupancy of each moment.

Conclusions and Outlook

- In this talk we have derived the first analytical solution of the Boltzmann equation for an expanding system using the method of moments.

- We considered massless particles, classical statistics, and a simple constant cross section interaction.

- For arbitrary initial conditions, we have shown how the general solution of the Boltzmann equation can be obtained once the moments are numerically computed.

- We found an intriguing equivalence between radically different systems:

Non-relativistic Maxwellian molecules x Massless particles in FRW spacetime

Even though the underlying symmetries are different, the dynamical moment equations are the same and, thus, these systems approach equilibrium in the same way.

Conclusions and Outlook

Our results can be generalized in many ways:

- It would be interesting to compare the differences between systems with different cross sections (e.g., massless scalar x constant cross section).

- Inclusion of quantum statistic effects.

- Inclusion of mass effects (this switches on bulk viscosity).

- Generalize approach to include the case of rapidly expanding anisotropic systems (e.g., Bjorken or Gubser expanding gases).

- Multi-particle systems (applications in cosmology and condensed matter systems).

- Inclusion of external fields (and possibly other background geometries).

- QED and QCD-like cross sections.

- More formal mathematical aspects about the relativistic Boltzmann equation.

EXTRA SLIDES