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# Kosterlitz-Thouless transition and chiral rotation in external electromagnetic field

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[PhysRevD.90.056005 \(2014\)](#) , [Phys. Lett. B757.1 \(2016\)](#)

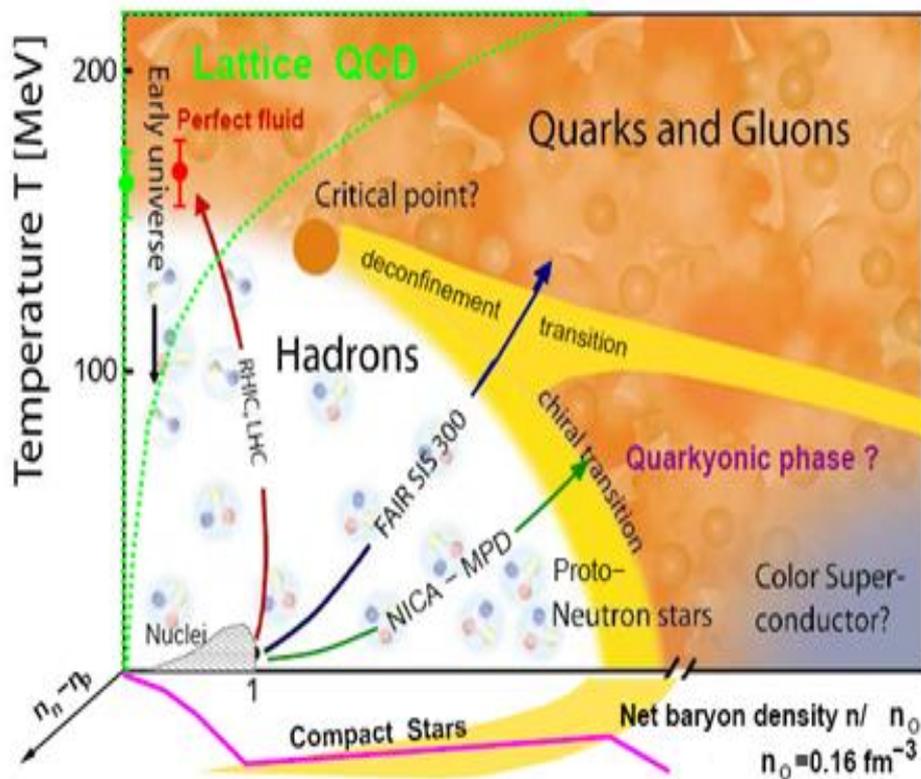
# Outline

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- QCD transitions in strong EM field
- NJL model, Schwinger and Ritus approaches
- KT transition and dHvA oscillation in pure magnetic field
- Chiral rotation and  $\pi_0$  condensation in parallel EM field
- Conclusions

# $T - \mu$ phase diagram of QCD



Crossover at low chemical potential-- $T_c \sim 165 \text{ MeV}$ ;

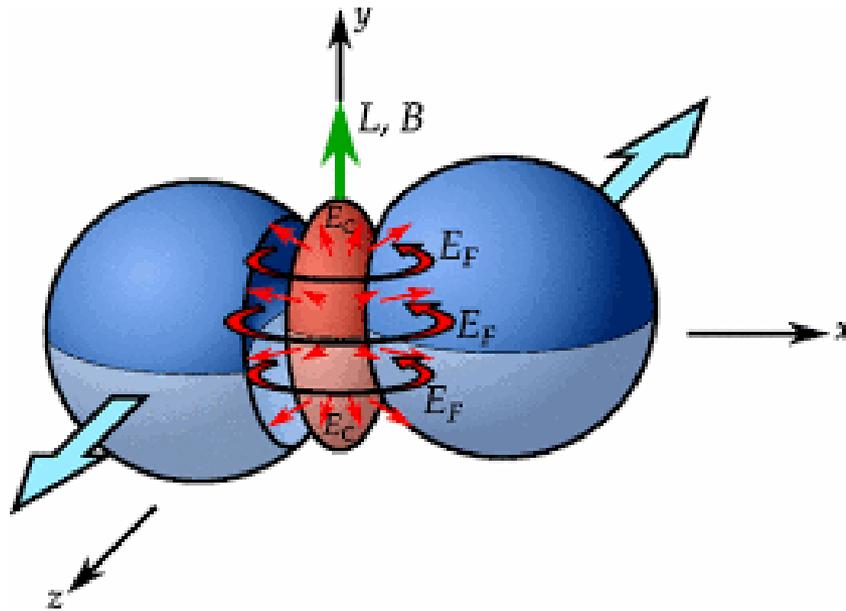
First-order transition at moderate chemical potential to quarkyonic matter--might be covered by inhomogeneous LOFF state;

Color superconductor at high chemical potential-- crystalline pairing structure.

**New parameters, new features!**

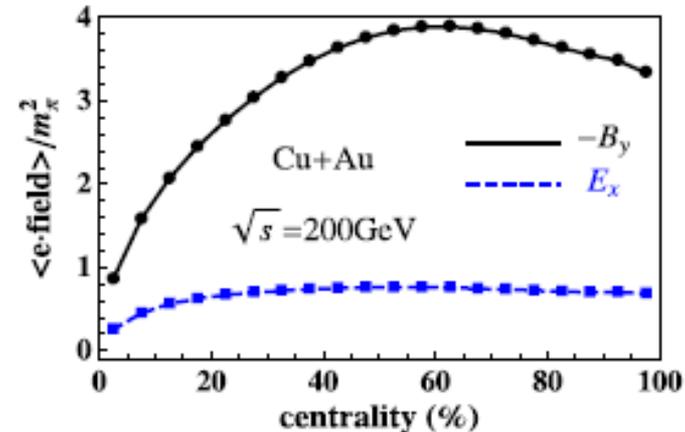
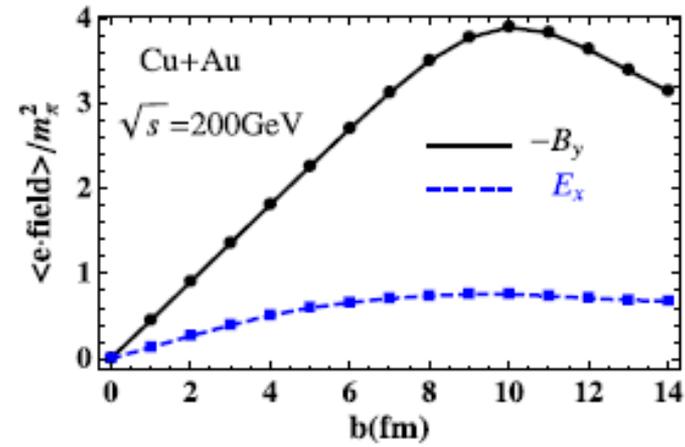
Pion superfluid at finite  $\mu_I$ .

# Strong EM field in HIC



In LHC, magnetic field is even larger  
 $\sim 1 \text{ GeV}^2$ .

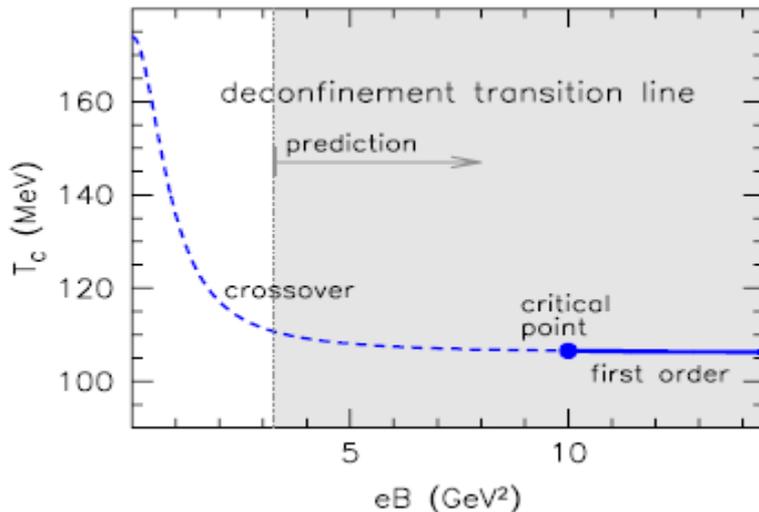
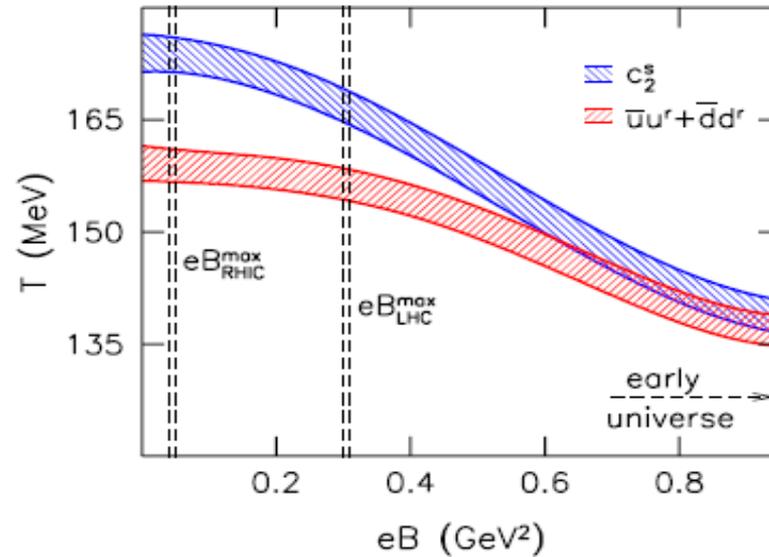
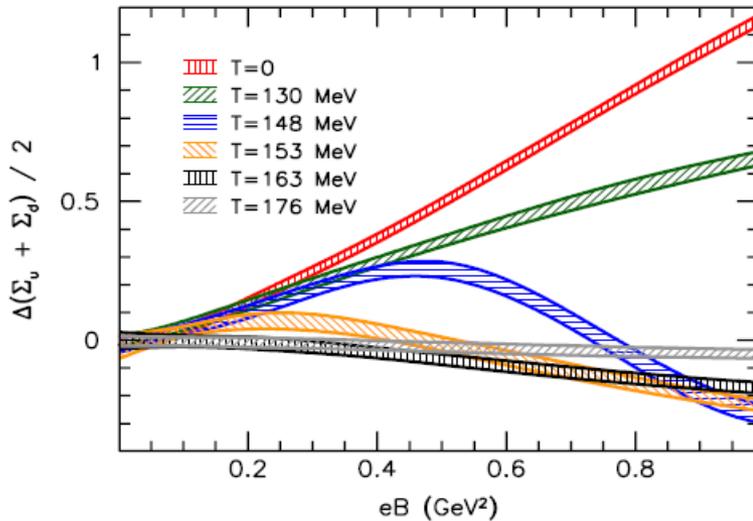
**QCD phase diagram in  
 strong EM field ?**



W.-T. Deng, X.-G. Huang, *PLB* 742, 296 (2015).



# Inverse magnetic catalysis (LQCD)



- (1) At  $T = 0$ ,  $\Sigma$  increases with  $B$ ; around  $T_c$ ,  $\Sigma$  decreases with  $B$ ;
- (2)  $T_c$  monotonically decreases with  $B$ ;
- (3)  $T_c$  converges to a finite value at large enough  $B$  and the crossover becomes first order.

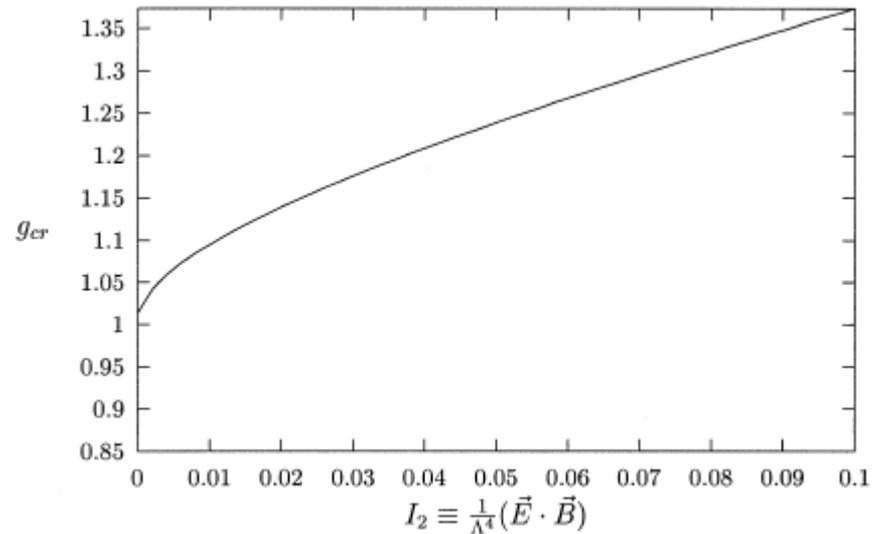
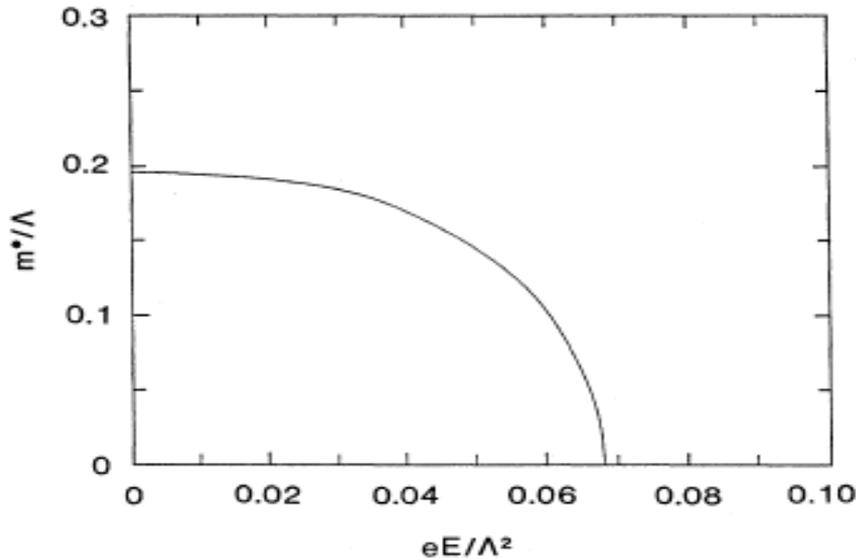
[G. Bali, et al., JHEP 1202 \(2012\) 044;](#)

[Phys.Rev. D86 \(2012\) 071502\(R\);](#)

[G. Endrődi, JHEP 1507 \(2015\) 173 .](#)



# The effect of electric field



- (1) Mass gap decreases monotonically with  $E$ ;
- (2) Chiral transition is second order.
- (1) Critical coupling increases with  $I_2$ , the effect is more E-like;
- (2) The transition is second order.

[S. P. Klevansky and R. H. Lemmer, Phys. Rev. D 39, 3478 \(1989\).](#)

[A. Y. Babansky, E. V. Gorbar and G. V. Shchepanyuk, Phys. Lett. B 419, 272 \(1998\).](#)



# Nambu—Jona-Lasinio model

The effective Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\mathcal{D} - m_0 - \mu\gamma^0)\psi + G \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2 \right]$$

$$\psi(x) = (u(x), d(x))^T, D_\mu = \partial_\mu + iQA_\mu$$

Introducing auxiliary fields  $\sigma, \pi$  (Hubbard-Stratonovich transformation), the partition function becomes:

$$\mathcal{Z} = \int [\mathcal{D}\hat{\sigma}][\mathcal{D}\hat{\pi}_0][\mathcal{D}\hat{\pi}_\pm] \exp \left\{ - \int dx \left[ \frac{(m - m_0)^2 + \hat{\sigma}^2 + \hat{\pi}_0^2 + \hat{\pi}_\pm^2}{4G} \right] \right. \\ \left. + \text{Tr} \ln \left[ i\mathcal{D} - m - \hat{\sigma} - i\gamma_5 (\tau_3\hat{\pi}_0 + \tau_\pm\hat{\pi}_\pm) - i\mu\gamma^4 \right] \right\},$$

In mean field approximation, the thermodynamic potential is

$$\Omega(m) = \frac{(m - m_0)^2}{4G} - \frac{1}{\beta V} \text{Tr} \ln \left[ i\mathcal{D} - m - i\mu\gamma^4 \right]$$



# Nambu—Jona-Lasinio model

Gap equation 
$$\frac{m - m_0}{2G} - \frac{1}{\beta V} \text{Tr} \mathcal{S}(x, x') = 0,$$

Collective modes give higher order contribution to  $\Omega$

$$\begin{aligned} i\mathcal{D}_M^{-1}(x, x') &= \frac{\delta(x - x')}{2G} + \Pi_M(x, x') \\ &= \frac{e^{iq_M \int_x^x A dx}}{2G} \delta(x - x') + \text{Tr} \mathcal{S}(x, x') \Gamma_M \mathcal{S}(x', x) \Gamma_M^* \end{aligned}$$

$$\Gamma_M = \begin{cases} 1, & M = \hat{\sigma} \\ i\gamma_5 \tau_+, & M = \hat{\pi}_+ \\ i\gamma_5 \tau_-, & M = \hat{\pi}_- \\ i\gamma_5 \tau_3, & M = \hat{\pi}_0 \end{cases}; \quad \Gamma_M^* = \begin{cases} 1, & M = \hat{\sigma} \\ i\gamma_5 \tau_-, & M = \hat{\pi}_+ \\ i\gamma_5 \tau_+, & M = \hat{\pi}_- \\ i\gamma_5 \tau_3, & M = \hat{\pi}_0 \end{cases}$$

For simplicity, we only study the properties of the Goldstone mode  $\hat{\pi}_0$ .



# Schwinger approach

J. Schwinger, Phys. Rev. 82, 664 (1951)

## EM current interaction

$$\begin{aligned}\delta W^{(1)} &= \int (dx) \delta A_\mu(x) \langle j_\mu(x) \rangle = ie \text{Tr} \gamma \delta A G \\ &= -\text{Tr} \delta(\gamma \Pi) \gamma \Pi \int_0^\infty ds \exp[-i(m^2 - (\gamma \Pi)^2)s] \\ &= \delta \left[ \frac{1}{2} i \int_0^\infty ds s^{-1} \exp[-i(m^2 - (\gamma \Pi)^2)s] \right],\end{aligned}$$

Equivalent to introduce an effective Lagrangian

$$\mathcal{L}^{(1)}(x) = \frac{1}{2} i \int_0^\infty ds s^{-1} \exp(-im^2s) \text{tr}(x | U(s) | x),$$

$$U(s) = \exp(-i\mathcal{H}Cs),$$

**U(s) is the time-evolution operator**  $(x' | U(s) | x'') = (x(s)' | x(0)'')$



# Schwinger approach

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For an uniform field , the fermion propagator is

$$G(x', x'') = i \int_0^\infty ds \exp(-im^2s) [ - (x(s)' | \Pi_\mu(0) | x(0)'') \gamma_\mu + m(x(s)' | x(0)'') ]$$

The evolution of the operators are

$$(x(s)' | \Pi(0) | x(0)'') = \frac{1}{2} [ eF \coth(eFs) - eF ] \times (x' - x'') (x(s)' | x(0)'')$$

$$(x(s)' | x(0)'') = -i(4\pi)^{-2} \Phi(x', x'') e^{-L(s)} s^{-2} \\ \times \exp[ i \frac{1}{4} (x' - x'') eF \coth(eFs) (x' - x'') ] \cdot \exp[ i \frac{1}{2} e\sigma F s ]$$



# Ritus approach

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V. I. Ritus, Ann. Phys. (Berlin) 69, 555 (1972);

V. I. Ritus, Zh. Eksp. Teor. Fiz. 75, 1560 (1978).

For simplicity, we refer to

J. Warringa, Phys. Rev. D86, 085029 (2012).

**Dirac Equation**  $(i\gamma^\mu D_\mu - m)\psi(x) = 0, A_\mu = (0, 0, B_z x, A_z(t))$

**Set the solutions to have the forms**

$$\psi_{ps}^+(x) = F_p^+(t)G_p(x)e^{ip_y y + ip_z z}u_s(\tilde{p}_+),$$

$$\psi_{ps}^-(x) = F_p^-(t)G_p(x)e^{ip_y y + ip_z z}v_s(\tilde{p}_-).$$

**and satisfy**

$$[i\gamma^0 \partial_t - \gamma^3(p_z - qA_z(t))]F_p^\pm(t) = \pm \kappa F_p^\pm(t)\gamma^0,$$

$$[i\gamma^1 \partial_x - \gamma^2(p_y - qB_z x)]G_p(x) = \lambda G_p(x)\gamma^2.$$



# Ritus approach

**Dirac equation becomes a free-Fermion equation**

$$(\tilde{\not{p}}_+ - m)u_s(\tilde{p}_+) = 0, \quad (\tilde{\not{p}}_- + m)v_s(\tilde{p}_-) = 0. \quad \tilde{p}_\pm^\mu = (\kappa, 0, \mp\lambda, 0).$$

**Final solution of  $G_p(x)$**

$$G_p(x) = \frac{1 + i\text{sgn}(qB_z)\gamma^1\gamma^2}{2}\phi_k(x - \frac{p_y}{qB_z}) + \frac{1 - i\text{sgn}(qB_z)\gamma^1\gamma^2}{2}\phi_{k-1}(x - \frac{p_y}{qB_z})$$

**Second quantization**

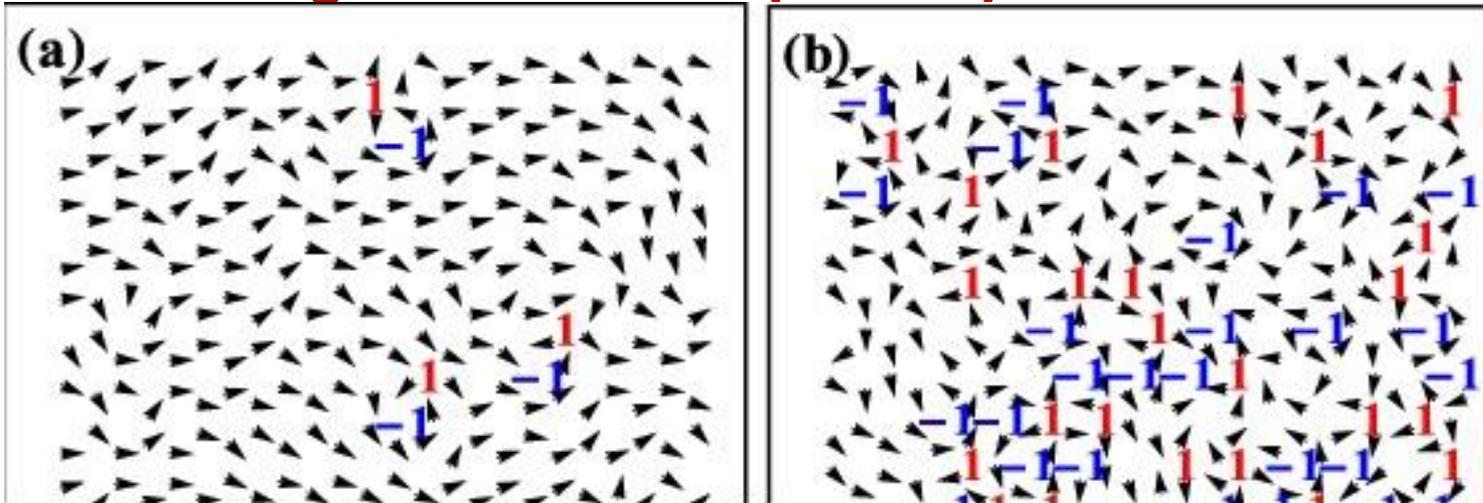
$$\Psi(x) = \sum_{s=\pm} \sum_p \frac{1}{\sqrt{2\kappa_p}} \left[ b_{ps} \psi_{ps}^+(x) + d_{-ps}^\dagger \psi_{ps}^-(x) \right],$$

**The propagators of particle and antiparticle**

$$S^\pm(x, x') = \sum_p \frac{1}{2\kappa_p} e^{ip_y(y-y') + ip_z(z-z')} F_p^\pm(t) G_p(x) \\ \times (\tilde{\not{p}}_\pm \pm m) \gamma^0 F_p^\pm(t')^\dagger \gamma^0 G_p(x').$$

# Kosterlitz-Thouless transition

Coleman-Mermin-Wagner theorem forbids spontaneous breaking of continuous symmetry at  $T \neq 0$  in 2+1D



(a) Metallic State

(b) Insulator State

KT transition temperature

$$T_{KT} = \frac{\pi}{2} J$$

stiffness

$$J = NM^2 \xi_1(M, T) = \frac{NM}{4\pi} \tanh \frac{M}{2T}$$

[E. Babaev, Phys. Lett. B497, 323 \(2001\).](#)



# Kosterlitz-Thouless transition

Introduce chiral chemical potential:

$$\mathcal{L} = \bar{\psi}(i\not{D} + \mu_5\gamma^0\gamma^5)\psi + \frac{G}{2N} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right]$$

Utilizing Ritus's method, the Fermion propagator is

$$\begin{aligned} \mathcal{S}_F(x, x') = & \sum_{s=\pm} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \frac{i}{2[p_0^2 - (\varepsilon_n^s)^2]} e^{-ip_0(x_0-x'_0)+ip_2(x_2-x'_2)} \\ & \times \left\{ G_n\left(x_1 - \frac{p_2}{eB}\right) G_n\left(x'_1 - \frac{p_2}{eB}\right) [p_0\gamma^0 + (\mu_5 + s\lambda_n)\gamma^0\gamma^5 + M] \right. \\ & \left. + G_n\left(x_1 - \frac{p_2}{eB}\right) \gamma^1 G_n\left(x'_1 - \frac{p_2}{eB}\right) (\gamma^1)^\dagger [p_0\gamma^0 - (\mu_5 + s\lambda_n)\gamma^0\gamma^5 - M] (-is\gamma^1\gamma^3) \right\}. \end{aligned}$$

Excitation energy  $\varepsilon_n^s = \sqrt{(\lambda_n + s\mu_5)^2 + M^2}$ .



# Kosterlitz-Thouless transition

Gap equation can be derived to

$$\frac{M^2}{\pi} - \frac{MM_0}{\pi} \text{sgn}(G - G_c) - \frac{eB}{2\pi} \frac{\eta f_{3/2}(\eta)}{\sqrt{\pi}} - \frac{eB}{4\pi} \sum_{s=\pm} \sum_{n=0}^{\infty} \alpha_n M \left[ \frac{1 - 2n_F(\epsilon_n^s)}{\epsilon_n^s} - \frac{1}{\epsilon_n} \right] = 0$$

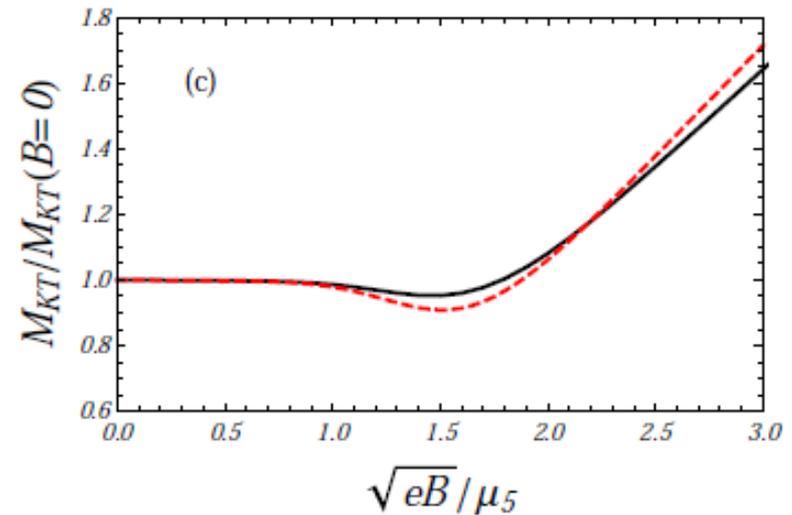
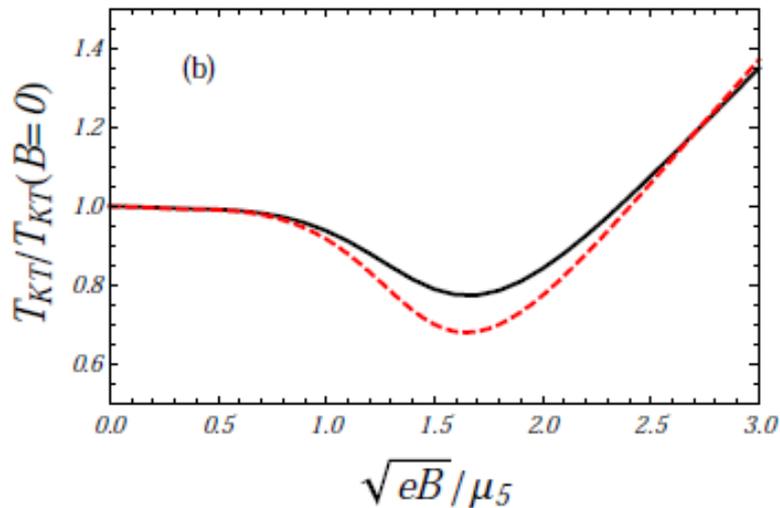
$$f_n(\eta) = \int_0^{\infty} \frac{dx}{x^n} e^{-\eta^2 x} \left( \frac{x}{\tanh x} - 1 \right) \quad \eta = M/(eB)^{1/2}$$

The stiffness is complex:

$$\begin{aligned} \frac{J}{N} = & \frac{M^2}{4\pi} \frac{\mu_5^2 - eB}{2\mu_5^2 - eB} \frac{1}{\epsilon_0} \tanh \frac{\epsilon_0}{2T} - \frac{M^2}{8\pi} \sum_{s=\pm} \frac{\lambda_1 + s\mu_5}{\lambda_1 + 2s\mu_5} \frac{1}{\epsilon_1^s} \tanh \frac{\epsilon_1^s}{2T} \\ & + \frac{M^2}{16\pi} \sum_{s=\pm} \sum_{n=1}^{\infty} \left\{ \frac{2n(eB)^2 + eB\mu_5(2\mu_5 + s\lambda_n) - 8n\mu_5^2(\mu_5 + s\lambda_n)^2}{(eB)^2 - 4\mu_5^2(\mu_5 + s\lambda_n)^2} \frac{2}{\epsilon_n^s} \tanh \frac{\epsilon_n^s}{2T} \right. \\ & \left. - \frac{(\mu_5 + s\lambda_{n+1})[2(2n+1)\mu_5 + s\lambda_{n+1}]}{2\mu_5^2 + 2s\mu_5\lambda_{n+1} + eB} \frac{1}{\epsilon_{n+1}^s} \tanh \frac{\epsilon_{n+1}^s}{2T} - \frac{(\mu_5 + s\lambda_{n-1})[2(2n-1)\mu_5 - s\lambda_{n-1}]}{2\mu_5^2 + 2s\mu_5\lambda_{n-1} - eB} \frac{1}{\epsilon_{n-1}^s} \tanh \frac{\epsilon_{n-1}^s}{2T} \right\}. \end{aligned}$$

The properties of the eigenfunctions of harmonic oscillator are used

# Kosterlitz-Thouless transition



1. Weak B ( $0 < \sqrt{eB}/\mu_5 < 1/\sqrt{2}$ ), determined by  $\mu_5$ ;
2. Intermediate B ( $1/\sqrt{2} < \sqrt{eB}/\mu_5 < 2$ ), a valley shows up—de Hass-van Alphen Effect;
3. Strong B ( $\sqrt{eB}/\mu_5 > 2$ ), T and M increases linearly with B—lowest Landau level approximation;
4. The inverse magnetic catalysis effect shows up in the lower B.



# $\pi_0$ condensation in parallel EM field

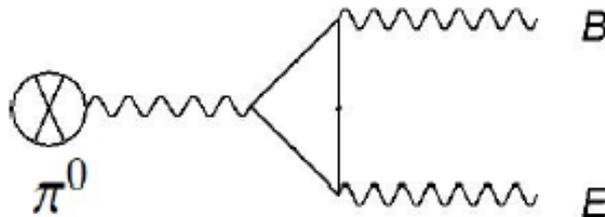
- (1)  $\pi_0$  condensate can inhomogeneously exist in the vacuum through chiral density wave:

$$\langle \bar{\psi}\psi \rangle = \Delta \cos(\mathbf{q} \cdot \mathbf{r}), \quad \langle \bar{\psi}i\gamma_5\tau_3\psi \rangle = \Delta \sin(\mathbf{q} \cdot \mathbf{r}),$$

**Can homogeneous  $\pi_0$  condensate exist?**

- (2)  $\pi_0$  is a neutral pseudo-scalar field and the second Lorentz invariant  $I_2 = \mathbf{B} \cdot \mathbf{E}$  is also a neutral pseudo-scalar.

In principal,  $\pi_0$  can be generated through triangle anomaly for nonzero  $I_2$ :





# NJL model calculation

Considering both  $\sigma$  and  $\pi_0$  condensation,  $\Omega$  becomes:

$$\Omega = \frac{(m - m_0)^2 + (\pi^0)^2}{4G} - \frac{1}{V_4} \text{Tr} \ln S^{-1},$$

Within Schwinger approach, the fermion propagators are modified:

$$\begin{aligned} S_f(x, x') = & \frac{-i}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-iq_f \int_{x'}^x A \cdot dx} \left[ -\frac{1}{2} \gamma(q_f F \coth(q_f F s) - q_f F)(x - x') + m - \text{sgn}(q_f) i \gamma^5 \pi^0 \right] \\ & \times \exp \left\{ -i [m^2 + (\pi^0)^2] s + \frac{i}{4} (x - x') q_f F \coth(q_f F s)(x - x') + \frac{i}{2} q_f \sigma F s \right\} \frac{-(q_f s)^2 I_2}{\text{Im} \cosh(i q_f s (I_1 + 2i I_2)^{1/2})}, \end{aligned}$$

The coupled gap equations

$$\begin{aligned} \frac{m - m_0}{2G} = & \frac{m N_c}{4\pi^2} \sum_{f=u,d} \int_0^\infty ds e^{-[m^2 + (\pi^0)^2] s} \frac{q_f^2 I_2 \text{Re} \cosh[q_f s \sqrt{I_1 + 2i I_2}]}{\text{Im} \cosh[q_f s \sqrt{I_1 + 2i I_2}]} - \frac{N_c}{4\pi^2} \frac{\pi^0}{m^2 + (\pi^0)^2} (q_u^2 - q_d^2) I_2, \\ \frac{\pi^0}{2G} = & \frac{\pi^0 N_c}{4\pi^2} \sum_{f=u,d} \int_0^\infty ds e^{-[m^2 + (\pi^0)^2] s} \frac{q_f^2 I_2 \text{Re} \cosh[q_f s \sqrt{I_1 + 2i I_2}]}{\text{Im} \cosh[q_f s \sqrt{I_1 + 2i I_2}]} + \frac{N_c}{4\pi^2} \frac{m}{m^2 + (\pi^0)^2} (q_u^2 - q_d^2) I_2. \end{aligned}$$



# NJL model calculation

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An universal relation for small  $I_2$ :

$$\pi_0 = \frac{N_c}{4\pi^2} \frac{2G}{m_0} (q_u^2 - q_d^2) I_2.$$

Gell-Mann—Oakes—Renner relation:  $m_\pi^2 f_\pi^2 = m_0 m^* (2G)^{-1}$

The final result for the whole range of  $I_2$  ( $m^* = \sqrt{(\pi^0)^2 + m^2}$ ):

$$\frac{\pi^0}{m^*} = \begin{cases} \frac{N_c}{4\pi^2 f_\pi^2 m_\pi^2} (q_u^2 - q_d^2) \mathbf{E} \cdot \mathbf{B} & \text{for } |I_2| < I_2^c, \\ \text{sgn}(I_2) & \text{for } |I_2| > I_2^c, \end{cases}$$

$I_2^c$  is the point where chiral rotation ends.

# Chiral perturbation theory calculation



The Lagrangian with Wess-Zumino-Witten term  $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{WZW}}$ ,

$$\mathcal{L}_0 = \frac{f_\pi^2}{4} \text{tr} \left[ D_\mu U^\dagger D^\mu U + m_\pi^2 (U + U^\dagger) \right] \quad \mathcal{L}_{\text{WZW}} = \frac{N_c}{48\pi^2} A_\mu \epsilon^{\mu\nu\alpha\beta} [\text{tr} (Q L_\nu L_\alpha L_\beta + Q R_\nu R_\alpha R_\beta) - i F_{\alpha\beta} T_\nu]$$

Choosing Weinberg parameterization  $U = \frac{1}{f_\pi} (s + i\tau \cdot t)$  and setting

$$s = f_\pi \cos \phi, t_3 = f_\pi \sin \phi, t_1 = t_2 = 0,$$

the thermodynamic potential becomes

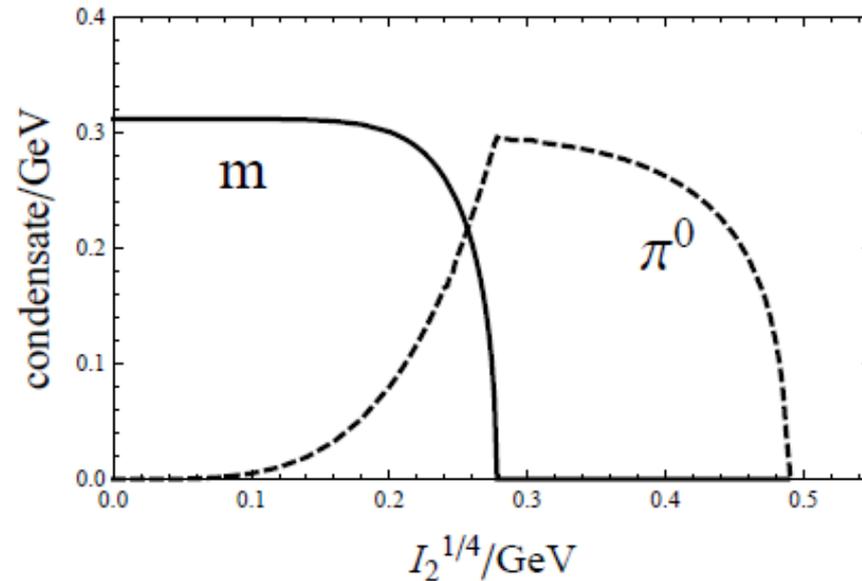
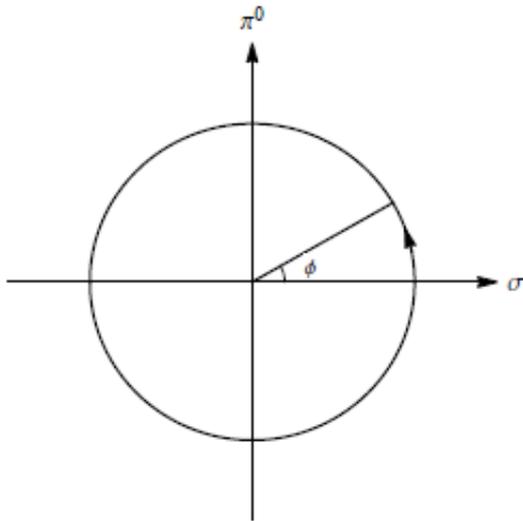
$$-\Omega(\phi) = f_\pi^2 m_\pi^2 \cos \phi + \frac{N_c I_2}{4\pi^2} \text{tr}(Q^2 \tau_3) \phi$$

The minimal point gives the same result as in NJL model.

Consistent with the linear sigma model in study of DCC

[M. Asakawa et. al., Phys. Rev. D 58, 094011 \(1998\).](#)

# Chiral rotation in parallel EM field



- (1)  $\pi_0$  condensate initially increases with  $I_2$  and the decreasing feature is due to chiral restoration;
- (2) During the chiral rotation,  $m^*$  is almost a constant;
- (3) A jump in chiral limit.



# The robustness of $\pi_0$ condensation

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## (1) Invalidation of Vafa-Witten Theorem:

$$D = \gamma_\mu(\partial_\mu - ig\mathcal{A}_\mu) + Q\gamma_4 A_0 + iQ\gamma_i A_i + M,$$

The static electric field is given by  $A_0$  which acts like  $\mu$  and the positivity of  $\text{Det } D$  is lost.

## (2) The stability related to Schwinger mechanism (mainly $E \rightarrow \pi^+ + \pi^-$ ):

The QCD vacuum is like insulator and QED system lies on top of it.

Stable QED system  $|eE| \ll m_\pi^2 + |eB|, \rightarrow m_{\pi^\pm}^2$

# Conclusions

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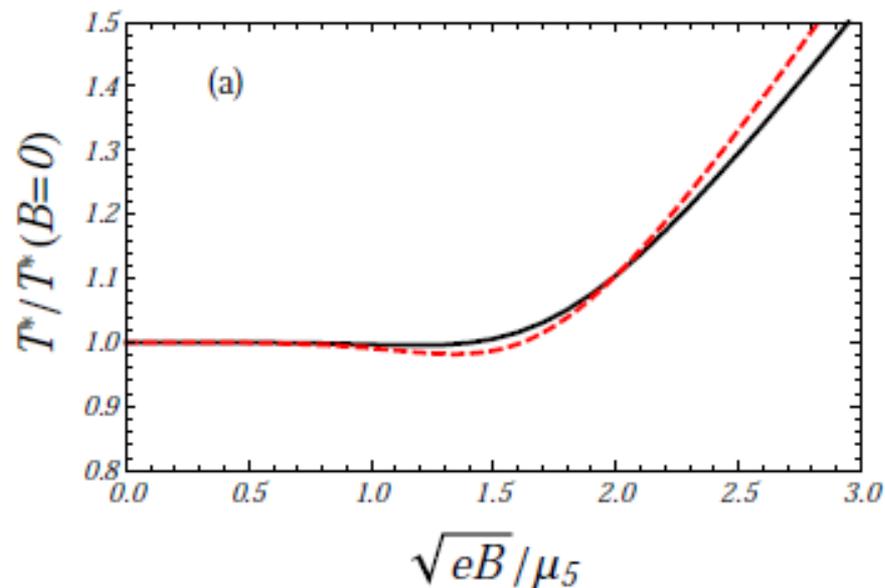


- Kosterlitz–Thouless transition and chiral rotation are studied in NJL model with Ritus and Schwinger approach, respectively;
- When  $\mu_5$  is introduced, inverse magnetic catalysis will show up in the KT transition temperature.
- Neutral pion condensation is found in parallel EM field due to triangle anomaly and is robust to Vafa–Witten theorem and Schwinger mechanism.



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*Thank you very much!*



$T^*$  — the temperature at which  $M$  vanishes  
show a similar behavior.