

# A kinetic regime of hydrodynamic fluctuations and long time tails for a Bjorken expansion

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QCD in Finite Temperature and Heavy-Ion Collisions  
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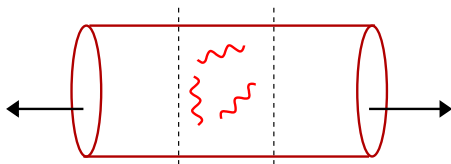
## Hydrodynamics with noise

- ▶ Initial state fluctuations
- ▶ Thermal fluctuations – for example Landau-Lifshitz

$$N_{gg}(t, k) \equiv \underbrace{\langle g^i(t, k) g^{j*}(t, k) \rangle}_{\text{momentum, } g^i \equiv T^{0i}} = \underbrace{(e + p) T \delta^{ij}}_{\text{equilibrium}}$$

1. Conceptually important (required by the FDT)
2. Larger in smaller systems:  $N_{\text{particle}} \sim 10000$  in the heavy-ions
3. Essential near a critical point

How do thermal fluctuations evolve during a Bjorken expansion?  
How do thermal fluctuations change the Bjorken expansion?



## Kinetic regime of hydrodynamic fluctuations – a new scale $k_*$

1. For hydrodynamic fluctuations with wavenumber  $k$ :

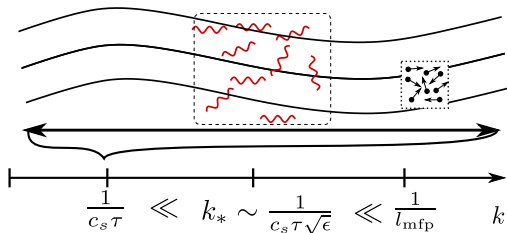
- ▶ Equilibration rate  $\sim \gamma_\eta k^2$  ( $\gamma_\eta \equiv \eta/(e+p)$ )
- ▶ Expansion rate  $\sim 1/\tau$  for a Bjorken expansion

2. Compete at a critical scale:

$$k_* \sim \frac{1}{\sqrt{\gamma_\eta \tau}}$$

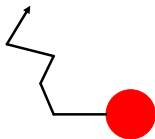
3. Derivative expansion controlled by  $\epsilon \equiv \gamma_\eta/\tau \ll 1$

$$\frac{1}{c_s \tau} \ll \underbrace{k_* \sim \frac{1}{c_s \tau \sqrt{\epsilon}}}_{k_* \text{ is hard !}}$$



We derive an effective description for the kinetic regime  $k_*$

## Hydro-kinetic equation: an analogy with Brownian motion



### 1. Langevin equation

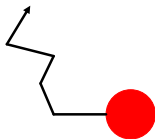
$$\frac{dp}{dt} = \underbrace{-\gamma p}_{\text{drag}} + \underbrace{\xi}_{\text{noise}}, \quad \langle \xi(t)\xi(t') \rangle = 2TM\gamma\delta(t-t')$$

### 2. Calculate how $\langle p^2(t) \rangle$ evolves through Langevin process

$$\frac{d}{dt}\langle p^2 \rangle = \underbrace{-2\gamma [\langle p^2 \rangle - MT]}_{\text{equilibration}}$$

Follow the same steps for hydrodynamics with external forcing

## Hydro-kinetic equation: an analogy with Brownian motion



### 1. Langevin equation

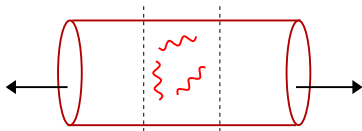
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## Hydro-Langevin equation



1. Linearized analysis in Bjorken:  $e = e_0 + \delta e$ ,  $\vec{g} = (e_0 + p_0)\vec{v}$

$$\phi_a(t, \vec{k}) \equiv (c_s \delta e, \vec{g})$$

2. Hydro-Langevin equation for  $\phi_a(t, \vec{k})$

$$-\dot{\phi}(t, \vec{k}) = \underbrace{i\mathcal{L}\phi}_{\text{ideal}} + \underbrace{\mathcal{D}\phi + \xi}_{\text{viscous}} + \underbrace{\mathcal{P}(t)\phi}_{\text{expansion}}$$

3. Four eigenmodes of  $\mathcal{L}$ :  $\phi_+$ ,  $\phi_-$ ,  $\phi_{T_1}$ ,  $\phi_{T_2}$

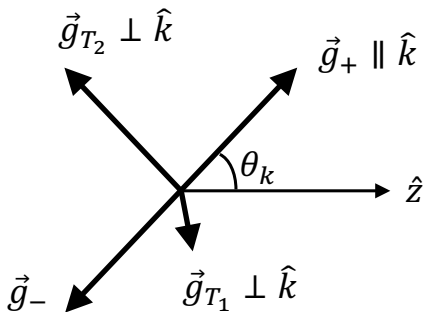
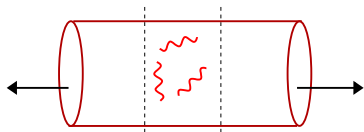
$$\underbrace{\text{left moving sound}}_{\lambda_- = -c_s k}$$

$$\underbrace{\text{right moving sound}}_{\lambda_+ = c_s k}$$

$$\underbrace{\text{transverse modes}}_{\lambda_T = 0}$$

The Hydro-Langevin equation in eigen basis is similar to Brownian motion

## Eigenbasis in the Bjorken expansion



## Hydro-kinetic equation

1. Analyze the square of the eigenmodes – for example

$$N_{++}(t, \vec{k}) \equiv \langle \phi_+(t, \vec{k}) \phi_+^*(t, \vec{k}) \rangle$$

2. Hydro-kinetic equations for  $N_{++}$  and  $N_{T_2 T_2}$

$$\dot{N}_{++} = -\frac{4}{3} \gamma_\eta k^2 [N_{++} - T_0(e_0 + p_0)] - \frac{1}{\tau} [2 + c_s^2 + \cos^2 \theta_k] N_{++}$$

$$\dot{N}_{T_2 T_2} = \underbrace{-2\gamma_\eta k^2 [N_{T_2 T_2} - T_0(e_0 + p_0)]}_{\text{equilibration}} - \underbrace{\frac{2}{\tau} [1 + \sin^2 \theta_k] N_{T_2 T_2}}_{\text{external forcing}}$$

3. Neglect off-diagonal components of density matrix

$$N_{+-} \sim \underbrace{e^{-i(\lambda_+ - \lambda_-)t}}_{\text{rotating wave approx}} \sim 0$$

No mixing between  $T_1$  and  $T_2$  because of rotational symmetry in  $xy$  plane

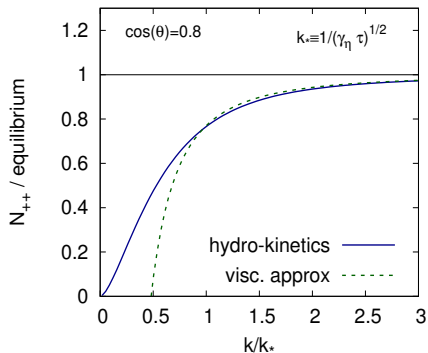
Hydro fluctuations are driven out of equilibrium at  $k_*$



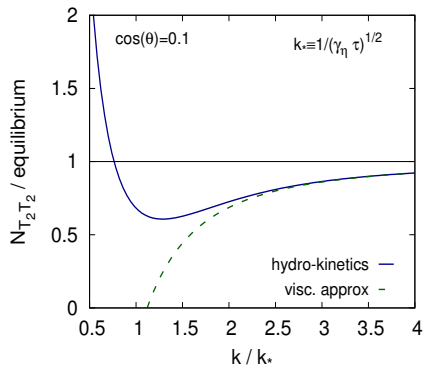
# Solution of the hydro-kinetic equation

## Bjorken expansion at late times

### Sound Modes



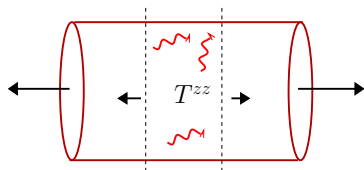
### Transverse modes



$k_*$  is the critical scale: For larger  $k \gg k_*$ , closer to equilibrium

$$N_{AA} \sim N_{\text{eq}} \left[ 1 + \frac{\#}{\gamma_\eta \tau k^2} + \dots \right]$$

## Evolution of the background



Hydrodynamic equation for a Bjorken expansion:

$$\frac{d}{d\tau} (\tau T^{\tau\tau}) = -T^{zz}$$

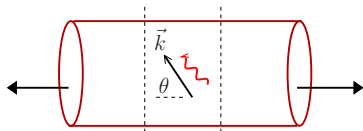
- ▶ Without hydrodynamic fluctuations:

$$T^{zz} = \underbrace{p_0}_{\text{ideal}} - \underbrace{\frac{4\eta_0}{3\tau}}_{\text{1st order}} + \underbrace{(\lambda_1 - \eta\tau\pi)}_{\text{2nd order}} \frac{8}{9\tau^2}$$

- ▶ Hydrodynamic fluctuations give another contributions

$$T_{\text{fluct}}^{zz} = (e_0 + p_0) \langle u^z u^z \rangle = \frac{\langle g^z g^z \rangle}{e_0 + p_0}$$

## Nonlinear contributions from $k_*$ to the background



### 1. Compute the contribution from fluctuation

$$\langle (g^z(t, \vec{x}))^2 \rangle = \int^\Lambda d^3k \underbrace{[N_{++} \cos^2 \theta_k + N_{T_2 T_2} \sin^2 \theta_k]}_{\sim 1 + \# / (\gamma_\eta \tau k^2) + \dots \text{ for } k \gg k_*}$$

- ▶ Regularize cubic and linear UV divergences by a cutoff  $\Lambda$

### 2. Renormalize the divergences c.f. Kovtun-Moore-Romatschke (11)

$$T^{zz} = \underbrace{p_0(\Lambda)}_{\equiv p_{\text{phys}}} + \frac{\Lambda^3 T}{6\pi^2} - \frac{4}{3\tau} \underbrace{\left[ \eta_0(\Lambda) + \frac{17\Lambda T}{120\pi^2} \frac{e_0 + p_0}{\eta_0} \right]}_{\equiv \eta_{\text{phys}}} + \text{finite}$$

The cutoff dependence is absorbed by renormalization of  $p_0$  and  $\eta_0$

## Finite contributions: Long-time tails

Evaluate the finite parts after renormalization

$$T^{zz} = \underbrace{p}_{\text{ideal}} - \underbrace{\frac{4\eta}{3\tau}}_{\text{1st order}} + \underbrace{1.08318 T \left( \frac{1}{4\pi\gamma_\eta\tau} \right)^{3/2}}_{\text{long-time tail}} + \dots$$

Simple understanding of the scaling

$$T_{\text{fluct}}^{zz} \sim \underbrace{\frac{1}{2}k_B T}_{\text{equipart}} \underbrace{\int d^3k}_{\text{\# of modes}} \sim T k_*^3 \sim T \left( \frac{1}{\gamma_\eta\tau} \right)^{3/2}$$

The finite contribution from  $k_*$  gives the long-time tails

## Implications for heavy-ion collisions

### Plugging in typical values

$$\frac{T^3}{s} \simeq \frac{1}{13.5}, \quad \frac{\lambda_1 - \eta\tau_\pi}{e+p} \simeq -0.8 \left( \frac{\eta}{e+p} \right)^2$$

### Compute $4\langle T^{zz} \rangle / (e+p)$

$$\frac{\eta}{s} = \frac{1}{4\pi} : \quad 1 - 0.092 \left( \frac{4.5}{\tau T} \right) + 0.034 \left( \frac{4.5}{\tau T} \right)^{3/2} - 0.00085 \left( \frac{4.5}{\tau T} \right)^2$$
$$\frac{\eta}{s} = \frac{2}{4\pi} : \quad \underbrace{1}_{\text{ideal}} - \underbrace{0.185 \left( \frac{4.5}{\tau T} \right)}_{\text{1st order}} + \underbrace{0.013 \left( \frac{4.5}{\tau T} \right)^{3/2}}_{\text{1.5th order}} - \underbrace{0.0034 \left( \frac{4.5}{\tau T} \right)^2}_{\text{2nd order}}$$

Thermal fluctuation is practically larger than 2nd order viscous correction

## A few remarks on transverse pressure

There is an extra contribution from initial-time fluctuations:

$$T^{xx} = \underbrace{p}_{\text{ideal}} + \underbrace{\frac{2\eta}{3\tau}}_{\text{1st order}} - \underbrace{0.273836 T \left( \frac{1}{4\pi\gamma_\eta\tau} \right)^{3/2}}_{\text{long-time tail}} + \underbrace{\left[ \frac{\chi_{\tau_0}^{gg} + \delta\chi_{\tau_0}^{gg}}{\tau^2(e+p)^2} \right]}_{\text{initial time contributions from } T_1} \frac{1}{12\pi\gamma_\eta\tau}$$

Origin of the initial time contributions:

- ▶ Initial fluctuations with a long-range rapidity correlation ( $\chi_{\tau_0}^{gg}$ )
- ▶ Initial-time thermal fluctuations diffused in rapidity ( $\delta\chi_{\tau_0}^{gg}$ )

$$\Delta\eta \sim \sqrt{\frac{6\gamma_\eta(\tau_0)}{\tau_0}}$$

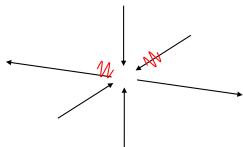
Initial and initial-time thermal fluctuations of  $T_1$  mode contribute to  $T^{xx}$

# Comparison to previous diagrammatic calculation in static systems

c.f. Kovtun-Moore-Romatschke (11), Kovtun-Yaffe (03)

1. Apply a shear perturbation:

$$h_{xy}(t) = h_{xy}e^{-i\omega t}$$



2. Compute how  $N_{++}$  evolves away from equilibrium:

$$\dot{N}_{++} = -\gamma_{\eta}k^2[N_{++} - N_{\text{eq}}] + \dot{h}(t)N_{++}$$

3. Compute fluctuation contribution to stress in the linear order of  $h$ :

$$T^{xy} = p_0 h^{xy} + \eta_0 \dot{h}^{xy} + \frac{\langle g^x g^y \rangle}{e_0 + p_0}$$

4. Separate divergent and finite contributions in  $\langle g^x g^y \rangle$ :

$$\langle g^x g^y \rangle \sim \int_k \frac{i\omega}{-i\omega + \gamma_{\eta}k^2}$$

Follow the same procedure with the Bjorken expansion

## Result for shear response function

### Linear response of $T^{xy}(\omega)$ to $h_{xy}(\omega)$

c.f. Kovtun-Moore-Romatschke (11), Kovtun-Yaffe (03)

$$G_R(\omega) \equiv \frac{T^{xy}(\omega)}{h^{xy}(\omega)} = \underbrace{\left( p_0 + \frac{\Lambda^3 T}{6\pi^2} \right)}_{\equiv p_{\text{phys}}} - i \underbrace{\left( \eta_0 + \frac{17\Lambda T}{120\pi^2 \gamma_\eta} \right)}_{\equiv \eta_{\text{phys}}} \omega + \underbrace{(1+i) \frac{\left(\frac{3}{2}\right)^{3/2} + 7}{240\pi} T \left(\frac{\omega}{\gamma_\eta}\right)^{3/2}}_{\text{long-time tail}}$$

### Comments:

- ▶ Diagrams: Computation in a flat space
- ▶ Kinetic: Computation in a curved space

The hydro-kinetic theory reproduced the previous result with diagrams



## Summary & Outlook

- ▶ Hydro-kinetic equation for  $k_*$ , advantageous in expanding systems
- ▶ Universal renormalization of pressure  $p_0(\Lambda)$  and viscosity  $\eta_0(\Lambda)$
- ▶ Background-dependent long-time tails  $\propto \tau^{-3/2}, \omega^{3/2}$
- ▶ **Alternative way to solve the hydrodynamics with noise**

$$d_\mu T^{\mu\nu} = 0, \quad \dot{N}_{AA} = \dots, \quad T^{\mu\nu} = T_{\text{bkg}}^{\mu\nu} + \int_k N_{AA},$$

- ▶ **Bulk viscosity renormalization for nonconformal fluid**

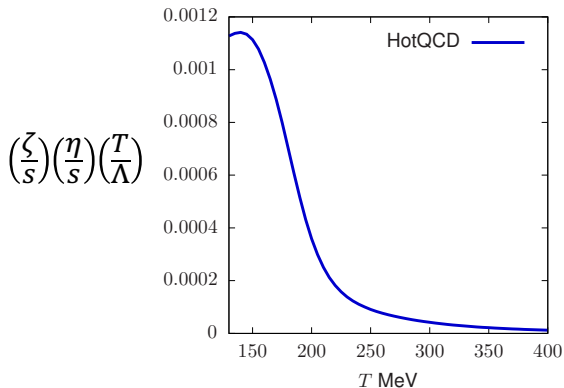
YA-Mazeliauskas-Teaney, in preparation

$$\zeta = \zeta_0(\Lambda) + \frac{\Lambda T^2}{2\pi^2} \left[ \left( \frac{1}{3} + \frac{T}{2} \frac{dc_s^2}{dT} - c_s^2 \right)^2 \frac{s}{\frac{4}{3}\eta} + \left( \frac{1}{3} - c_s^2 \right)^2 \frac{2s}{\eta} \right]$$

- ▶ **Application to critical dynamics** YA-Teaney-Yan-Yin, in progress

## Bulk viscosity from thermal fluctuations

- ▶ Use lattice EoS by Hot QCD Collaboration
- ▶ For  $\zeta_0(\Lambda) = 0$  case



Bulk viscosity from thermal fluctuations enhances near  $T_c^*$

*Thank you for your attention!*