Hydrodynamics Beyond the Gradient Expansion: Resurgence and Resummation

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(see also **1302.0697 [hep-th]** and **1409.5087 [hep-th]**)

Introduction

Relativistic viscous hydrodynamics…

… has been playing a very prominent role in the last 15 years. In particular:

- it describes the quark-gluon plasma in heavy ion collisions at RHIC and LHC
- it appears in the AdS/CFT as a concrete realization of the membrane paradigm

These highly successful applications of relativistic viscous hydrodynamics make it easy to forget that it is a theory that deserves understanding on its own.

This is precisely the topic of my talk.

Relativistic hydrodynamics as an EFT

hydrodynamics is an EFT of the slow evolution of conserved currents in collective media close to equilibrium

As any EFT it is based on the idea of the gradient expansion

DOFs: always local energy density ϵ and local flow velocity u^μ $\quad (u_\nu u^\nu = -1)$

EOMs: conservation eqns $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$ for $\langle T^{\mu\nu} \rangle$ systematically expanded in gradients

terms carrying 2 and more gradients

$$\langle T^{\mu\nu} \rangle = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\} - \eta(\epsilon)\, \sigma^{\mu\nu} - \zeta(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}(\nabla \cdot u) + \ldots$$

microscopic input:

EoS $\left(P(\epsilon) = \frac{1}{3}\epsilon \text{ for CFTs} \right)$

shear viscosity

bulk viscosity (vanishes for CFTs)

Dissipation: $\nabla_\mu \left\{ \frac{\epsilon + P(\epsilon)}{T} \cdot u^\mu + \ldots \right\} = \frac{\eta}{2T} \sigma_{\alpha\beta} \sigma^{\alpha\beta} + \frac{\xi}{T} (\nabla \cdot u)^2 + \ldots \geq 0$

Hydrodynamics and AdS/CFT

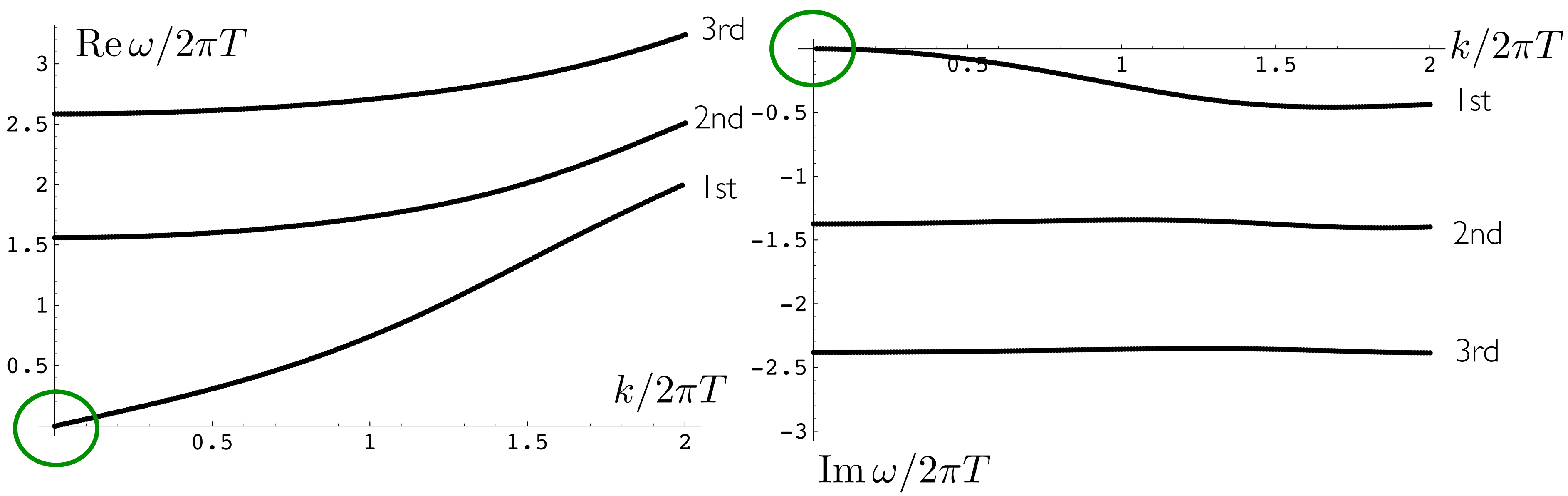
see, e.g. Kovtun & Starinets [hep-th/0506184]

Consider small amplitude perturbations ($\delta T_{\mu\nu}/{N_c}^2 \ll T^4$) on top of a holographic plasma

$$T_{\mu\nu} = \frac{1}{8}\pi^2 N_c^2 T^4 \,\mathrm{diag}\,(3,1,1,1)_{\mu\nu} + \delta T_{\mu\nu} \quad (\sim e^{-i\,\omega(k)\,t + i\,\vec{k}\cdot\vec{x}})$$

Dissipation leads to modes with complex $\omega(k)$, which look like

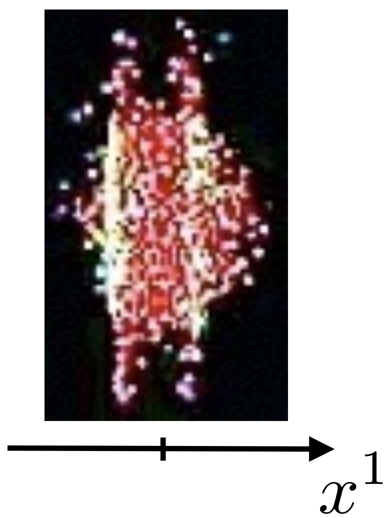
$$\left.\frac{\partial\omega}{\partial k}\right|_{k\to 0} = c_{sound}$$



$\omega(k) \to 0$ as $k \to 0$: slowly dissipating modes (hydrodynamic sound waves)

all the rest: far from equilibrium (QNM) modes damped over $t_{therm} = \mathcal{O}(1)/T$

Hydrodynamics and QGP [Bjorken 1982]

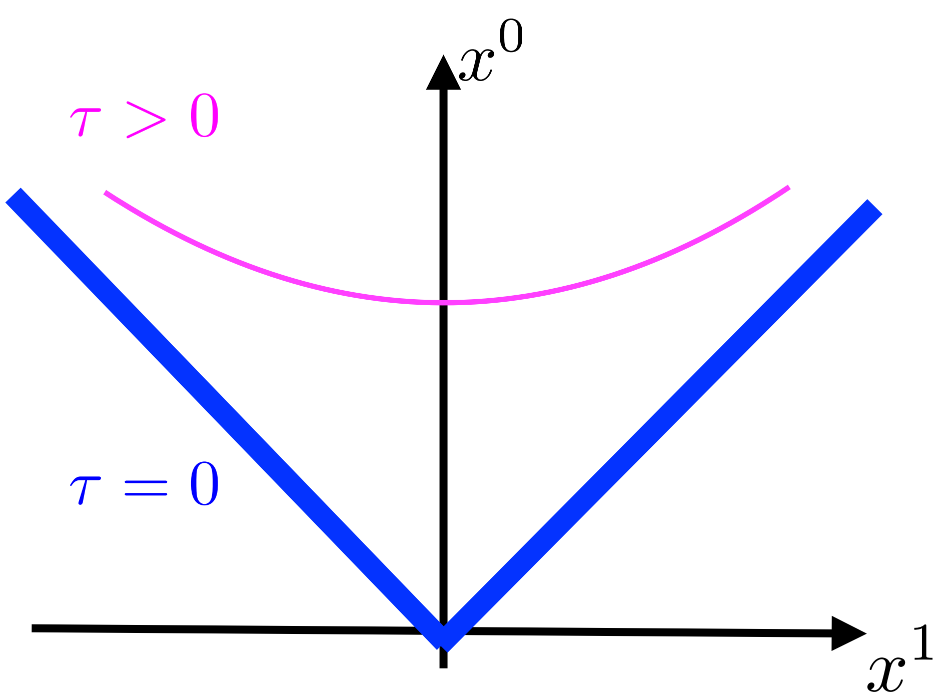


The "spherical cow" of heavy ion collision is the **boost-invariant flow** with **no transverse expansion.**

In Bjorken scenario dynamics depends only on proper time $\tau = \sqrt{(x^0)^2 - (x^1)^2}$

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_2^2 + dx_3^2$$

and stress tensor (for a CFT) is entirely expressed in terms of local energy density



$$\langle T^\mu_{\ \nu} \rangle = \mathrm{diag}\{-\epsilon(\tau), \frac{\epsilon(\tau)}{3} - \phi(\tau), \frac{\epsilon(\tau)}{3} + \frac{\phi(\tau)}{2}, \frac{\epsilon(\tau)}{3} + \frac{\phi(\tau)}{2}\}^\mu_{\ \nu}$$

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0 \longrightarrow \tau\,\dot{\epsilon}(\tau) = -\frac{4}{3}\epsilon(\tau) + \phi(\tau)$$

Hydrodynamic gradient expansion is divergent

Hydrodynamic gradient expansion: $\epsilon(\tau)$ and $\phi(\tau)$ are expressed as series in

$$\frac{1}{w} \equiv \frac{1}{\tau} \cdot \frac{1}{T(\tau)}$$

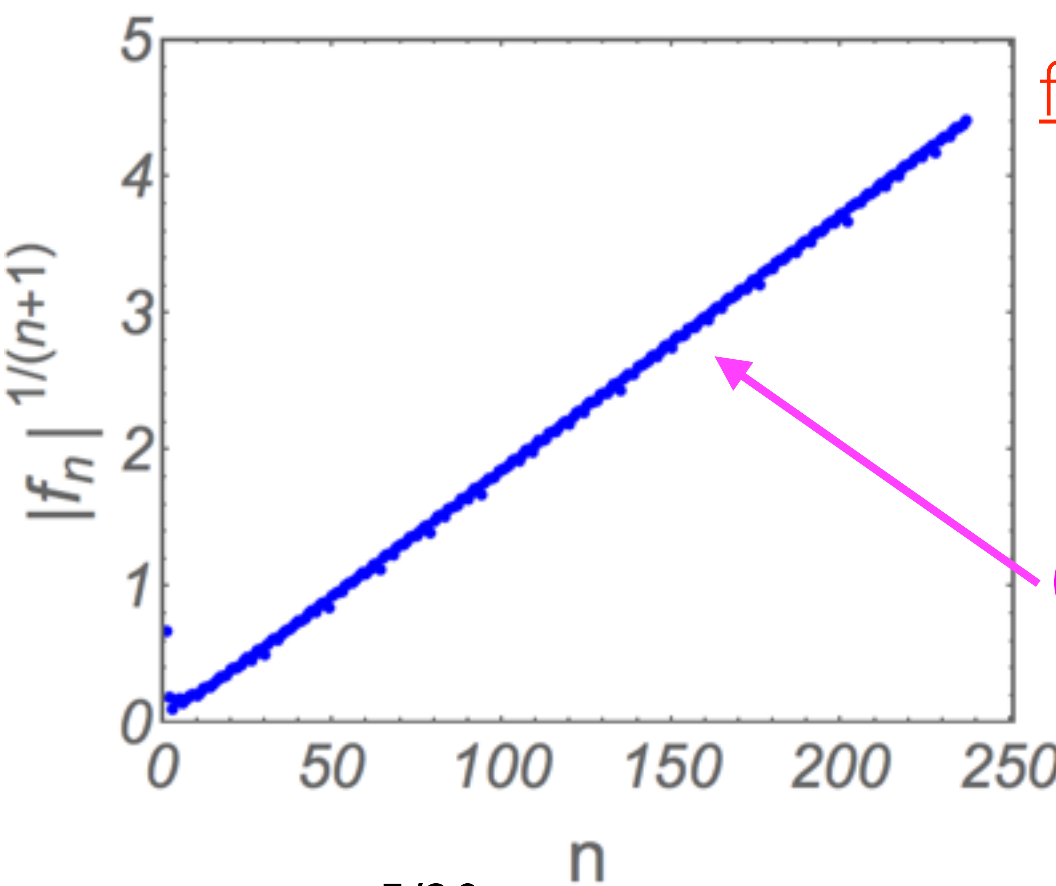
at strong coupling the temperature sets the microscopic scale: $\frac{\eta}{\epsilon} \sim \frac{1}{T}$

size of the gradient based on dimensional analysis:

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_2^2 + dx_3^2$$

In 1302.0697 [hep-th] we computed $f\left(\tau\, T(\tau)\right) \equiv \frac{2}{3} + \frac{1}{4}\frac{\phi(\tau)}{\epsilon(\tau)}$ up to $O(w^{-240})$:

$$f(w) = \sum_{n=0}^{\infty} f_n w^{-n} = \frac{2}{3} + \frac{1}{9\pi} w^{-1} + \ldots$$

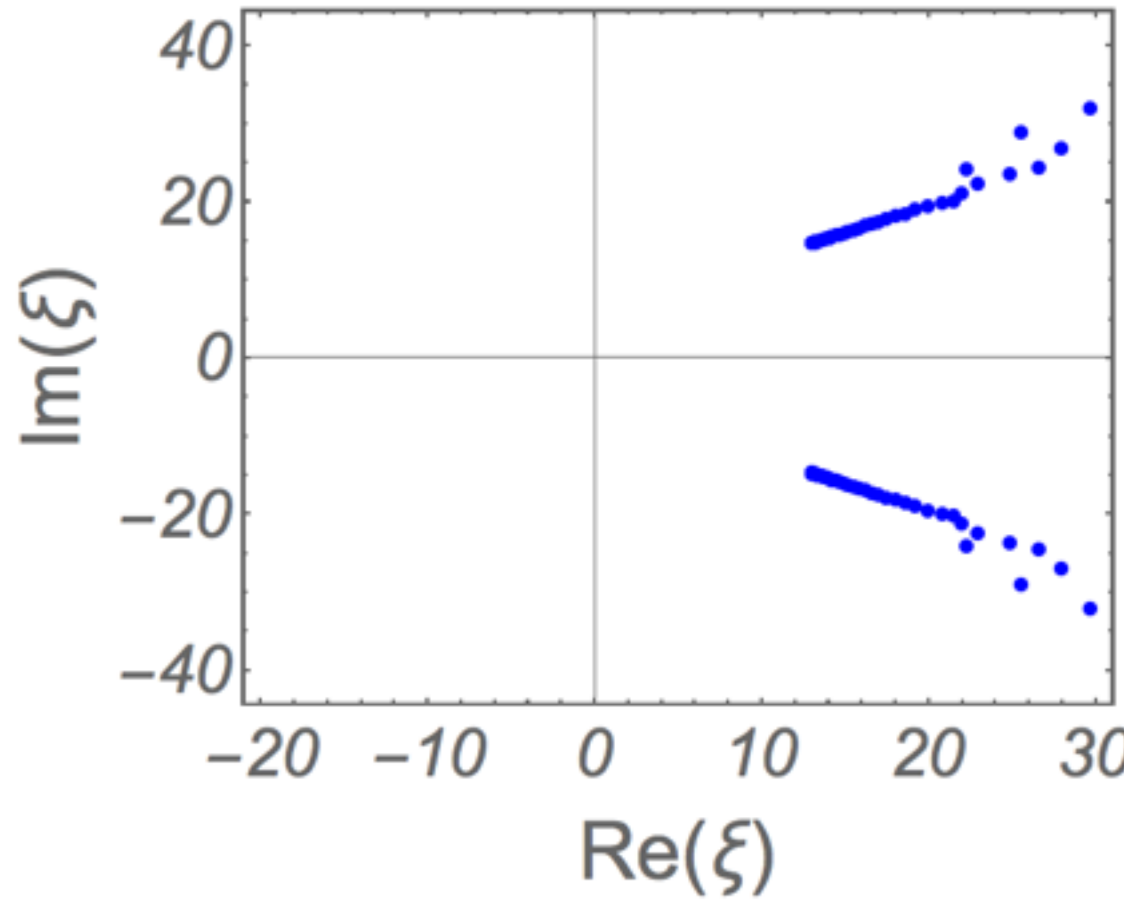


for N=4 SYM

$$(n!)^{1/(n+1)}\Big|_{n\to\infty} \approx \frac{1}{e} \cdot n$$

Hydrodynamics and QNMs

Analytic continuation (using symmetric Pade approximant) of $f_B(\xi) \approx \sum_{n=0}^{240} \frac{1}{n!} f_n \, \xi^n$ revealed the following analytic structure of $f_B(\xi)$:



for N=4 SYM

Branch cut singularities start at $\frac{3}{2} \, i \, \omega_{QNM_1}$!

Question behind this work

How to make sense of the divergent hydrodynamic gradient expansion?

Evolution equations for relativistic viscous fluids

Evolution equations for relativistic viscous fluids

Naive viscous hydrodynamics: $\nabla_\mu \{ \epsilon\, u^\mu u^\nu + P(\epsilon) \{ g^{\mu\nu} + u^\mu u^\nu \} - \eta(\epsilon)\, \sigma^{\mu\nu} \} = 0$

leads to the diffusion equation for $\delta u_z(t,x)$ perturbation on top of $T = \text{const}$ and $u^t = 1$

$$\partial_t\, \delta u_z - D\, \partial_x^2\, \delta u_z = 0 \text{ with } D = \frac{\eta}{T\, s}$$

This equation does not have a well-posed initial value problem in boosted frames.

One known remedy: promote $\Pi^{\mu\nu} = \langle T^{\mu\nu} \rangle - (\epsilon\, u^\mu u^\nu + P(\epsilon)\{g^{\mu\nu} + u^\mu u^\nu\})$

to an independent dynamical field. The prototypical example is:

$$(\tau_\Pi \mathcal{D} + 1)\, \Pi^{\mu\nu} = -\eta \sigma^{\mu\nu}$$

Instead of the diffusion equation, one now gets the Maxwell-Cattaneo equation

$$\partial_t^2 \delta u_z - \frac{\eta}{s} \frac{1}{\tau_\Pi T} \partial_x^2 \delta u_z + \frac{1}{\tau_\Pi} \partial_t \delta u_z = 0 : \text{okay* as long as } \tau_\Pi T \geq \frac{\eta}{s}$$

Formulation of hydrodynamics

Equation $\partial_t^2 \delta u_z - \frac{\eta}{s}\frac{1}{\tau_\Pi T}\partial_x^2 \delta u_z + \frac{1}{\tau_\Pi}\partial_t \delta u_z = 0$ derived from $\left(\tau_\Pi \mathcal{D} + 1\right)\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$ has two modes as solutions:

$$\omega = -i\frac{\eta}{sT}k^2 + \ldots \quad \text{and} \quad \omega = -i\frac{1}{\tau_\Pi} + i\frac{\eta}{sT}k^2 + \ldots$$

hydrodynamics — purely imaginary "quasinormal mode"

On top of this, $\left(\tau_\Pi \mathcal{D} + 1\right)\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$ leads to an infinite gradient expansion.

Conclusion: theories like this are certain "UV-completion" of hydrodynamic modes, much like the full *N*=4 SYM is a "UV-completion" of its hydrodynamic sector.

As an aside: 1409.5087 [hep-th] showed that $\left((\frac{1}{T}\mathcal{D})^2 + 2\omega_I \frac{1}{T}\mathcal{D} + |\omega|^2 \right)\Pi^{\mu\nu} = -\,\eta|\omega|^2\sigma^{\mu\nu}$ is a structure coupling hydrodynamics to the lowest quasinormal mode.

In the following we will address questions about the hydrodynamic gradient expansion raised in *N*=4 SYM within the framework of $(\tau_\Pi \mathcal{D} + 1)\, \Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$.

To make contact with the literature, I will consider a slightly more general theory:

0712.2451 [hep-th]: $(\tau_\Pi \mathcal{D} + 1)\, \Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \dfrac{\lambda_1}{\eta^2} \Pi^{\langle\mu}{}_{\alpha} \Pi^{\nu\rangle\alpha}$

The attractor in

$$(\tau_\Pi \mathcal{D} + 1)\, \Pi^{\mu\nu} = -\eta \sigma^{\mu\nu} + \frac{\lambda_1}{\eta^2} \Pi^{\langle\mu}{}_{\alpha} \Pi^{\nu\rangle\alpha}$$

Boost-invariant generalized hydrodynamics

For the boost-invariant flow $\Pi^{\mu\nu}$ has only one independent component: $\Pi^{y}{}_{y} = -\phi(\tau)$

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0 \longrightarrow \tau \dot{\epsilon} = -\frac{4}{3}\epsilon + \phi$$

$$(\tau_\Pi \mathcal{D} + 1)\,\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \frac{\lambda_1}{\eta^2}\Pi^{\langle\mu}{}_\alpha \Pi^{\nu\rangle\alpha} \longrightarrow \tau_\Pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{\lambda_1 \phi^2}{2\eta^2} - \frac{4\tau_\Pi \phi}{3\tau} - \phi$$

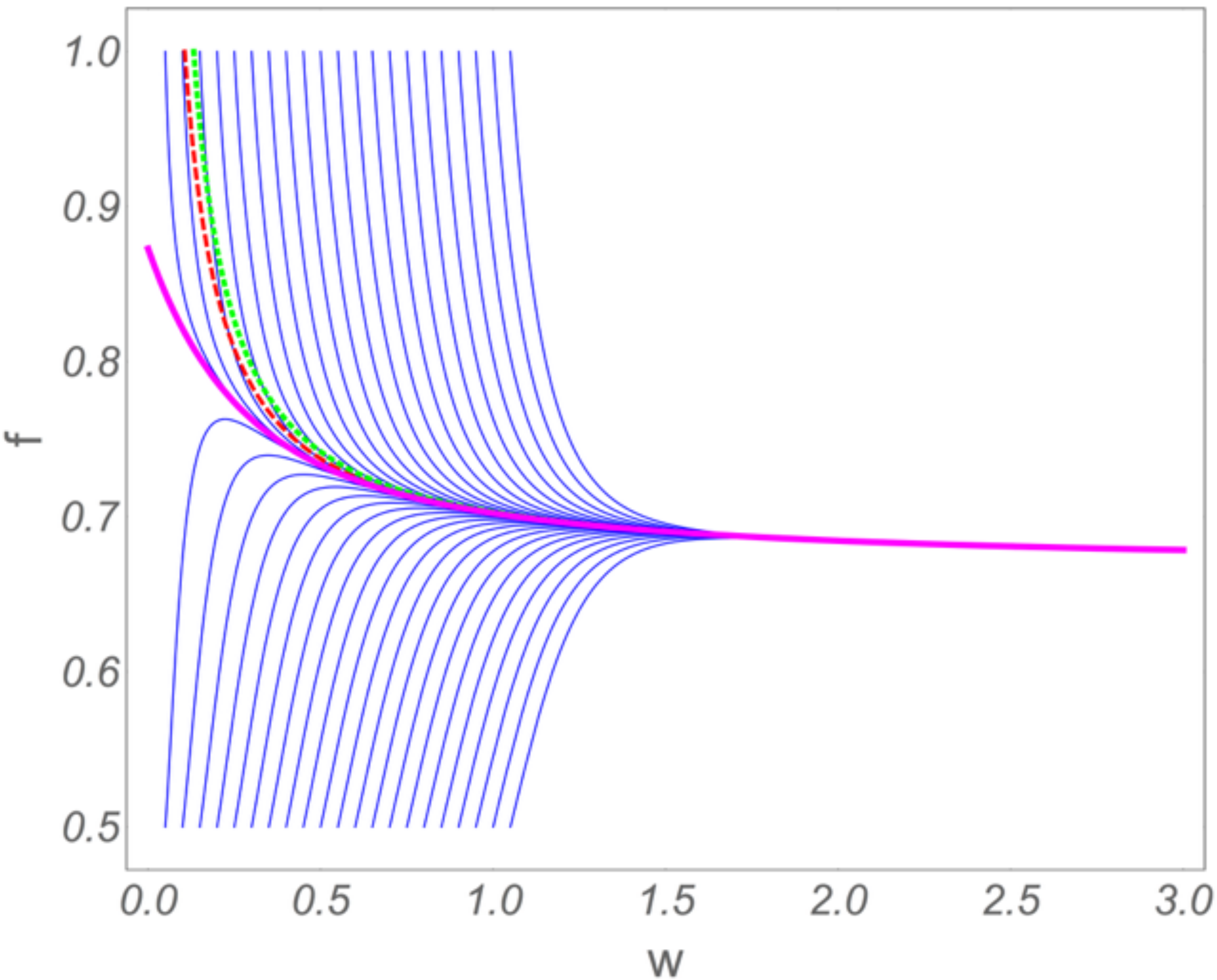
In a CFT $\tau_\Pi = \frac{C_{\tau_\Pi}}{T}$, $\lambda_1 = C_{\lambda_1}\frac{\eta}{T}$, $\eta = C_\eta\, s$ and set *N*=4 SYM gradient expansion values.

Introducing the scale invariant combination $w = \tau\, T(\tau)$ one equivalently* gets

$$\frac{1}{w}\tau\,\dot{w} = f(w) \quad \left(= \frac{2}{3} + \frac{1}{4}\frac{\phi(\tau)}{\epsilon(\tau)} \right)$$

$$C_{\tau_\Pi}\left(f\, f' + \frac{4f^2}{w} - \frac{16f}{3w} + \frac{16}{9w} \right) + C_{\lambda_1}\left(\frac{3f^2}{2C_\eta} - \frac{2f}{C\eta} + \frac{2}{3C_\eta} \right) + f - \frac{2}{3} - \frac{4C_\eta}{9w} = 0$$

The attractor



attractor

different solutions

$f(w) = f_0 + f_1 w^{-1}$

$f(w) = f_0 + f_1 w^{-1} + f_2 w^{-2}$

early time

late time

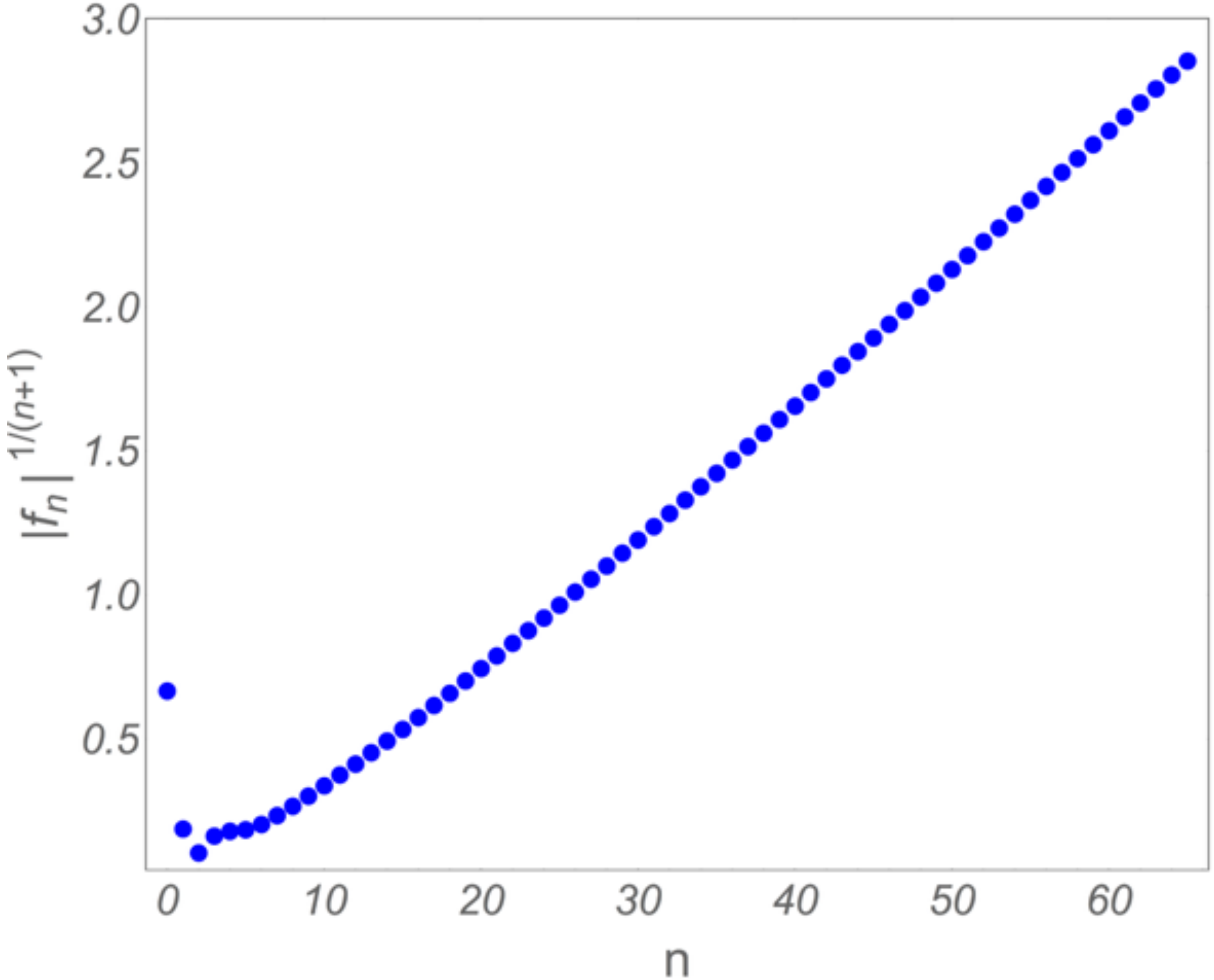
Beyond the gradient expansion in

$$(\tau_\Pi \mathcal{D} + 1)\,\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \frac{\lambda_1}{\eta^2}\Pi^{\langle\mu}{}_{\alpha}\Pi^{\nu\rangle\alpha}$$

Divergent gradient expansion

$$f = \sum_{n=0}^{\infty} f_n w^{-n} = \frac{2}{3} + \frac{4\,C_\eta}{9} w^{-1} + \frac{8\,C_\eta (C_{\tau_\Pi} - C_{\lambda_1})}{27} w^{-2} + \ldots =$$

$$= \frac{2}{3} + \frac{1}{9\pi} w^{-1} + \frac{1 - \log 2}{27\pi^2} w^{-2} + \ldots$$



Quasinormal mode

Let's now consider small perturbations on top of hydrodynamics

$$f = \frac{2}{3} + \frac{4\,C_\eta}{9} w^{-1} + \frac{8\,C_\eta (C_{\tau_\Pi} - C_{\lambda_1})}{27} w^{-2} + \ldots + \boxed{\delta f}$$

The solution takes the form

$$\delta f \sim \boxed{\exp\left(-\frac{3}{2C_{\tau_\Pi}} w\right)} w^{\frac{C_\eta - 2C_{\lambda_1}}{C_{\tau_\Pi}}} \boxed{\left\{1 + \left(\frac{2C_\eta^2}{3C_{\tau_\Pi}} - \frac{2C_\eta \mathrm{C}\lambda 1}{3C_{\tau_\Pi}} + 4C_\eta - \frac{4C_{\lambda_1}^2}{3C_{\tau_\Pi}} + \frac{4C_{\lambda_1}}{3}\right) w^{-1} + \ldots\right\}}$$

exponential dampening (QNM)

divergent series

Analogy with QFT:

$$\frac{1}{w} \sim \text{coupling constant}$$

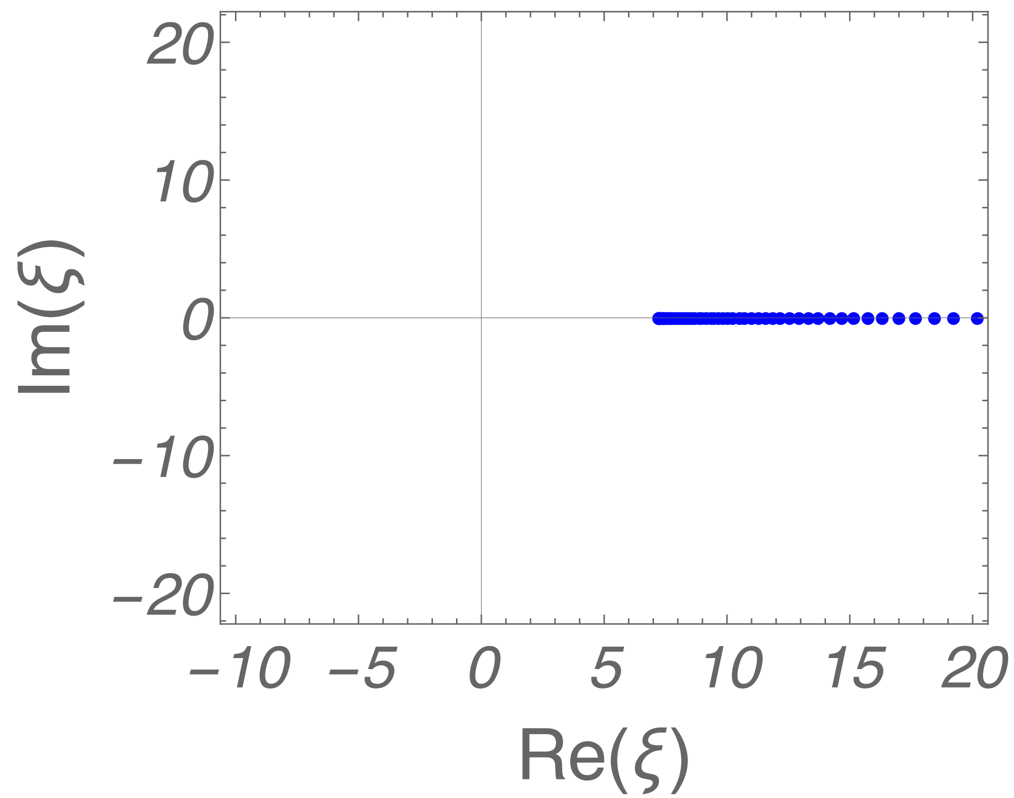
$$\text{expansion in powers of } \frac{1}{w} \sim \text{perturbative expansion}$$

QNM ~ nonperturbative object (instanton*)

Singularities of the Borel transform

Analytic continuation of Borel transform reveals dense series of poles on real axis

$$f_B(\xi) \approx \sum_{n=0}^{200} \frac{1}{n!} f_n\, \xi^n = \sum_{n=0}^{200} \tilde{f}_n\, \xi^n$$



Assuming the leading singularity of the form $(\xi - \xi_0)^\gamma$ we obtain the following relation

$$\frac{\tilde{f}_n}{\tilde{f}_{n+1}} = \xi_0 \frac{n}{n+1}\left(1 + \frac{1+\gamma}{n} + O(\frac{1}{n^2})\right)$$

Applying it to our data, we got $\xi_0 = 7.21181$ and $\gamma = 1.1449$.

Ambiguities in Borel summation

We assume the following analytic structure of $f_B(\xi)$ with $h_i(\xi)$ analytic for $\mathrm{Re}(\xi) \geq 0$

$$\tilde{f}_B(\xi) = h_0(\xi) + (\xi_0 - \xi)^\gamma h_1(\xi) + (2\xi_0 - \xi)^{2\gamma} h_2(\xi) + \ldots$$

Cuts lead to ambiguities in the Borel summation

$$f_R(w) = \int_C d\xi\, e^{-\xi}\, f_B(\xi/w) = w \int_C d\xi\, e^{-w\xi}\, f_B(\xi)$$

as C is a contour connecting 0 with ∞. For each cut we get the following ambiguity

$$\delta f_R(w) = e^{i\pi k p \gamma} w \int_{k\xi_0}^{\infty} d\xi\, e^{-w\,\xi} (\xi - k\xi_0)^{k\gamma} h_k(\xi)\text{, where } p \in \mathbb{Z}_{odd}$$

Evaluating the ambiguity at large w we obtain the k-th power of the QNM

$$\delta f_R(w) \approx e^{i\pi k p \gamma} \Gamma(k\gamma + 1) h_k(k\xi_0) \left(w^{-\gamma} e^{-w\xi_0}\right)^k$$

Transseries

The analysis of ambiguities leads us to the transseries for $f(w)$

$$f(w) = \sum_{m=0}^{\infty} c^m \Omega(w)^m \sum_{n=0}^{\infty} a_{m,n} w^{-n}, \text{ where } \Omega \equiv w^{-\gamma} \exp(-w\xi_0)$$

The coefficients $a_{m,n}$ are fixed uniquely up to $a_{1,0}$, which can be reabsorbed in c

Constant c is called the transseries parameter and we assume it is complex.

Hydrodynamic gradient expansion corresponds to $m = 0$ subseries.

For each m, corresponding subseries in n is divergent. We expect their Borel transforms to each have a sequence of branch-cuts starting at $k\,\xi_0$.

The sum of the transseries, defined using Borel summation, should be real and unambiguous, up to a single real integration constant. This leads to the resurgence.

Ambiguity cancellation in the lowest order

Let's consider the transseries with Borel sums performed over n index

$$f(w) = \sum_{m=0}^{\infty} c^m \Omega(w)^m \sum_{n=0}^{\infty} a_{m,n} w^{-n} \xrightarrow[\text{resummation}]{} f_R(w) = f_R^{(0)}(w) + c\,\Omega(w) f_R^{(1)}(w) + \ldots$$

Let's investigate the leading large-w behavior of the resummed expressions:

$$f_R^{(0)} = \frac{2}{3} + \Omega b_1^{(0)} + \Omega^2 b_2^{(0)} + \ldots$$

$$f_R^{(1)} = 1 + \Omega b_1^{(1)} + \ldots$$

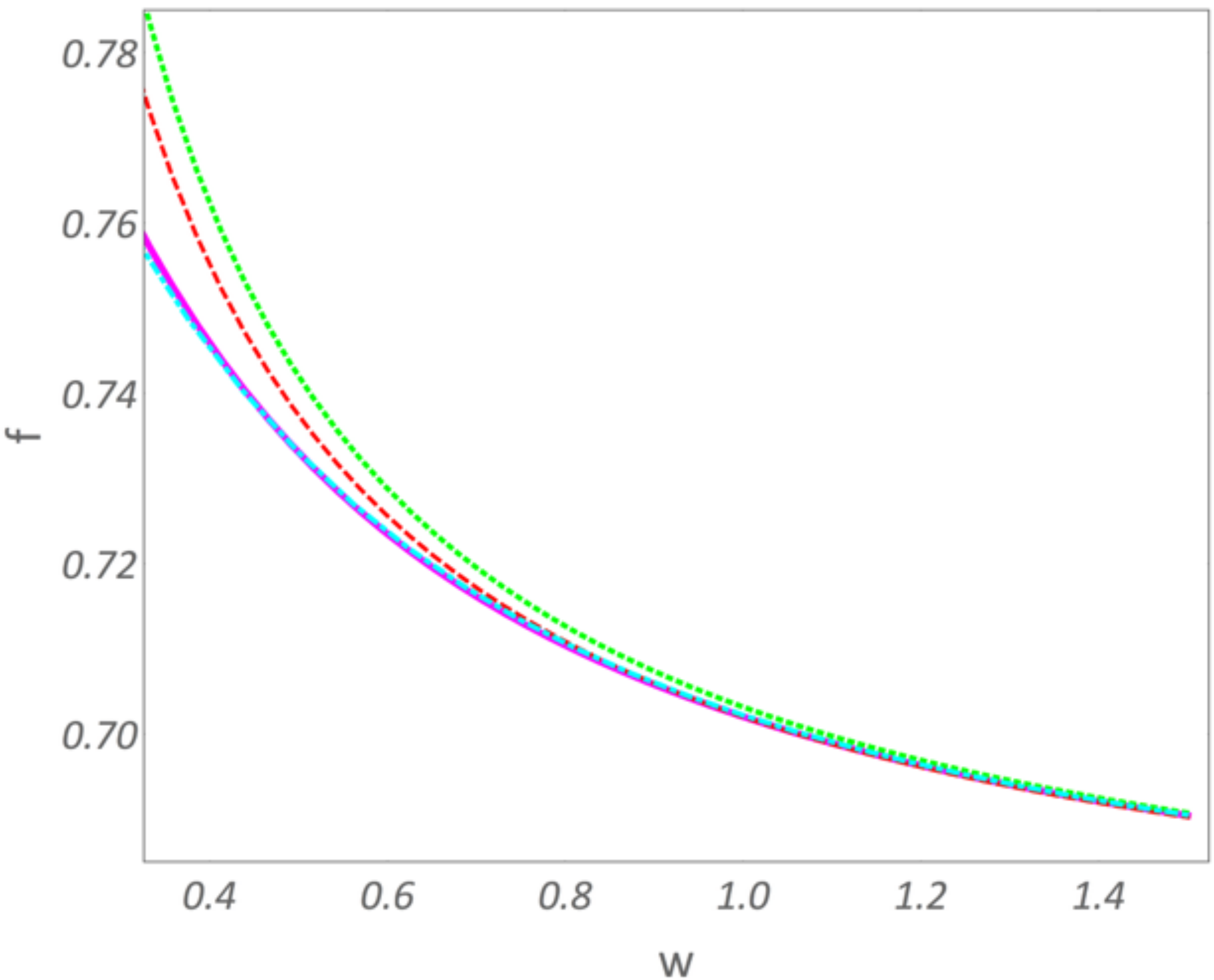
Note that $b_1^{(0)} = e^{i\pi\gamma p}\Gamma(\gamma+1)h_1^{(0)}(\xi_0)$ is ambiguous.

The only way to cancel ambiguities at the order Ω^1 is to demand

$$c = r - e^{i\pi\gamma p}\Gamma(\gamma+1)h_1^{(0)}(\xi_0)$$

r is a physical integration constant specifying a particular solution $f(w)$

Resummed hydrodynamics and the attractor



attractor

resummed transseries

$f(w) = f_0 + f_1 w^{-1}$

$f(w) = f_0 + f_1 w^{-1} + f_2 w^{-2}$

Note that matching to the attractor required choosing r = 0.049 (not 0 !).

Summary

Summary

1503.07514 [hep-th]

Hydrodynamic gradient expansion is divergent.

Our proposal: hydrodynamics beyond the gradient expansion = attractor.

Gradient expansion needs to be supplemented with "QNM" $\longrightarrow$ transseries

Analogy with QFT: gradient expansion = perturbative expansion; "QNM" = instanton

Resurgence: resummation seems to be free from ambiguities, up to integration const.

Support: ambiguities at NLO

$$f(w) = \sum_{m=0}^{\infty} c^m \Omega(w)^m \sum_{n=0}^{\infty} a_{m,n} w^{-n} \xrightarrow[\text{resummation}]{} f_R(w) = f_R^{(0)}(w) + c\,\Omega(w) f_R^{(1)}(w) + \ldots$$

The leading large-w behavior of the resummed expressions:

$$f_R^{(0)} = \frac{2}{3} + \Omega b_1^{(0)} + \Omega^2 b_2^{(0)} + \ldots$$

$$f_R^{(1)} = 1 + \Omega b_1^{(1)} + \ldots$$

$$f_R^{(2)} = -\frac{3}{2} + \ldots$$

where $b_1^{(0)} = e^{i\pi\gamma p}\Gamma(\gamma+1)h_1^{(0)}(\xi_0),\ b_2^{(0)} = e^{2i\pi\gamma p}\Gamma(2\gamma+1)h_2^{(0)}(2\xi_0),\ b_1^{(1)} = e^{i\pi\gamma p}\Gamma(\gamma+1)h_1^{(1)}(\xi_0)$

At order Ω^1 : $c = r - e^{i\pi\gamma p}\Gamma(\gamma+1)h_1^{(0)}(\xi_0)$

Now, at order Ω^2:

$$h_1^{(1)}(\xi_0) + 3h_1^{(0)}(\xi_0) = 0$$

$$2\Gamma(2\gamma+1)h_2^{(0)}(2\xi_0) + 3\Gamma(\gamma+1)^2 h_1^{(0)}(\xi_0)^2 = 0$$

We were able to numerically verify only the first relation.