Derivation of the Hydrodynamic Equation from the Quantum Transport Equation

YUTA KIKUCHI (Department of Physics, Kyoto University)

collaborators

KYOSUKE TSUMURA (Analysis Technology Center, Fujifilm Corporation)
TEIJI KUNIHIRO (Department of Physics, Kyoto University)

We derived

$$\left\{\begin{array}{c}\textbf{relativistic},\\ \textbf{second order},\\ \textbf{multi-component}\end{array}\right\}\text{ hydrodynamic equation}$$

with **no ansatz!!**

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1. Introduction and Motivation

We are interested in the hydrodynamic analyses of the following two similar phenomena:

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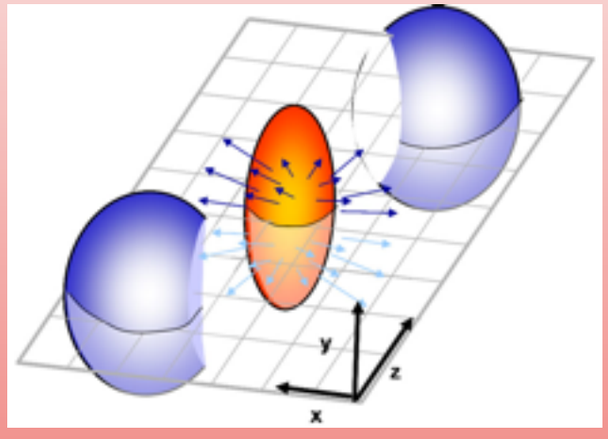
Quark-Gluon Plasma

Relativistic hydrodynamics is useful.



Fundamental problems

- Ambiguity in the definition of the flow velocity
- Unphysical instabilities of the equilibrium state
- Lack of causality

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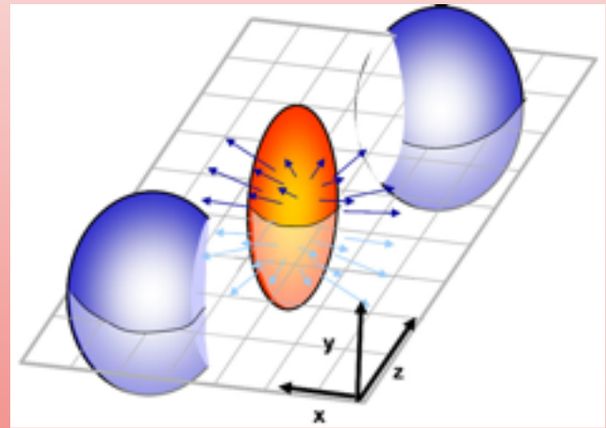

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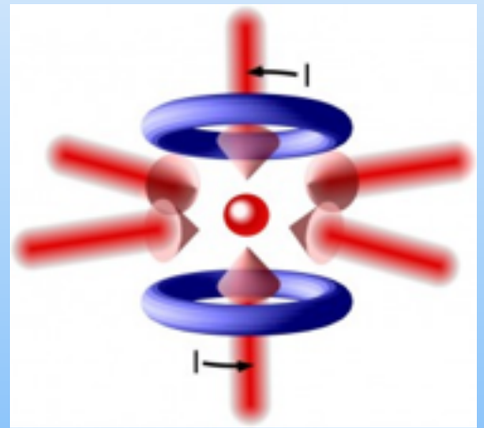
Unitary Fermi Gas

Expanding gas behaves **hydrodynamically**.



T. Schafer (2014)

Problem

- Two regions: hydrodynamic core and dilute corona
- How to describe the transition between these regions
- Consider a relaxation of dissipative currents

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Second-Order (Mesoscopic) Hydrodynamic Equation is Needed!!

Relativistic hydrodynamic equation

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$\nabla_\mu = \Delta^\nu_\mu \partial_\nu \,, \quad \frac{\partial}{\partial \tau} = u^\mu \partial_\mu$$

$$\Delta^{\mu\nu\rho\sigma} = \frac{1}{2}\left(\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right)$$

0th order

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0, & T^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} \\ \partial_\mu N^\mu = 0, & N^\mu = n u^\mu \end{cases}$$

Relativistic hydrodynamic equation

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0th order ⟶ 1st order

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0, & T^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} \\ \partial_\mu N^\mu = 0, & N^\mu = n u^\mu + \left(\frac{nT}{\varepsilon + P}\right) Q^\mu \end{cases}$$

Π : bulk pressure

$\pi^{\mu\nu}$: stress tensor

Q^μ : heat flow

$$\begin{cases} \Pi = -\zeta \nabla_\mu u^\mu \\ Q^\mu = T\lambda\left(\nabla_\mu \ln T - \frac{1}{\varepsilon + P}\nabla_\mu P\right) \\ \pi^{\mu\nu} = 2\eta \Delta^{\mu\nu\rho\sigma}\nabla_\rho u_\sigma \end{cases}$$

Acausal (Diffusion) eq.

Relativistic hydrodynamic equation

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$\nabla_\mu = \Delta^\nu_\mu \partial_\nu \,, \quad \frac{\partial}{\partial \tau} = u^\mu \partial_\mu$$

$$\Delta^{\mu\nu\rho\sigma} = \frac{1}{2}\left(\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right)$$

0th order → 1st order → 2nd order

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0, & T^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} \\ \partial_\mu N^\mu = 0, & N^\mu = n u^\mu + \left(\dfrac{nT}{\varepsilon + P}\right) Q^\mu \end{cases}$$

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$$\begin{cases} \Pi = -\zeta \nabla_\mu u^\mu - \tau_\Pi \dfrac{\partial}{\partial \tau}\Pi + (\text{other terms}) \\ Q^\mu = T\lambda\left(\nabla_\mu \ln T - \dfrac{1}{\varepsilon + P}\nabla_\mu P\right) - \tau_Q \Delta^{\mu\nu}\dfrac{\partial}{\partial \tau} Q_\nu + (\text{other terms}) \\ \pi^{\mu\nu} = 2\eta \Delta^{\mu\nu\rho\sigma}\nabla_\rho u_\sigma - \tau_\pi \Delta^{\mu\nu\rho\sigma}\dfrac{\partial}{\partial \tau}\pi_{\rho\sigma} + (\text{other terms}) \end{cases}$$

Acausal (Diffusion) eq. → **Causal (Wave) eq.**

Moment method

Relativistic Boltzmann equation

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x)$$

$$\sum_p \frac{1}{p^0} C[f]_p(x) = 0, \quad \sum_p \frac{1}{p^0} p^\mu C[f]_p(x) = 0$$

Moment equations

$$\partial_\mu N^\mu(x) = \partial_\mu \left[\sum_p \frac{1}{p^0} p^\mu f_p(x) \right] = 0$$

$$\partial_\mu T^{\mu\nu}(x) = \partial_\mu \left[\sum_p \frac{1}{p^0} p^\mu p^\nu f_p(x) \right] = 0$$

Balance equation

$$f_p(x) = f_p^{\mathrm{eq}}(x)(1 + \delta\Phi_p(x))$$

Ambiguous

$$\partial_\mu \left[\sum_p \frac{1}{p^0} p^\mu p^{\nu_1} \cdots p^{\nu_2} f_p(x) \right] = \sum_p \frac{1}{p^0} p^{\nu_1} \cdots p^{\nu_2} C[f]_p(x)$$

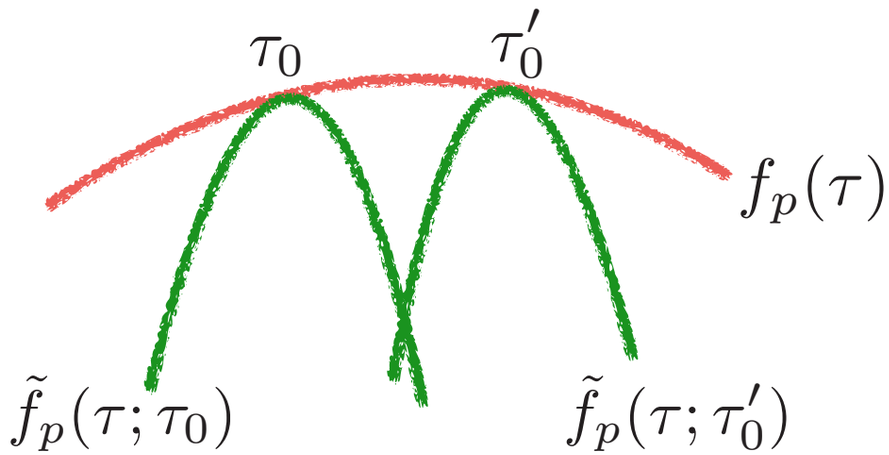
$$\delta\Phi_p(x) = a(x) + b_\mu(x) p^\mu + c_{\mu\nu}(x) p^\mu p^\nu$$

Israel and Stewart (1979)

Theoretical foundation is needed!

2. Renormalization Group Method

Renormalization group method is used to derive an infrared effective dynamics of underlying microscopic theories.



$f_p(\tau)$ ⋯ global solution

$\tilde{f}_p(\tau;\tau_0)$ ⋯ perturbative solution

basic concept $\tilde{f}_p(\tau;\tau_0) = \tilde{f}_p(\tau;\tau_0')$

(condition for constructing envelope = RG eq.)

Chen, Goldenfeld, and Oono (1994)
Kunihiro (1995)
Boyanovski, et.al. (1999)
Ei, Fujii, and Kunihiro(2000)

Taking the limit $\tau_0' \to \tau_0$

$$\left.\frac{\mathrm{d}\tilde{f}_p(\tau;\tau_0)}{\mathrm{d}\tau_0}\right|_{\tau_0=\tau} = 0$$

Coarse-graining condition

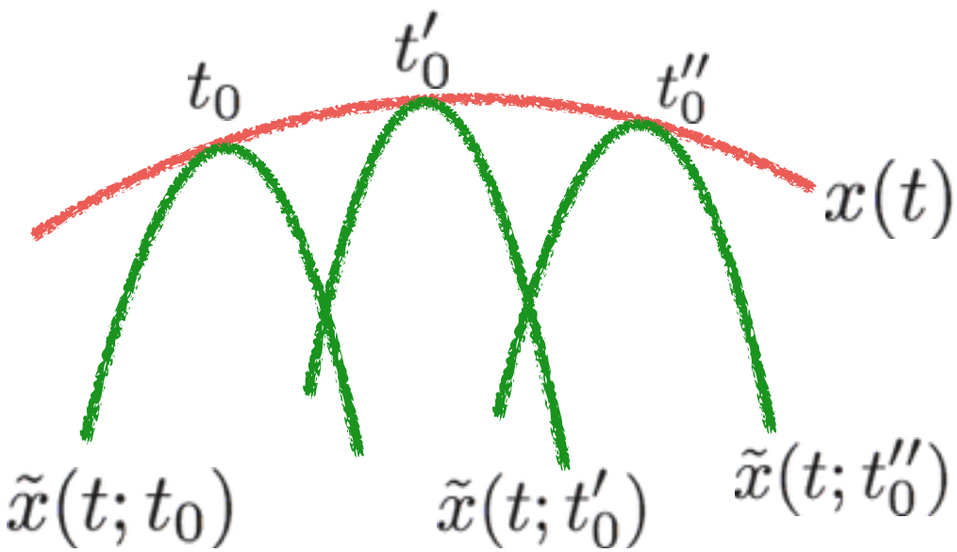
This RG eq. describes the slow dynamics!!

Ex. Van der Pol equation

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}$$

$\tilde{x}(t; t_0)$ $\cdots$ local solution

$x(t)$ $\cdots$ global solution



Local solution is represented as perturbation series

$$\tilde{x}(t; t_0) = \tilde{x}_0(t; t_0) + \epsilon\tilde{x}_1(t; t_0) + \epsilon^2\tilde{x}_2(t; t_0) + \cdots$$

Initial value is taken as the exact solution

$$\begin{aligned} \tilde{x}(t_0; t_0) &= x(t_0) \\ &= \tilde{x}_0(t_0; t_0) + \epsilon\tilde{x}_1(t_0; t_0) + \epsilon^2\tilde{x}_2(t_0; t_0) + \cdots \end{aligned}$$

2. Renormalization Group Method

0th order equation $\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+1\right)\tilde{x}_0=0$

0th order solution $\tilde{x}(t;t_0)=A(t_0)\cos(t+\theta(t_0))$ with $\tilde{x}(t_0;t_0)=A(t_0)\cos(t_0+\theta(t_0))$

$A(t_0)$ and $\theta(t_0)$ are integration constants.

1st order equation $\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+1\right)\tilde{x}_1=-A\left(1-\frac{A^2}{4}\right)\sin(t+\theta(t_0))+\frac{A^3}{4}\sin 3(t+\theta(t_0))$

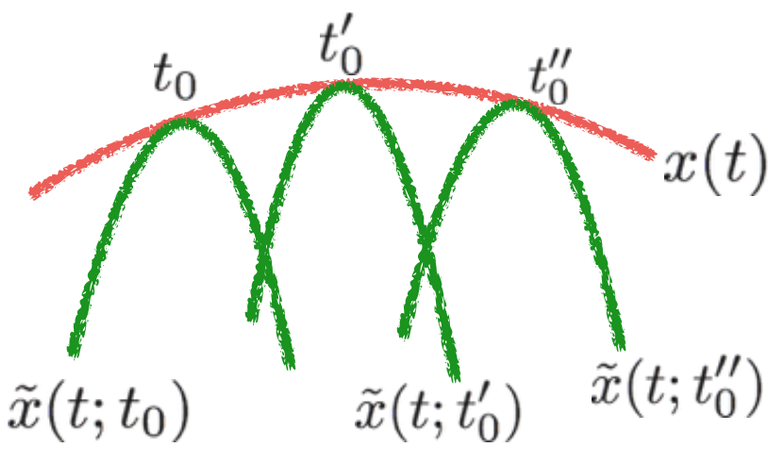
1st order solution $\tilde{x}(t;t_0)=(t-t_0)\frac{A}{2}\left(1-\frac{A^2}{4}\right)\sin(t+\theta(t_0))-\frac{A^3}{32}\sin 3(t+\theta(t_0))$

secular divergence

Solve the renormalization group equation to obtain the global solution

$$\left.\frac{d\tilde{x}}{dt_0}\right|_{t_0=t}=0 \longrightarrow \boxed{\dot{A}=\epsilon\frac{A}{2}\left(1-\frac{A^2}{4}\right),\quad \dot{\theta}=0}$$

Equations for slow variables



The global solution

$$x_E(t)\equiv\tilde{x}(t;t_0=t)=A(t)\cos\phi(t)-\epsilon\frac{A^3(t)}{32}\sin 3\phi(t)$$

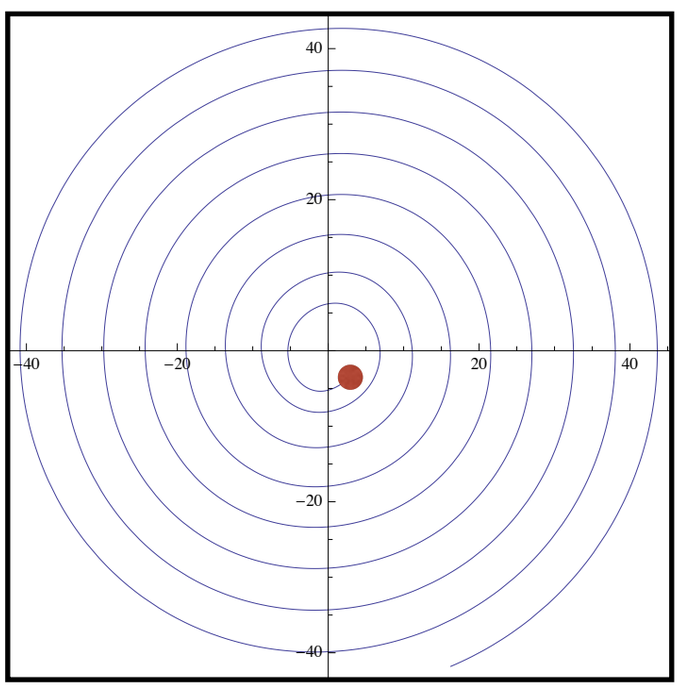
The solution of Van der Pol equation

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}$$

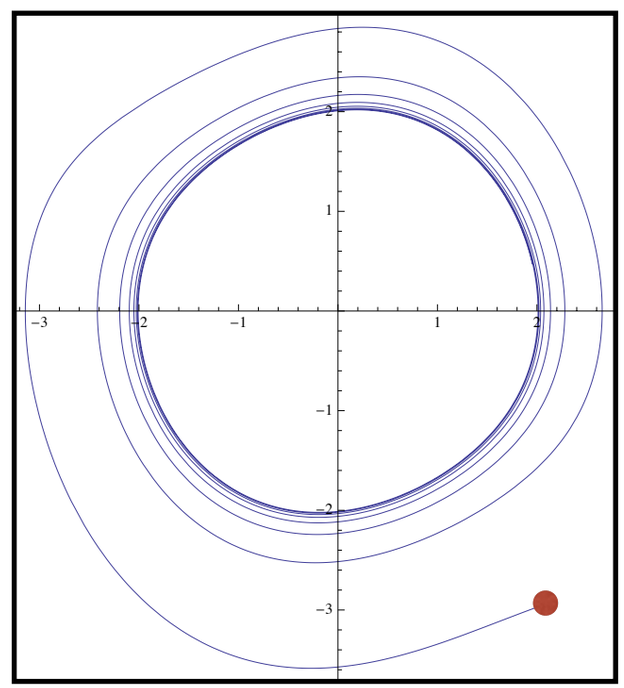
horizontal axis : x
vertical axis : $\dot{x}$
$\epsilon = 0.1$

limit cycle

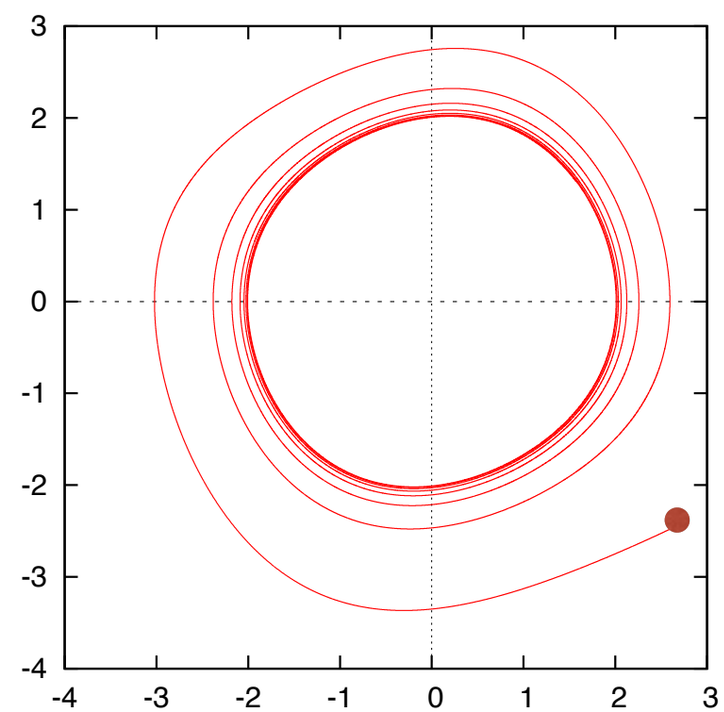
cf. Van der Pol oscillator



Perturbative solution



RG-improved solution



Numerical solution

3. Derivation of Hydrodynamic Equation

Kadanoff-Baym eq.

$$\left(\partial_x^2 + m^2 + \Sigma^\delta(x)\right) G^<(x,y) = -\int_{-\infty}^{\infty} \mathrm{d}^4 z \left[\underbrace{\Sigma_R(x,z)}_{\text{self energy}} G^<(z,y) + \Sigma^<(x,z) \underbrace{G_A(z,y)}_{\text{Green's function}}\right]$$

Derivative expansion

Boltzmann eq.

$$(k\cdot\partial_x)G^< + \frac{1}{2}\left(\partial_x^\mu \Sigma^\delta\right)\partial_\mu^k G^< = -\left(G^>\Sigma^< + \Sigma^> G^<\right)$$

Quasi-particle approximation : $G^<(k,x) \propto \delta(k^2 - m^2) f_k(x)$

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x)$$

$$C[f]_p(x) = \sum_{p_1,p_2,p_3} w(p,p_1;p_2,p_3)[f_p f_{p_1}(1+af_{p_2})(1+af_{p_3}) - (1+af_p)(1+af_{p_1})f_{p_2}f_{p_3}]$$

$a = +1$: boson, -1: fermion

Perturbative calculation w.r.t Knudsen number and **renormalization-group** resummation

Tsumura, Kunihiro, and Ohnishi (2007) : 1st order in classical case

Tsumura and Kunihiro (2012) : 2nd order in classical case

Hydrodynamic eq.

Derivation of the first-order hydrodynamics

Hatta and Kunihiro (2002)
Kunihiro and Tsumura (2006)
Tsumura, Kunihiro, and Ohnishi (2007)

Relativistic Boltzmann equation

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x)$$

$$C[f]_p(x) = \sum_{p_1,p_2,p_3} w(p,p_1;p_2,p_3)[f_p f_{p_1}(1+af_{p_2})(1+af_{p_3}) - (1+af_p)(1+af_{p_1})f_{p_2}f_{p_3}]$$

$$\Rightarrow \quad \sum_p \frac{1}{p^0} C[f]_p(x) = 0, \quad \sum_p \frac{1}{p^0} p^\mu C[f]_p(x) = 0$$

Balance equation

$$\partial_\mu N^\mu(x) \equiv \partial_\mu \left[\sum_p \frac{1}{p^0} p^\mu f_p(x) \right] = \sum_p \frac{1}{p^0} C[f]_p(x) = 0$$

$$\partial_\mu T^{\mu\nu}(x) \equiv \partial_\mu \left[\sum_p \frac{1}{p^0} p^\mu p^\nu f_p(x) \right] = \sum_p \frac{1}{p^0} p^\nu C[f]_p(x) = 0$$

3. Derivation of Hydrodynamic Equation

Relativistic Boltzmann eq.

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x)$$

$$\frac{\partial}{\partial \tau} f_p(x) = \frac{1}{p \cdot u} C[f]_p(x) - \epsilon \frac{1}{p \cdot u} p \cdot \nabla f_p(x)$$

$$\frac{\partial}{\partial \tau} \equiv u^\mu \partial_\mu, \quad \nabla^\mu \equiv (g^{\mu\nu} - u^\mu u^\nu)\partial_\nu \equiv \Delta^{\mu\nu}\partial_\nu$$

u^μ : flow velocity

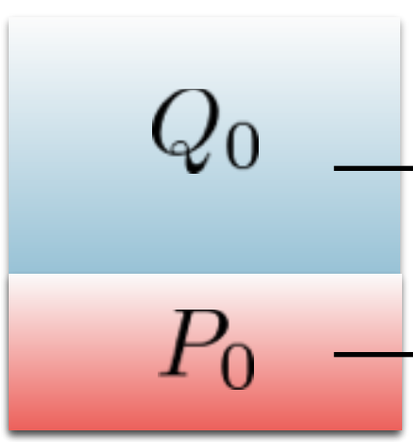
ϵ : measure of the inhomogeneity of fluid

Solve the relativistic Boltzmann eq. perturbatively

Initial condition : $\tilde{f}_p(\tau = \tau_0; \tau_0) = f_p(\tau_0)$

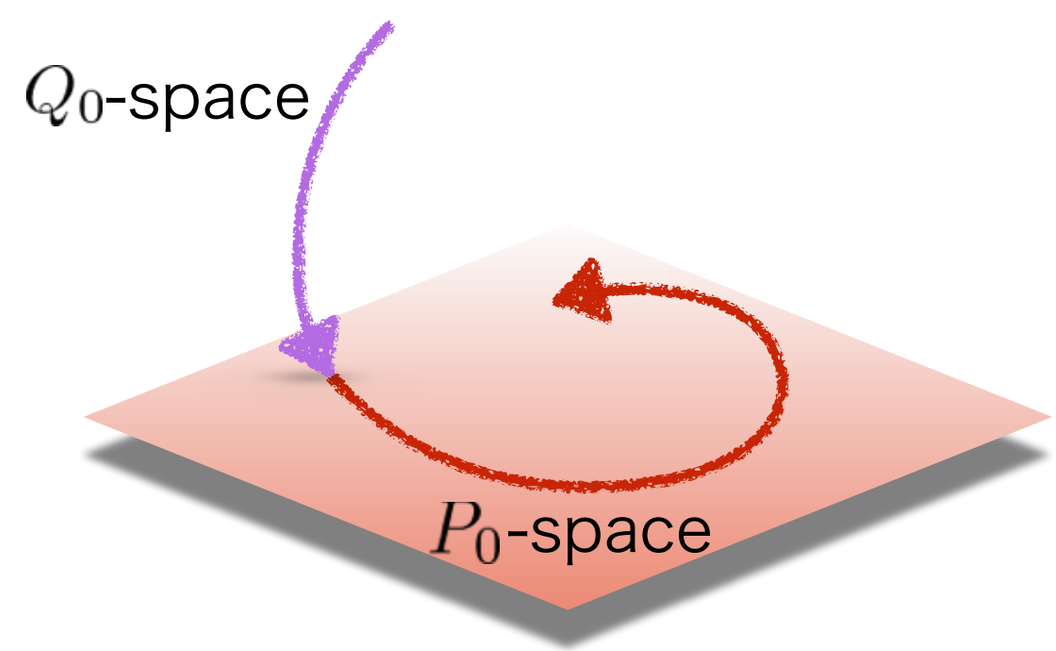
Perturbative solution: $\tilde{f}_p(\tau; \tau_0) = \tilde{f}_p^{(0)}(\tau; \tau_0) + \epsilon \tilde{f}_p^{(1)}(\tau; \tau_0) + \epsilon^2 \tilde{f}_p^{(2)}(\tau; \tau_0) + \cdots$

solution space



fast modes

slow modes
corresponding to
conserved quantities



3. Derivation of Hydrodynamic Equation

0th-order eq.

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)}(\tau;\tau_0) = \frac{1}{p\cdot u} C[\tilde{f}^{(0)}]_p(\tau;\tau_0)$$

0th-order solution

$$f_p^{(0)}(\tau,\tau_0) = f_p^{\mathrm{eq}}(\tau_0) \equiv \frac{1}{e^{\beta(\tau_0)\{p\cdot u(\tau_0)-\mu(\tau_0)\}} - a}$$

interested in the slow dynamics

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interested in the slow dynamics

1st order eq.

$$\frac{\partial}{\partial\tau}\tilde{f}^{(1)} = A\tilde{f}^{(1)} + F$$

$$A_{pq} \equiv \frac{\partial}{\partial f_q}\left(\frac{1}{p\cdot u}C[f]_p\right)\Big|_{f=f^{\mathrm{eq}}}, F_p \equiv -\frac{1}{p\cdot u}p\cdot\nabla f_p^{\mathrm{eq}}$$

2nd order eq.

$$\frac{\partial}{\partial\tau}\tilde{f}^{(2)} = A\tilde{f}^{(2)} + (\tau-\tau_0)^2 G + (\tau-\tau_0)H + I$$

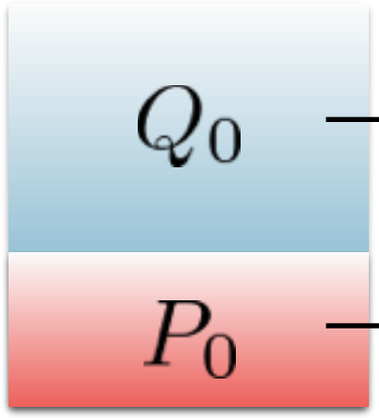
$$G_p \equiv \sum_{q,r} B_{pqr}[P_0F]_q[P_0F]_r$$

$$H_p \equiv \sum_{q,r} B_{pqr}\left([P_0F]_q[A^{-1}Q_0F]_r + [A^{-1}Q_0F]_q[P_0F]_r\right) - \frac{1}{p\cdot u}p\cdot\nabla[P_0F]_p$$

$$I_p \equiv \sum_{q,r} B_{pqr}[A^{-1}Q_0F]_q[A^{-1}Q_0F]_r - \frac{1}{p\cdot u}p\cdot\nabla[A^{-1}Q_0F]_p$$

$$B_{pqr} \equiv \frac{\partial^2}{\partial f_q\partial f_r}\left(\frac{1}{p\cdot u}C[f]_p\right)\Big|_{f=f^{\mathrm{eq}}}$$

$$P_0 + Q_0 = 1$$



fast modes

slow modes

5 zero-mode of A:

p^μ …energy and momenta

m …particle (mass) current

3. Derivation of Hydrodynamic Equation

0th-order eq.

$$\frac{\partial}{\partial\tau}\tilde{f}_p^{(0)}(\tau;\tau_0) = \frac{1}{p\cdot u}C[\tilde{f}^{(0)}]_p(\tau;\tau_0)$$

0th-order solution

$$f_p^{(0)}(\tau,\tau_0) = f_p^{\mathrm{eq}}(\tau_0) \equiv \frac{1}{e^{\beta(\tau_0)\{p\cdot u(\tau_0)-\mu(\tau_0)\}}-a}$$

interested in the slow dynamics

1st order eq.

$$\frac{\partial}{\partial\tau}\tilde{f}^{(1)} = A\tilde{f}^{(1)} + F$$

$$A_{pq} \equiv \frac{\partial}{\partial f_q}\left(\frac{1}{p\cdot u}C[f]_p\right)\Big|_{f=f^{\mathrm{eq}}}, F_p \equiv -\frac{1}{p\cdot u}p\cdot\nabla f_p^{\mathrm{eq}}$$

2nd order eq.

$$\frac{\partial}{\partial\tau}\tilde{f}^{(2)} = A\tilde{f}^{(2)} + (\tau-\tau_0)^2 G + (\tau-\tau_0)H + I$$

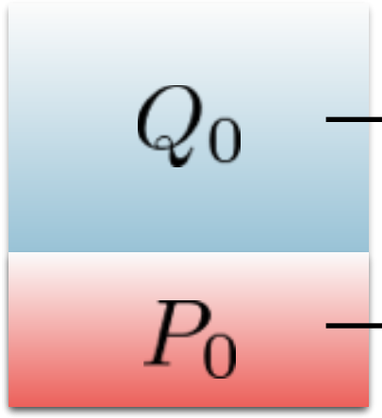
$$G_p \equiv \sum_{q,r} B_{pqr}[P_0F]_q[P_0F]_r$$

$$H_p \equiv \sum_{q,r} B_{pqr}\left([P_0F]_q[A^{-1}Q_0F]_r + [A^{-1}Q_0F]_q[P_0F]_r\right) - \frac{1}{p\cdot u}p\cdot\nabla[P_0F]_p$$

$$I_p \equiv \sum_{q,r} B_{pqr}[A^{-1}Q_0F]_q[A^{-1}Q_0F]_r - \frac{1}{p\cdot u}p\cdot\nabla[A^{-1}Q_0F]_p$$

$$B_{pqr} \equiv \frac{\partial^2}{\partial f_q\partial f_r}\left(\frac{1}{p\cdot u}C[f]_p\right)\Big|_{f=f^{\mathrm{eq}}}$$

$$P_0 + Q_0 = 1$$



fast modes

slow modes

5 zero-mode of A:

p^μ ⋯energy and momenta

m ⋯particle (mass) current

Perturbative solution up to 2nd order

$$\tilde{f}(\tau) = f^{\mathrm{eq}} + \epsilon\left((\tau-\tau_0)P_0F - A^{-1}Q_0F\right)$$
$$+\epsilon^2\Bigg(\frac{1}{3}(\tau-\tau_0)^3P_0G + \frac{1}{2}(\tau-\tau_0)^2\left(P_0H - 2A^{-1}Q_0G\right)$$
$$+(\tau-\tau_0)\left(P_0I - 2A^{-2}Q_0G - A^{-1}Q_0H\right) - 2A^{-3}Q_0G - A^{-2}Q_0H - A^{-1}Q_0I\Bigg)$$

secular divergences

3. Derivation of Hydrodynamic Equation

Write down the renormalization group equation

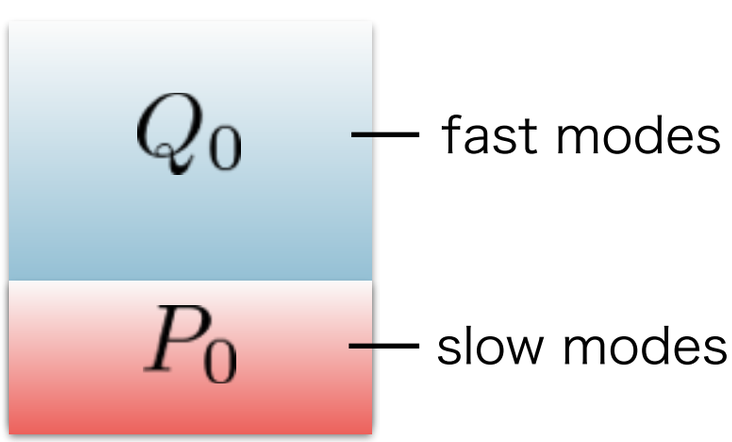
$$\left.\frac{\mathrm{d}\tilde{f}_p(\tau;\tau_0)}{\mathrm{d}\tau_0}\right|_{\tau_0=\tau} = 0$$

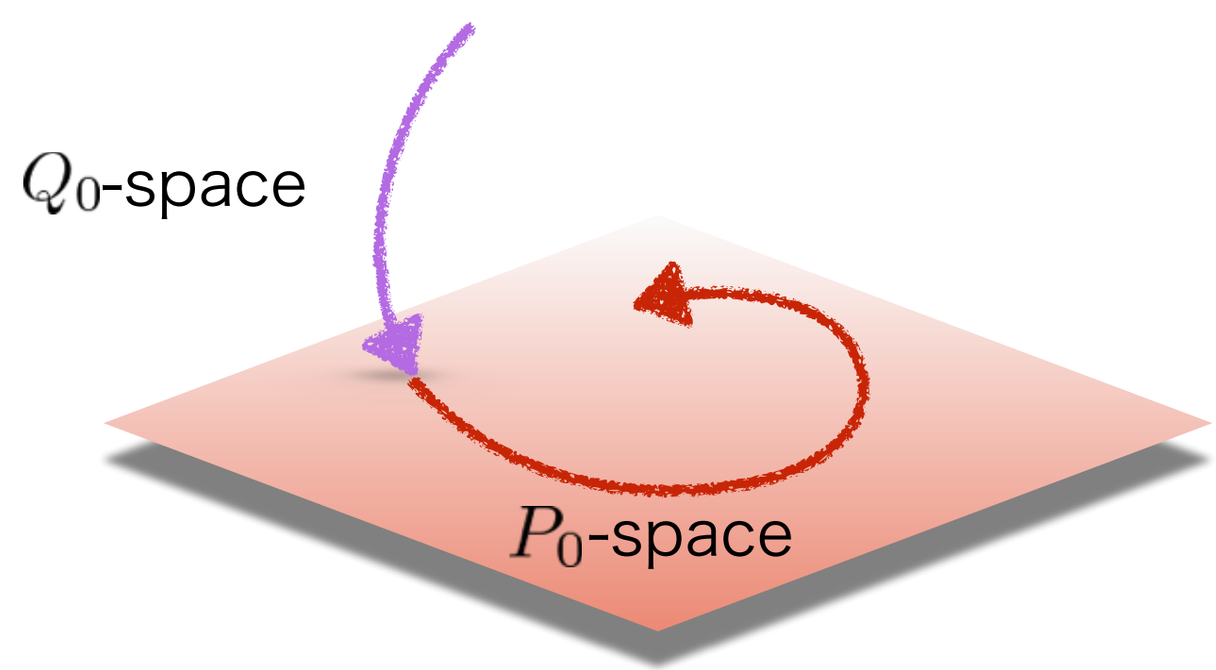
This equation corresponds to the hydrodynamic eq.!!

Projecting on P_0-space

$$\boxed{\begin{cases} \partial_\mu N^\mu = 0, & N^\mu = \sum_p \frac{1}{p^0} p^\mu m \left\{ f_p^{\mathrm{eq}} - [A^{-1} Q_0 F]_p \right\} \\ \partial_\mu T^{\mu\nu} = 0, & T^{\mu\nu} = \sum_p \frac{1}{p^0} p^\mu p^\nu \left\{ f_p^{\mathrm{eq}} - [A^{-1} Q_0 F]_p \right\} \end{cases}}$$

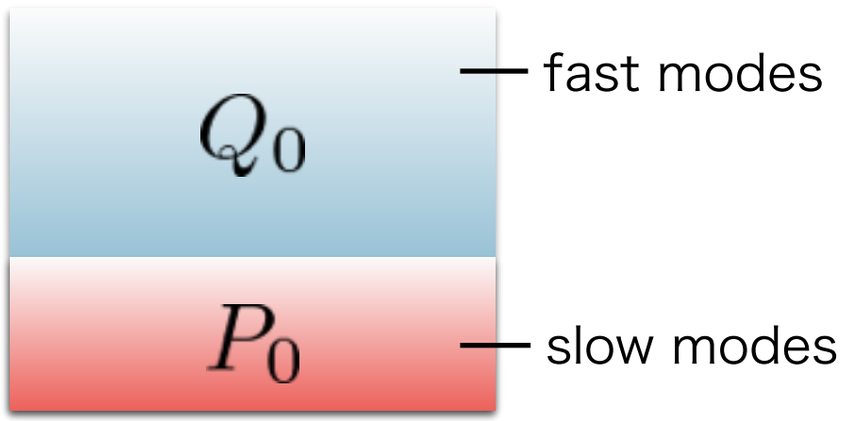
Solution space

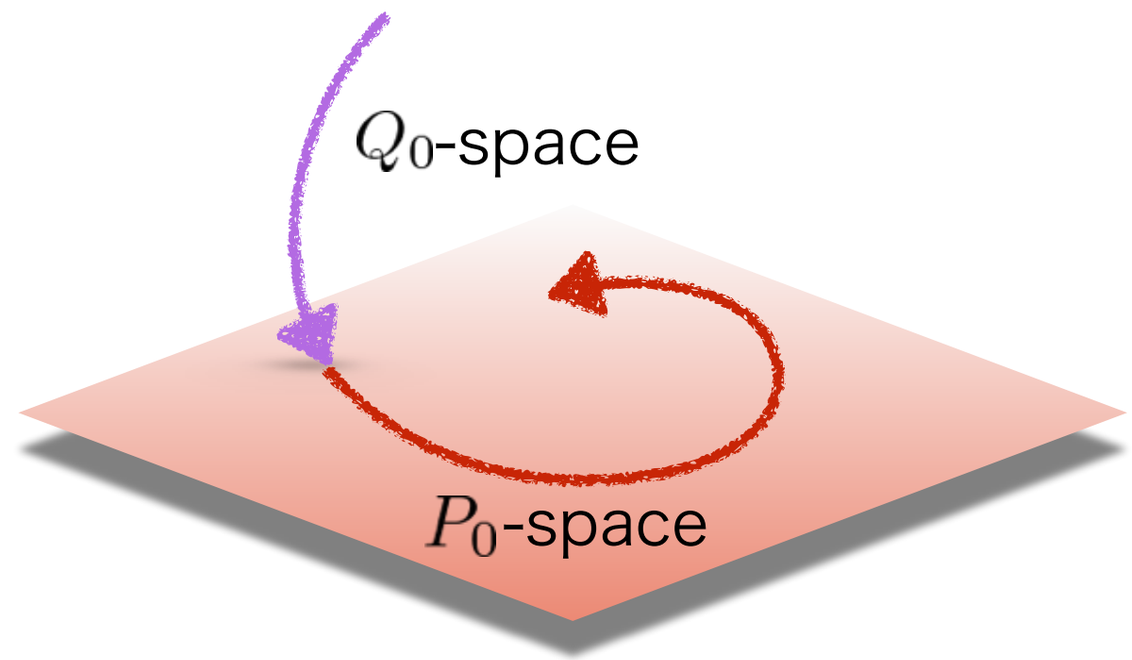




Extension to second order eq.

Solution space





Include the quasi-slow dynamics besides the slowest dynamics

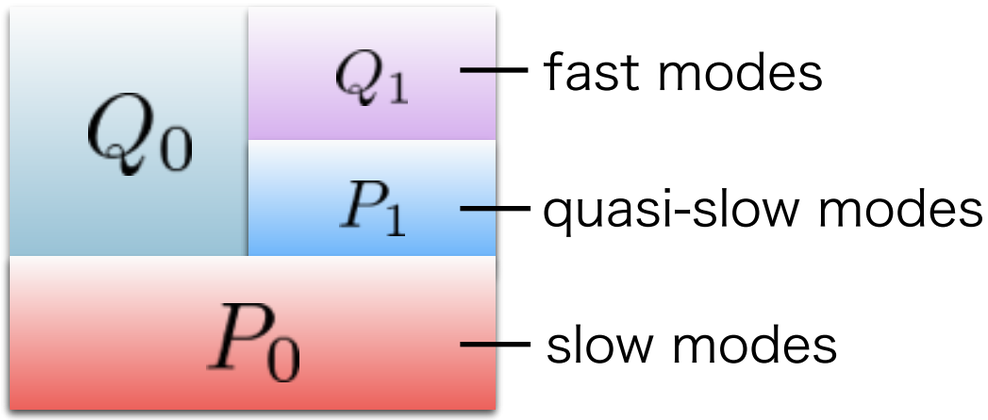
"Doublet scheme": Tsumura and Kunihiro (2013)

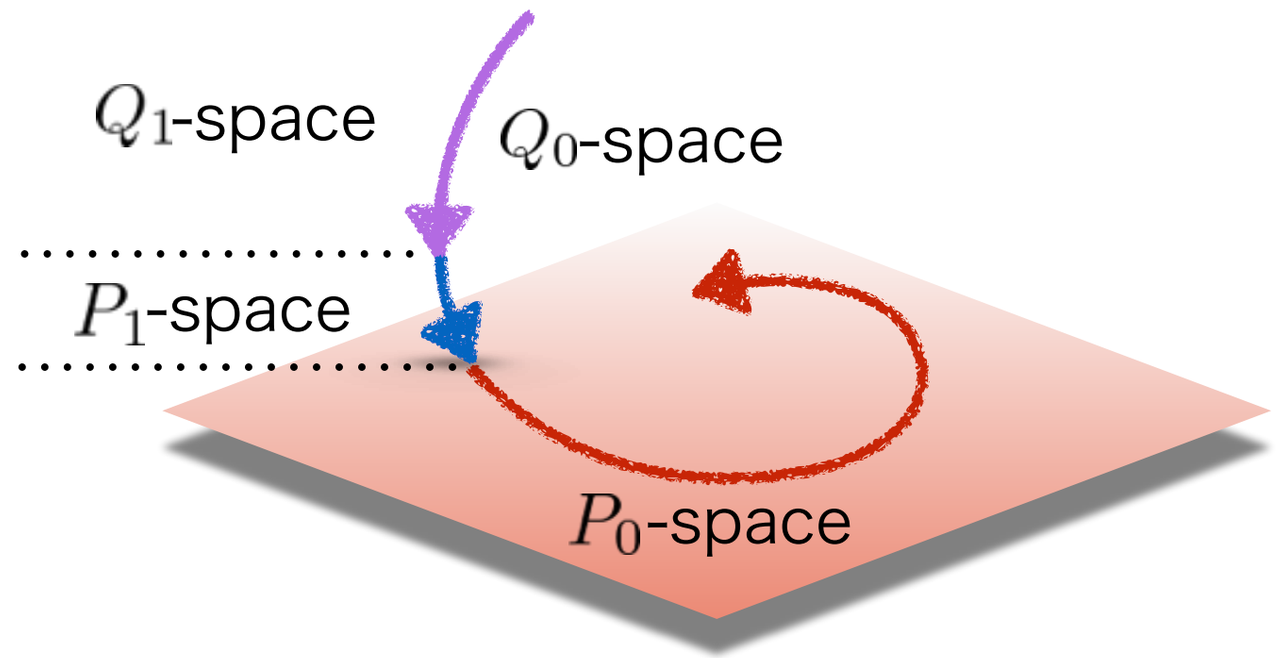
Enable us to describe the relaxation of dissipative currents:

Second order equation

Extension to second order eq.

Solution space





Include the quasi-slow dynamics besides the slowest dynamics

"Doublet scheme": Tsumura and Kunihiro (2013)

Enable us to describe the relaxation of dissipative currents:

Second order equation

4. Result: Hydrodynamic Equation

Second order hydrodynamic eq. (single-component)

Eq. of continuity

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = \varepsilon u^\mu u^\nu - (P - \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\partial_\mu N^\mu = 0, \qquad N^\mu = mnu^\mu + m\left(\frac{nT}{\varepsilon + P}\right) Q^\mu$$

Π : bulk pressure

$\pi^{\mu\nu}$: stress tensor

Q^μ : heat flow

Eq. of relaxation

$$\Pi = -\zeta \nabla_\mu u^\mu - \tau_\Pi \frac{\partial}{\partial \tau}\Pi + (\text{other terms})$$

$$Q^\mu = T\lambda\left(\nabla_\mu \ln T - \frac{1}{\varepsilon + P}\nabla_\mu P\right) - \tau_Q \Delta^{\mu\nu}\frac{\partial}{\partial \tau} Q_\nu + (\text{other terms})$$

$$\pi^{\mu\nu} = 2\eta \Delta^{\mu\nu\rho\sigma}\nabla_\rho u_\sigma - \tau_\pi \Delta^{\mu\nu\rho\sigma}\frac{\partial}{\partial \tau}\pi_{\rho\sigma} + (\text{other terms})$$

with **the microscopic expressions** :

$$\zeta \equiv \frac{1}{T}\int_0^\infty \mathrm{d}s \left\langle \hat{\Pi}(0)\hat{\Pi}(s)\right\rangle \qquad \lambda \equiv -\frac{1}{3T^2}\int_0^\infty \mathrm{d}s \left\langle \hat{Q}^\mu(0)\hat{Q}_\mu(s)\right\rangle \qquad \eta \equiv \frac{1}{10T}\int_0^\infty \mathrm{d}s \left\langle \hat{\pi}^{\mu\nu}(0)\hat{\pi}_{\mu\nu}(s)\right\rangle$$

$$\tau_\Pi \equiv \frac{\int_0^\infty \mathrm{d}s s \left\langle \hat{\Pi}(0)\hat{\Pi}(s)\right\rangle}{\int_0^\infty \mathrm{d}s \left\langle \hat{\Pi}(0)\hat{\Pi}(s)\right\rangle} \qquad \tau_Q \equiv \frac{\int_0^\infty \mathrm{d}s s \left\langle \hat{Q}^\mu(0)\hat{Q}_\mu(s)\right\rangle}{\int_0^\infty \mathrm{d}s \left\langle \hat{Q}^\mu(0)\hat{Q}_\mu(s)\right\rangle} \qquad \tau_\pi \equiv \frac{\int_0^\infty \mathrm{d}s s \left\langle \hat{\pi}^{\mu\nu}(0)\hat{\pi}_{\mu\nu}(s)\right\rangle}{\int_0^\infty \mathrm{d}s \left\langle \hat{\pi}^{\mu\nu}(0)\hat{\pi}_{\mu\nu}(s)\right\rangle}$$

$\hat{\Pi}(s) = e^{sL}\hat{\Pi}$, L: Linearized collision operator

Eq. of relaxation

$$
\begin{aligned}
\Pi &= -\zeta\,\nabla\cdot u - \tau_\Pi \frac{\partial}{\partial\tau}\Pi \\
&\quad - \ell_{\Pi J}\,\Delta^{ab}\,\nabla_a J_b \\
&\quad + \left(\kappa_{\Pi\Pi}\,\nabla\cdot u\right)\Pi \\
&\quad + \left(\kappa^{(1)}_{\Pi J}\,\Delta^{a\rho}\,T\,\nabla_a(1/T) + \kappa^{(2)}_{\Pi J}\,\Delta^{a\rho}\,\frac{\partial}{\partial\tau}u_a\right) J_\rho \\
&\quad + \left(\kappa_{\Pi\pi}\,\Delta^{ab\rho\sigma}\,\nabla_a u_b\right)\pi_{\rho\sigma}, \\
J^\mu &= \frac{\lambda}{\hat{h}^2}\,\nabla^\mu\frac{\mu}{T} - \tau_J\,\Delta^{\mu a}\,\frac{\partial}{\partial\tau}J_a \\
&\quad - \ell_{J\Pi}\,\nabla^\mu\Pi - \ell_{J\pi}\,\Delta^{\mu abc}\,\nabla_a\pi_{bc} \\
&\quad + \left(\kappa^{(1)}_{J\Pi}\,\Delta^{\mu a}\,T\,\nabla_a(1/T) + \kappa^{(2)}_{J\Pi}\,\Delta^{\mu a}\,\frac{\partial}{\partial\tau}u_a\right)\Pi \\
&\quad + \left(\kappa^{(0)}_{JJ}\,\Delta^{\mu\rho}\,\nabla\cdot u + \kappa^{(1)}_{JJ}\,\Delta^{\mu\rho ab}\,\nabla_a u_b + \kappa^{(2)}_{JJ}\,\omega^{\mu\rho}\right) J_\rho \\
&\quad + \left(\kappa^{(1)}_{J\pi}\,\Delta^{\mu a\rho\sigma}\,T\,\nabla_a(1/T) + \kappa^{(2)}_{J\pi}\,\Delta^{\mu a\rho\sigma}\,\frac{\partial}{\partial\tau}u_a\right)\pi_{\rho\sigma}, \\
\pi^{\mu\nu} &= 2\,\eta\,\Delta^{\mu\nu ab}\,\nabla_a u_b - \tau_\pi\,\Delta^{\mu\nu ab}\,\frac{\partial}{\partial\tau}\pi_{ab} \\
&\quad - \ell_{\pi J}\,\Delta^{\mu\nu ab}\,\nabla_a J_b \\
&\quad + \left(\kappa_{\pi\Pi}\,\Delta^{\mu\nu ab}\,\nabla_a u_b\right)\Pi \\
&\quad + \left(\kappa^{(1)}_{\pi J}\,\Delta^{\mu\nu a\rho}\,T\,\nabla_a(1/T) + \kappa^{(2)}_{\pi J}\,\Delta^{\mu\nu a\rho}\,\frac{\partial}{\partial\tau}u_a\right) J_\rho \\
&\quad + \left(\kappa^{(0)}_{\pi\pi}\,\Delta^{\mu\nu\rho\sigma}\,\nabla\cdot u + \kappa^{(1)}_{\pi\pi}\,\Delta^{\mu\nu ac}\,\Delta_c{}^{b\rho\sigma}\,\Delta_{abde}\,\nabla^d u^e + \kappa^{(2)}_{\pi\pi}\,\Delta^{\mu\nu ac}\,\Delta_c{}^{b\rho\sigma}\,\omega_{ab}\right)\pi_{\rho\sigma}.
\end{aligned}
$$

Vorticity: $\omega^{\mu\nu} \equiv \frac{1}{2}\left(\nabla^\mu u^\nu - \nabla^\nu u^\mu\right)$

Comparison with the other formalisms

RG : Renormalization group method,
CE : Chapmen-Enskog method,
IS : Moment method.
W. Israel and J. M. Stewart (1979)

Linearized collision operator:

$$L_{pq} \equiv f_p^{\mathrm{eq}-1}\underline{(1+af_p^{\mathrm{eq}})}^{-1}\left(\frac{1}{p\cdot u}\frac{\partial}{\partial f_q}C[f]_p\Big|_{f=f^{\mathrm{eq}}}\right)f_q^{\mathrm{eq}}\underline{(1+af_q^{\mathrm{eq}})}$$

Quantum statistical effect

Inner product:

$$\langle\phi,\psi\rangle \equiv \sum_p \frac{1}{p^0}(p\cdot u)f_p^{\mathrm{eq}}\underline{(1+af_p^{\mathrm{eq}})}\phi_p\psi_p$$

1st-order coefficients

$$\eta^{\mathrm{RG}} = -\frac{1}{10T}\langle\hat{\pi}^{\mu\nu}, L^{-1}\hat{\pi}_{\mu\nu}\rangle$$

$$\eta^{\mathrm{CE}} = -\frac{1}{10T}\langle\hat{\pi}^{\mu\nu}, L^{-1}\hat{\pi}_{\mu\nu}\rangle$$

$$\eta^{\mathrm{IS}} = -\frac{1}{10T}\frac{\langle\hat{\pi}^{\mu\nu},\pi_{\mu\nu}\rangle\langle\pi^{\rho\sigma},\hat{\pi}_{\rho\sigma}\rangle}{\langle\pi^{\alpha\beta},L\pi_{\alpha\beta}\rangle}$$

2nd-order coefficients

$$\tau_\pi^{\mathrm{RG}} = -\frac{\langle\hat{\pi}^{\mu\nu}, L^{-2}\hat{\pi}_{\mu\nu}\rangle}{\langle\hat{\pi}^{\rho\sigma}, L^{-1}\hat{\pi}_{\rho\sigma}\rangle} = \frac{\int_0^\infty \mathrm{d}s\, s\,\langle\hat{\pi}^{\mu\nu}(0),\hat{\pi}_{\mu\nu}(s)\rangle}{\int_0^\infty \mathrm{d}s\,\langle\hat{\pi}^{\rho\sigma}(0),\hat{\pi}_{\rho\sigma}(s)\rangle}$$

$$\tau_\pi^{\mathrm{IS}} = -\frac{\langle\pi^{\mu\nu},\pi_{\mu\nu}\rangle}{\langle\pi^{\rho\sigma},L\pi_{\rho\sigma}\rangle}$$

This expression is natural and plausible!!

RG is consistent with Kubo-formula!!
Jeon (1995), Jeon and Yaffe (1996),
Hidaka and Kunihiro (2011)

$$\pi_p^{\mu\nu} \equiv \Delta^{\mu\nu\rho\sigma}p_\rho p_\sigma, \quad \hat{\pi}_p^{\mu\nu} \equiv \pi_p^{\mu\nu}/(p\cdot u)$$

Extension to reactive multi-component systems

Setup

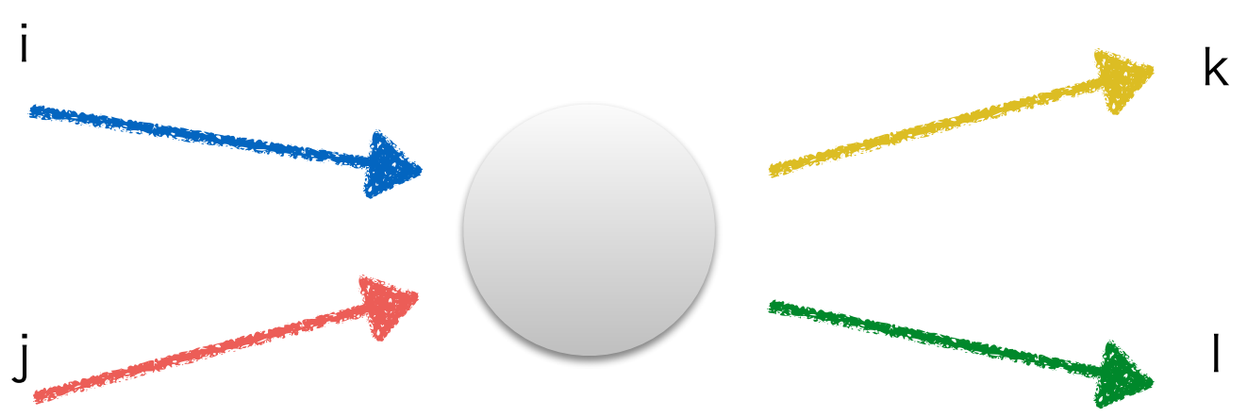
Prakash, Prakash, Venugopalan, and Welke (1993)
Monnai and Hirano (2010)

Number of components : N
Number of conserved currents : M

q_k^a ⋯ charge of k-th component associated with a-th current

(In case of elastic collisions : $q_k^a = m_k \delta_{k,a}$)

Collision process



$$q_i^a + q_j^a = q_k^a + q_l^a$$

Extension to reactive multi-component systems

Eq. of continuity

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = \varepsilon u^\mu u^\nu - (P - \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\partial_\mu N_a^\mu = 0, \qquad N_a^\mu = \sum_{k=1}^{N} q_k^a n_k u^\mu + Q_a^\mu + \sum_{l=1}^{N-1} I_{a,l}^\mu$$

Diffusion flow

Eq. of relaxation

$$\Pi = -\zeta \nabla_\mu u^\mu - \tau_\Pi \frac{\partial}{\partial \tau}\Pi + (\text{other terms})$$

$$L_{I_l Q} = \frac{1}{3T^2}\int_0^\infty \mathrm{d}s \left\langle \hat{I}_l^\mu(0), \hat{Q}_\mu(s) \right\rangle$$

$$Q_a^\mu = \left(\sum_{k=1}^{N} \frac{q_k^a n_k}{\varepsilon+P} T\lambda - \sum_{l=1}^{N-1}(q_l^a - q_N^a) T L_{I_l Q}\right)\left(\nabla_\mu \ln T - \frac{1}{\varepsilon+P}\nabla_\mu P\right)$$
$$-\tau_{QQ}\Delta^{\mu\nu}\frac{\partial}{\partial\tau}Q_\nu - \sum_{m=1}^{N-1}\tau_{QI_m}\Delta^{\mu\nu}\frac{\partial}{\partial\tau}I_{m,\nu} + (\text{other terms})$$

$$I_{a,l}^\mu = \left(\sum_{k=1}^{N} \frac{q_k^a n_k}{\varepsilon+P} T L_{QI_l} - \sum_{m=1}^{N-1}(q_l^a - q_N^a) T L_{I_l I_m}\right) T \sum_{b=1}^{M}\left(q_m^b - q_N^b\right)\nabla_\mu \frac{\mu_b}{T}$$
$$-\tau_{I_l Q}\Delta^{\mu\nu}\frac{\partial}{\partial\tau}Q_\nu - \sum_{m=1}^{N-1}\tau_{I_l I_m}\Delta^{\mu\nu}\frac{\partial}{\partial\tau}I_{n,\nu} + (\text{other terms})$$

$$\pi^{\mu\nu} = 2\eta\Delta^{\mu\nu\rho\sigma}\nabla_\rho u_\sigma - \tau_\pi \Delta^{\mu\nu\rho\sigma}\frac{\partial}{\partial\tau}\pi_{\rho\sigma} + (\text{other terms})$$

5. Summary and Outlook

Summary

- We have derived **relativistic second-order** hydrodynamic equation in **reactive multi-component system**.
- **No ansatz** is imposed in renormalization group method!!
- **New microscopic expressions** for 2nd-order transport coefficients have been obtained.

Outlook

- Reduce the chiral kinetic theory into hydrodynamics (chiral magnetic effect, chiral vortical effect, ⋯).
  - Include force terms (Lorentz force, mean field, ⋯) in the Boltzmann eq.
- Derive hydrodynamic eq. **from Kadanoff-Baym eq. directly**.
  - Include the higher loop and the off-shell effect.
- Apply the obtained eq. to phenomena (QGP, cold atom, ⋯).