Solving the NLO BK equation in coordinate space

T. Lappi

University of Jyväskylä, Finland

BNL, NT seminar, May 2015

Outline

This talk describes the results of the paper
"Direct numerical solution of the coordinate space Balitsky-Kovchegov equation at next to leading order,"
T. L., **H. Mäntysaari**, Phys. Rev. D **91** (2015) 7, 074016, [arXiv:1502.02400 [hep-ph]]

Outline of the talk:

- NLO BK equation
- Numerical result: $\ln r$ divergence
- Conformal dipole

Approach

Brute force solution of the coordinate space equation as it is written down by Balitsky & Chirilli in

Phys. Rev. D **77** (2008) 014019, [arXiv:0710.4330 [hep-ph]].

Motivation

Many ingredients available for NLO small-x calculations

- NLO BK equation
- NLO JIMWLK equation Balitsky, Chirilli, Kovner, Lublinsky, Grabovsky
- NLO γ^* impact factor for DIS Balitsky, Chirilli, Beuf
- NLO single inclusive cross section for forward pA
  Chirilli, Xiao, Yuan; Stasto, Zaslavsky, Kang, Vitev, Xing, Beuf, Altinoluk, Armesto, Kovner, Lublinsky ...

Want to start doing NLO phenomenology with these!

But first need to solve the equations and calculate the cross sections!

Dilute-dense scattering in CGC: Wilson line

Classical color field described as Wilson line

$$U(\mathbf{x}_T) = P\exp\left\{ig\int \mathrm{d}x^- A^+_{\mathrm{cov}}(\mathbf{x}_T, x^-)\right\} \quad \in \quad \mathrm{SU}(3)$$

Relation to color charge

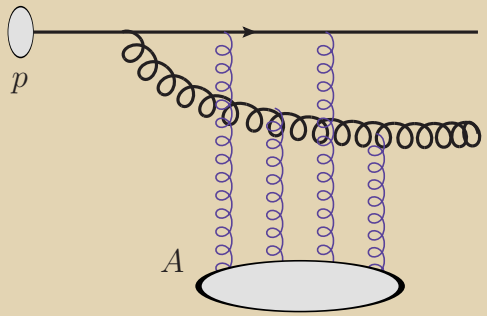
$$\nabla_T{}^2 A^+_{\mathrm{cov}}(\mathbf{x}_T, x^-) = -g\rho(\mathbf{x}_T, x^-)$$

Example of use: forward pA

- Quark from p (large x) , radiate gluon(s)
- Eikonal propagation $\Longrightarrow U(\mathbf{x}_T)$

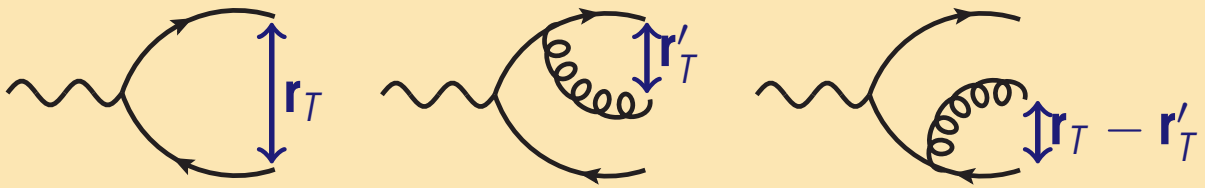
Need expectation values of operators:

$$\mathrm{Tr}\, U(\mathbf{x}_T)U^\dagger(\mathbf{y}_T) \quad \mathrm{Tr}\, U(\mathbf{x}_T)U^\dagger(\mathbf{y}_T)U(\mathbf{u}_T)U^\dagger(\mathbf{v}_T) \quad \ldots$$



BK equation

γ^* dipole scattering from dense color field



Integrate over gluon; leading soft divergence gives Balitsky-Kovchegov evolution equation $\sim$1995

$$\partial_y \mathcal{N}(\mathbf{r}_T) = \frac{\alpha_s N_c}{2\pi^2} \int d^2\mathbf{r}'_T \frac{\mathbf{r}_T{}^2}{\mathbf{r}'_T{}^2(\mathbf{r}'_T - \mathbf{r}_T)^2} \left[\mathcal{N}_{\mathbf{r}'_T} + \mathcal{N}_{\mathbf{r}_T - \mathbf{r}'_T} - \mathcal{N}_{\mathbf{r}'_T}\mathcal{N}_{\mathbf{r}_T - \mathbf{r}'_T} - \mathcal{N}_{\mathbf{r}_T}\right]$$

Notation

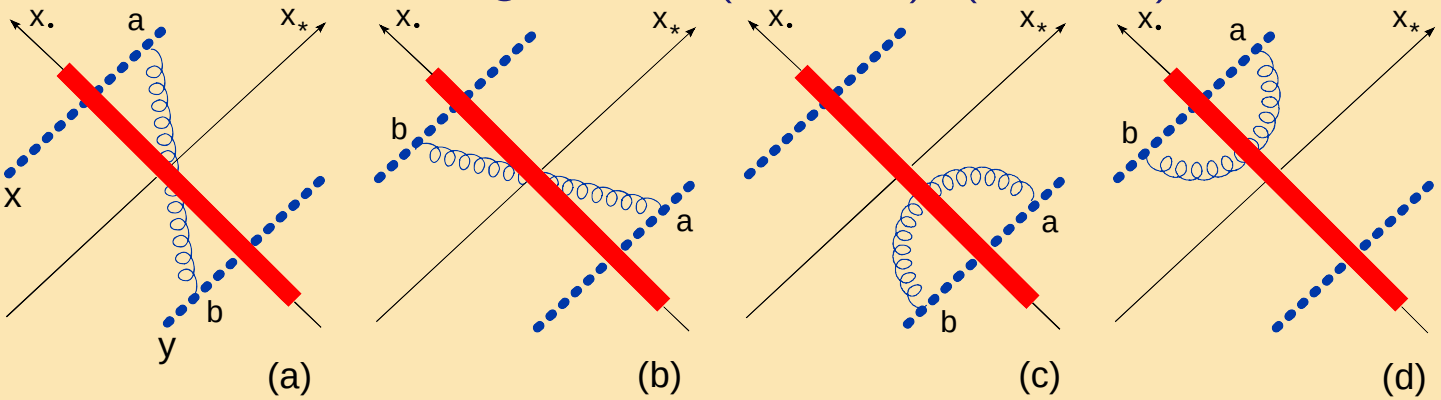
$$S(\mathbf{x}_T - \mathbf{y}_T) \equiv 1 - \mathcal{N}(\mathbf{x}_T - \mathbf{y}_T) \equiv \frac{1}{N_c}\langle U^\dagger(\mathbf{x}_T)U(\mathbf{y}_T)\rangle$$

- Linear limit: BFKL: LO & NLO known
- NLO BK equation Balitsky, Chirilli 2008
- BK is mean field approx for nonlinear term
  - Full correlations = Balitsky hierarchy = JIMWLK: LO 1997, NLO 2013

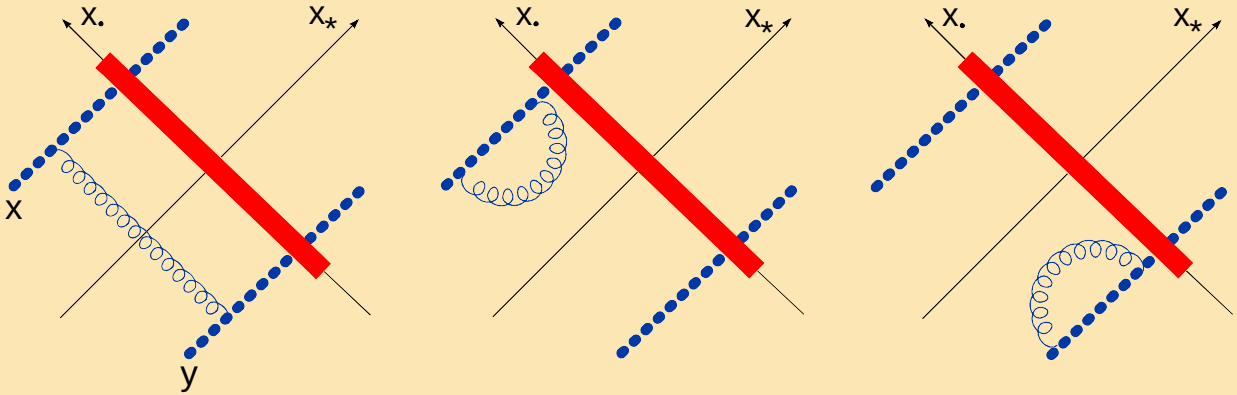
Diagrams LO

In the B&C diagrams, the leading order equation comes from:

Real diagrams: $S(\mathbf{x}_T - \mathbf{z}_T)S(\mathbf{z}_T - \mathbf{y}_T)$



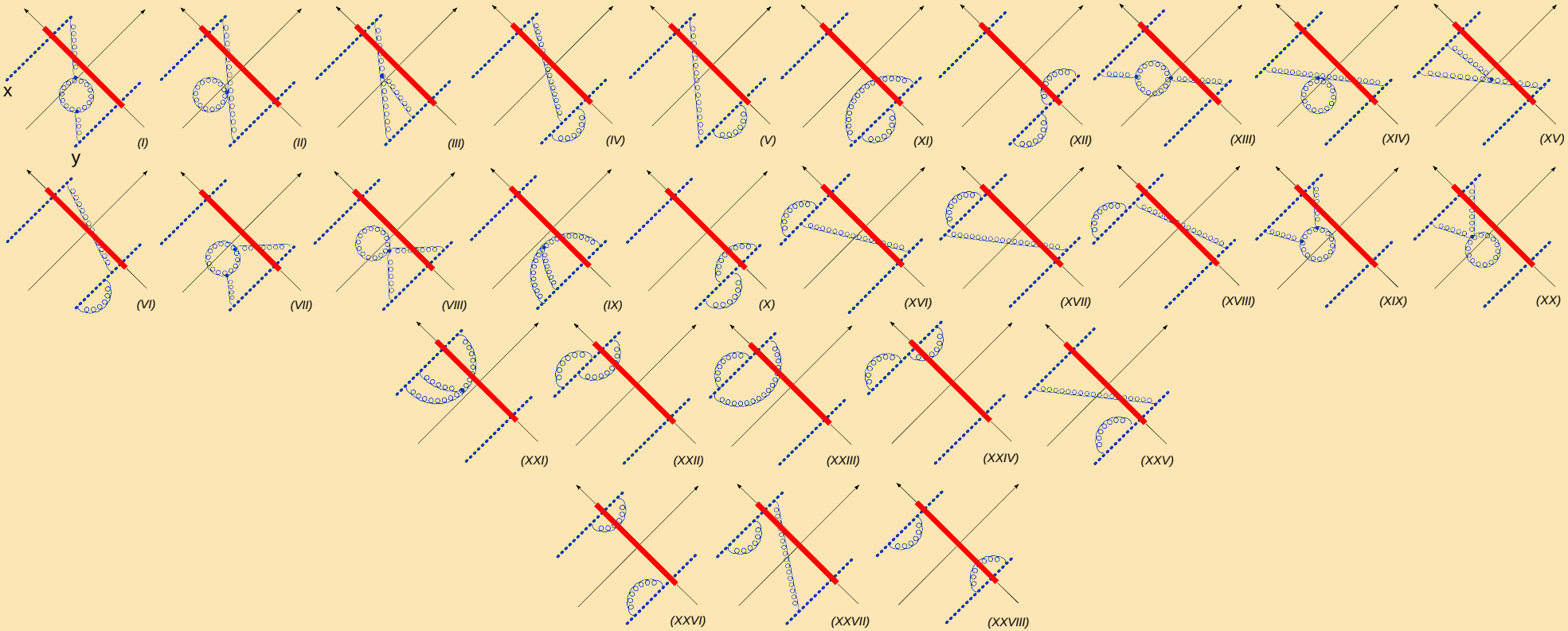
Virtual diagrams: $S(\mathbf{x}_T - \mathbf{y}_T)$



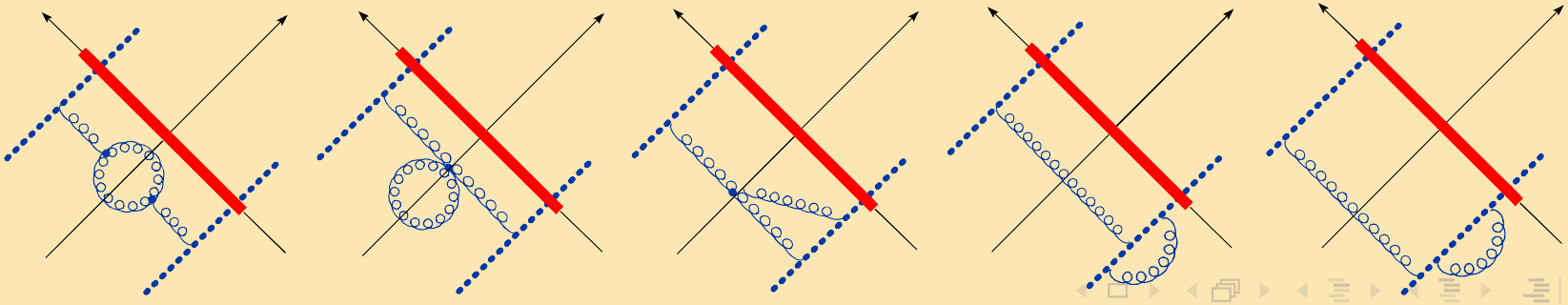
Diagrams NLO: original operator structure

As calculated by Balitsky & Chirilli

New diagrams with: $S(\mathbf{x}_T - \mathbf{z}_T)S(\mathbf{z}_T - \mathbf{y}_T)$, e.g.

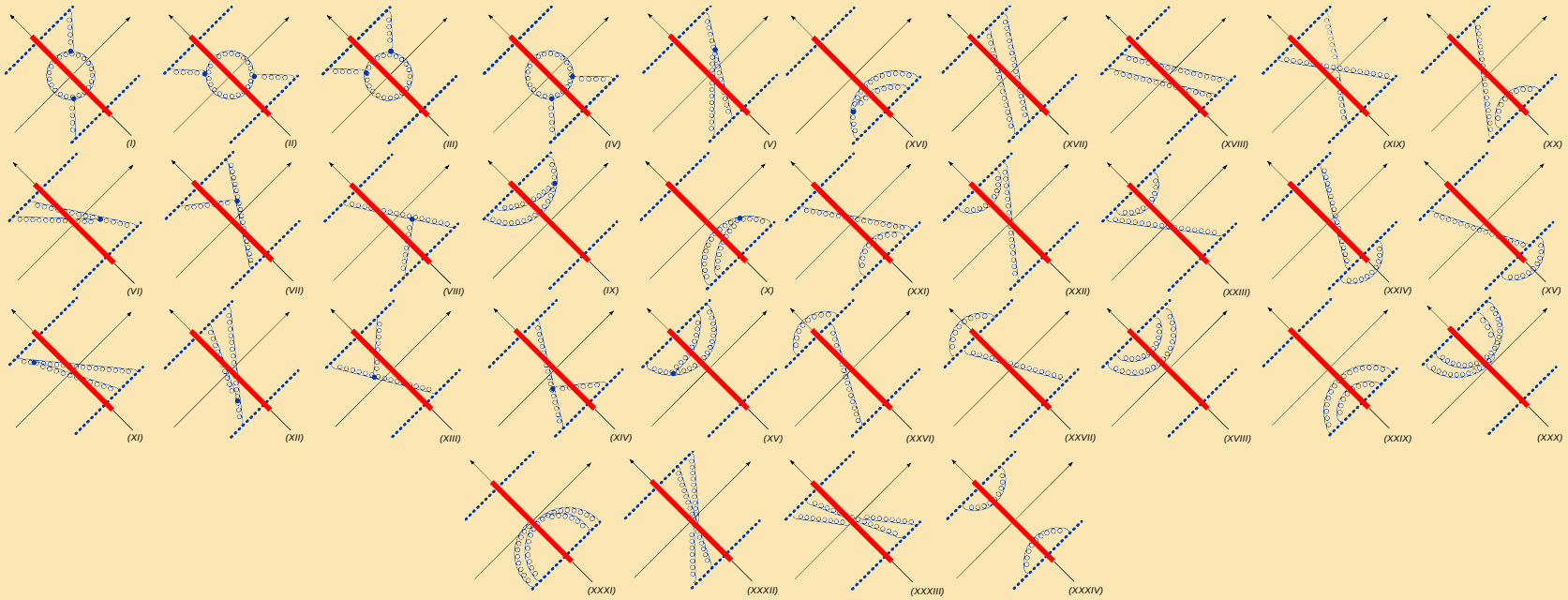


New diagrams with: $S(\mathbf{x}_T - \mathbf{y}_T)$, e.g.



Diagrams NLO: 2 gluon coordinates

Also diagrams with new structure $S(\mathbf{x}_T - \mathbf{u}_T)S(\mathbf{u}_T - \mathbf{v}_T)S(\mathbf{v}_T - \mathbf{y}_T)$, e.g.



Also diagrams with fermions.

The equation

Equation

$$\partial_y S(r) = \frac{\alpha_s N_c}{2\pi^2} K_1 \otimes [S(X)S(Y) - S(r)] + \frac{\alpha_s^2 N_F N_c}{8\pi^4} K_f \otimes S(Y)[S(X') - S(X)]$$
$$+ \frac{\alpha_s^2 {N_c}^2}{8\pi^4} K_2 \otimes [S(X)S(z-z')S(Y') - S(X)S(Y)]$$

Notations & approximations

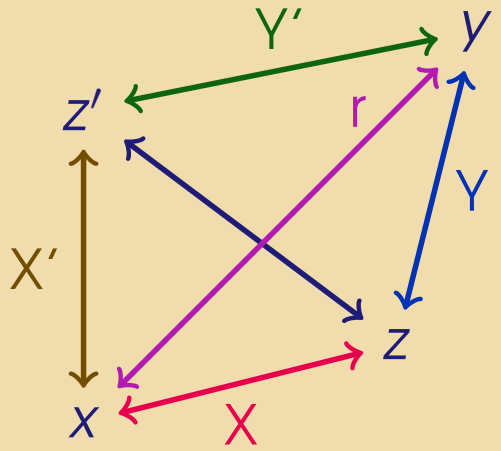
$$S(x-y) \equiv \langle \operatorname{Tr} U^\dagger(x) U(y) \rangle$$

$$\otimes = \int d^2 z \Big/ \int d^2 z\, d^2 z'$$

- Large N_c, but keep $N_F \sim N_c$
- Mean field (follows from large N_c) :

$$\langle \operatorname{Tr} U^\dagger U \operatorname{Tr} U^\dagger U \rangle \to \langle \operatorname{Tr} U^\dagger U \rangle \langle \operatorname{Tr} U^\dagger U \rangle$$

Coordinates



Kernels

$$K_1 = \frac{r^2}{X^2 Y^2}\left[1 + \frac{\alpha_s N_c}{4\pi}\left(\frac{\beta}{N_c}\ln r^2\mu^2 - \frac{\beta}{N_c}\frac{X^2 - Y^2}{r^2}\ln\frac{X^2}{Y^2}\right.\right.$$
$$\left.\left. + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{N_F}{N_c} - \ln\frac{X^2}{r^2}\ln\frac{Y^2}{r^2}\right)\right]$$

$$K_2 = -\frac{2}{(z-z')^4} + \left[\frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\right.$$
$$\left. + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z-z')^2}\right]\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

$$K_f = \frac{2}{(z-z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

Kernels

$$K_1 = \frac{r^2}{X^2 Y^2}\left[1 + \frac{\alpha_s N_c}{4\pi}\left(\frac{\beta}{N_c}\ln r^2\mu^2 - \frac{\beta}{N_c}\frac{X^2 - Y^2}{r^2}\ln\frac{X^2}{Y^2}\right.\right.$$
$$\left.\left. + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{N_F}{N_c} - \ln\frac{X^2}{r^2}\ln\frac{Y^2}{r^2}\right)\right]$$

$$K_2 = -\frac{2}{(z-z')^4} + \left[\frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\right.$$
$$\left. + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z-z')^2}\right]\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

$$K_f = \frac{2}{(z-z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

- K_1: original operator structure $SS - S$

Kernels

$$K_1 = \frac{r^2}{X^2 Y^2}\left[1 + \frac{\alpha_s N_c}{4\pi}\left(\frac{\beta}{N_c}\ln r^2\mu^2 - \frac{\beta}{N_c}\frac{X^2 - Y^2}{r^2}\ln\frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{N_F}{N_c} - \ln\frac{X^2}{r^2}\ln\frac{Y^2}{r^2}\right)\right]$$

$$K_2 = -\frac{2}{(z-z')^4} + \left[\frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z-z')^2}\right]\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

$$K_f = \frac{2}{(z-z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

- K_1: original operator structure $SS - S$
- K_2: new operator structure $SSS - SS$

Kernels

$$K_1 = \frac{r^2}{X^2 Y^2}\left[1 + \frac{\alpha_s N_c}{4\pi}\left(\frac{\beta}{N_c}\ln r^2\mu^2 - \frac{\beta}{N_c}\frac{X^2 - Y^2}{r^2}\ln\frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{N_F}{N_c} - \ln\frac{X^2}{r^2}\ln\frac{Y^2}{r^2}\right)\right]$$

$$K_2 = -\frac{2}{(z-z')^4} + \left[\frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z-z')^2}\right]\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

$$K_f = \frac{2}{(z-z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

- K_1: original operator structure $SS - S$
- K_2: new operator structure $SSS - SS$
- Leading order

Kernels

$$K_1 = \frac{r^2}{X^2 Y^2}\left[1 + \frac{\alpha_s N_c}{4\pi}\left(\frac{\beta}{N_c}\ln r^2\mu^2 - \frac{\beta}{N_c}\frac{X^2 - Y^2}{r^2}\ln\frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{N_F}{N_c} - \ln\frac{X^2}{r^2}\ln\frac{Y^2}{r^2}\right)\right]$$

$$K_2 = -\frac{2}{(z-z')^4} + \left[\frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z-z')^2}\right]\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

$$K_f = \frac{2}{(z-z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

- K_1: original operator structure $SS - S$
- K_2: new operator structure $SSS - SS$
- Leading order
- Running coupling (Terms with β function coefficient)

Kernels

$$K_1 = \frac{r^2}{X^2 Y^2}\left[1 + \frac{\alpha_s N_c}{4\pi}\left(\frac{\beta}{N_c}\ln r^2\mu^2 - \frac{\beta}{N_c}\frac{X^2 - Y^2}{r^2}\ln\frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{N_F}{N_c} - \ln\frac{X^2}{r^2}\ln\frac{Y^2}{r^2}\right)\right]$$

$$K_2 = -\frac{2}{(z-z')^4} + \left[\frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z-z')^2}\right]\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

$$K_f = \frac{2}{(z-z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

- K_1: original operator structure $SS - S$
- K_2: new operator structure $SSS - SS$
- Leading order
- Running coupling (Terms with β function coefficient)
- Conformal logs $\Longrightarrow$ vanish for $r = 0$ ($X = Y$ & $X' = Y'$)

Kernels

$$K_1 = \frac{r^2}{X^2 Y^2}\left[1 + \frac{\alpha_s N_c}{4\pi}\left(\frac{\beta}{N_c}\ln r^2\mu^2 - \frac{\beta}{N_c}\frac{X^2 - Y^2}{r^2}\ln\frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{N_F}{N_c} - \ln\frac{X^2}{r^2}\ln\frac{Y^2}{r^2}\right)\right]$$

$$K_2 = -\frac{2}{(z-z')^4} + \left[\frac{X^2 Y'^2 + X'^2 Y^2 - 4r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{r^2}{X^2 Y'^2(z-z')^2}\right]\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

$$K_f = \frac{2}{(z-z')^4} - \frac{X'^2 Y^2 + Y'^2 X^2 - r^2(z-z')^2}{(z-z')^4(X^2 Y'^2 - X'^2 Y^2)}\ln\frac{X^2 Y'^2}{X'^2 Y^2}$$

- K_1: original operator structure $SS - S$
- K_2: new operator structure $SSS - SS$
- Leading order
- Running coupling (Terms with β function coefficient)
- Conformal logs $\Longrightarrow$ vanish for $r = 0$ ($X = Y$ & $X' = Y'$)
- Nonconformal double log $\Longrightarrow$ blows up for $r = 0$

Running coupling

Absorb the β-terms into

- "Balitsky" running for LO term
- Parent dipole running for NLO terms $\Longrightarrow$ not determined by calculation at this order

What was before:

$$\frac{\alpha_s N_c}{2\pi^2} K_1 = \frac{\alpha_s N_c}{2\pi^2} \frac{r^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{\beta}{N_c} \ln r^2 \mu^2 - \frac{\beta}{N_c} \frac{X^2 - Y^2}{r^2} \ln \frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{N_F}{N_c} - \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right) \right]$$

is now:

$$\frac{\alpha_s N_c}{2\pi^2} K_1 = \frac{\alpha_s(r) N_c}{2\pi^2} \left[\frac{r^2}{X^2 Y^2} + \frac{1}{X^2} \left(\frac{\alpha_s(X)}{\alpha_s(Y)} - 1 \right) + \frac{1}{Y^2} \left(\frac{\alpha_s(Y)}{\alpha_s(X)} - 1 \right) \right] + \frac{\alpha_s(r)^2 {N_c}^2}{8\pi^3} \frac{r^2}{X^2 Y^2} \left[\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{N_F}{N_c} - 2 \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \right]$$

Initial condition

$$N(r) \equiv 1 - S(r) = 1 - \exp\left[-\frac{(r^2 Q_{s0}^2)^\gamma}{4} \ln\left(\frac{1}{r\Lambda_{QCD}} + e\right)\right],$$

2 tunable parameters

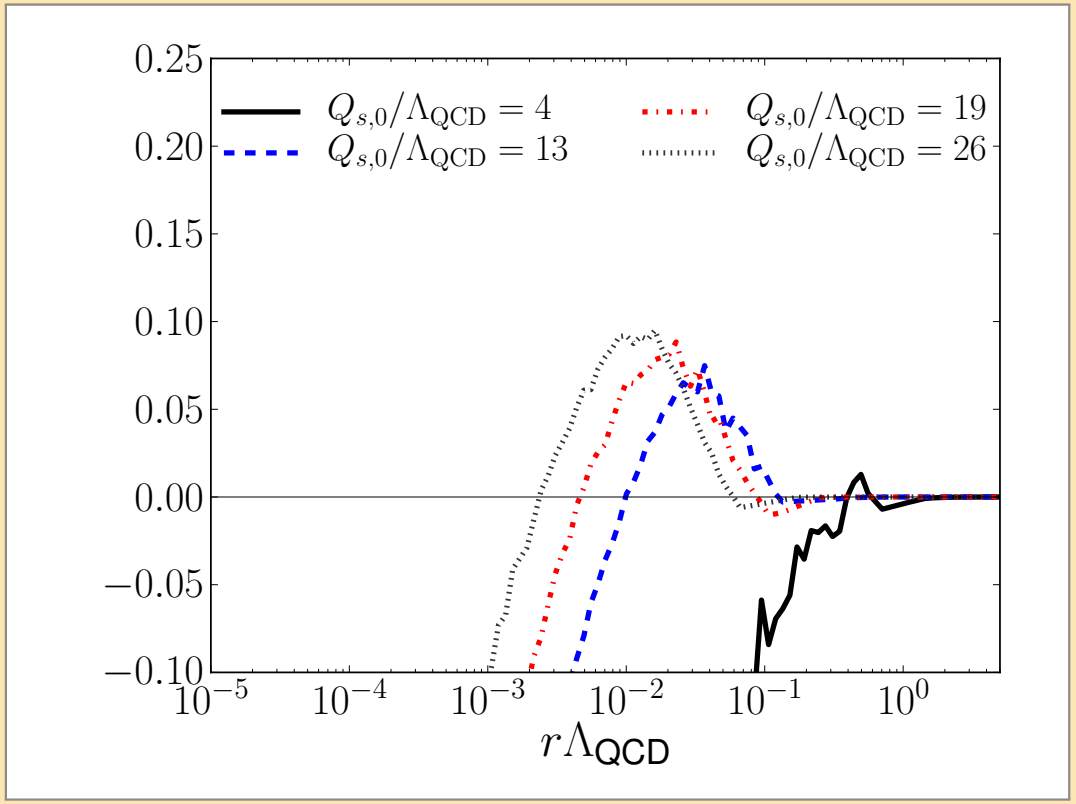
- $\frac{Q_{s0}}{\Lambda_{QCD}} \Longrightarrow$ basically determines value of α_s
- γ: anomalous dimension: shape
  - LO phenomenology prefers $\gamma \gtrsim 1$
  - This eventually evolves into $\gamma \sim 0.8$ (running α_s)

Evolution speed at initial condition

ln r divergence

$\gamma = 1$ (MV model)

- Small Q_s/Λ_{QCD}
  $\Longrightarrow$ large α_s
  NLO corrections big, amplitude decreases at all r
- For smaller α_s region around $r \sim 1/Q_s$ is ok.
- For small dipoles $\partial_y N/N \sim \ln r$



$\partial_y N/N$

$N < 0$ as a practical problem

LO equation

$$\partial_Y S(r) = \frac{\alpha_s N_c}{2\pi^2} \int_z \frac{r^2}{X^2 Y^2} [N(X) + N(Y) - N(r) - N(X)N(Y)]$$

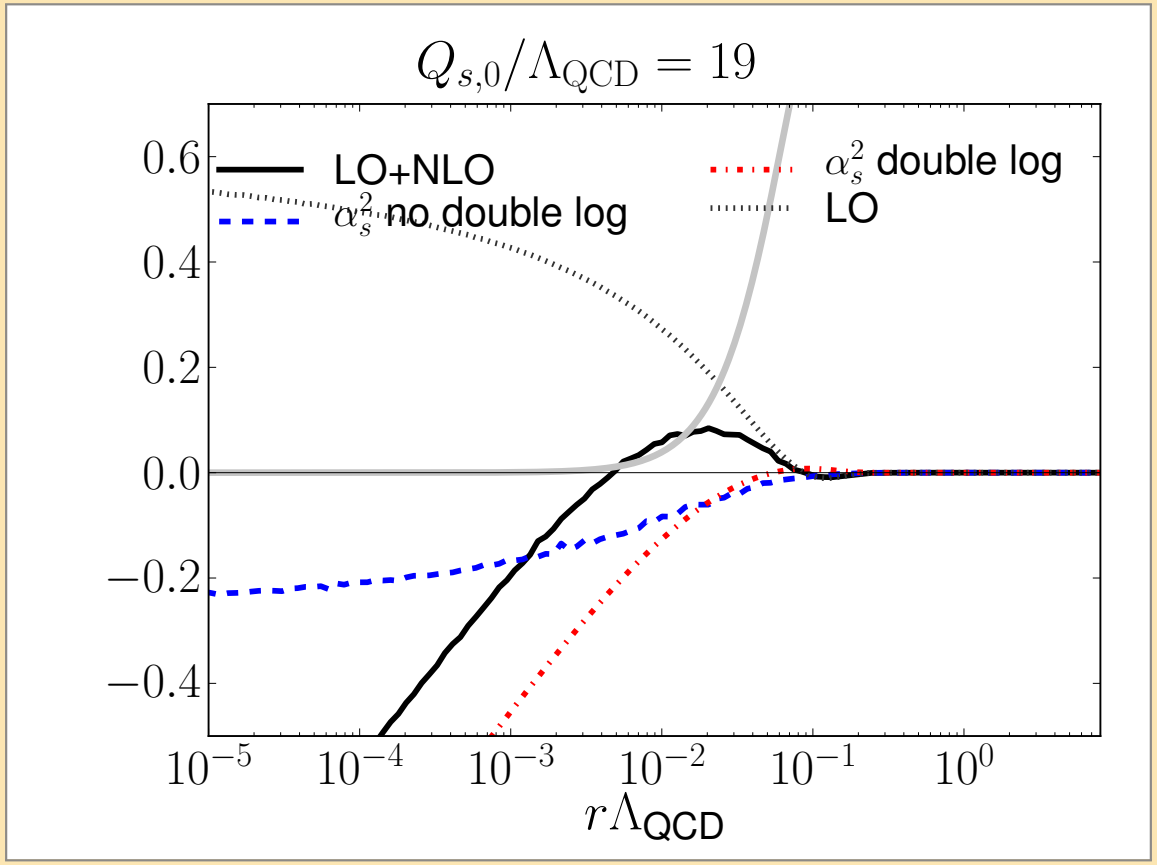
Consider a small but finite dy (as in a numerical solution)
A diverging $\partial_Y N/N$ makes $N < 0$ in one rapidity step
Convergence of the z-integral on the r.h.s. of the (LO!) BK equation requires $N(r) \to 0$ for $r \to 0$ (Limit $X \to 0$, $Y \to r$ in integral)

$\Longrightarrow$ If $N < 0$ for $r \to 0$:

- The equation blows up mathematically
- Solution is inconsistent with the definition $N(\mathbf{x}_T - \mathbf{y}_T) = 1 - \frac{1}{N_c} \mathrm{Tr}\, U^\dagger(\mathbf{x}_T) U(\mathbf{y}_T)$

In the numerics, we enforce $N(r) \geq 0$ by hand.

Origin of the negativity

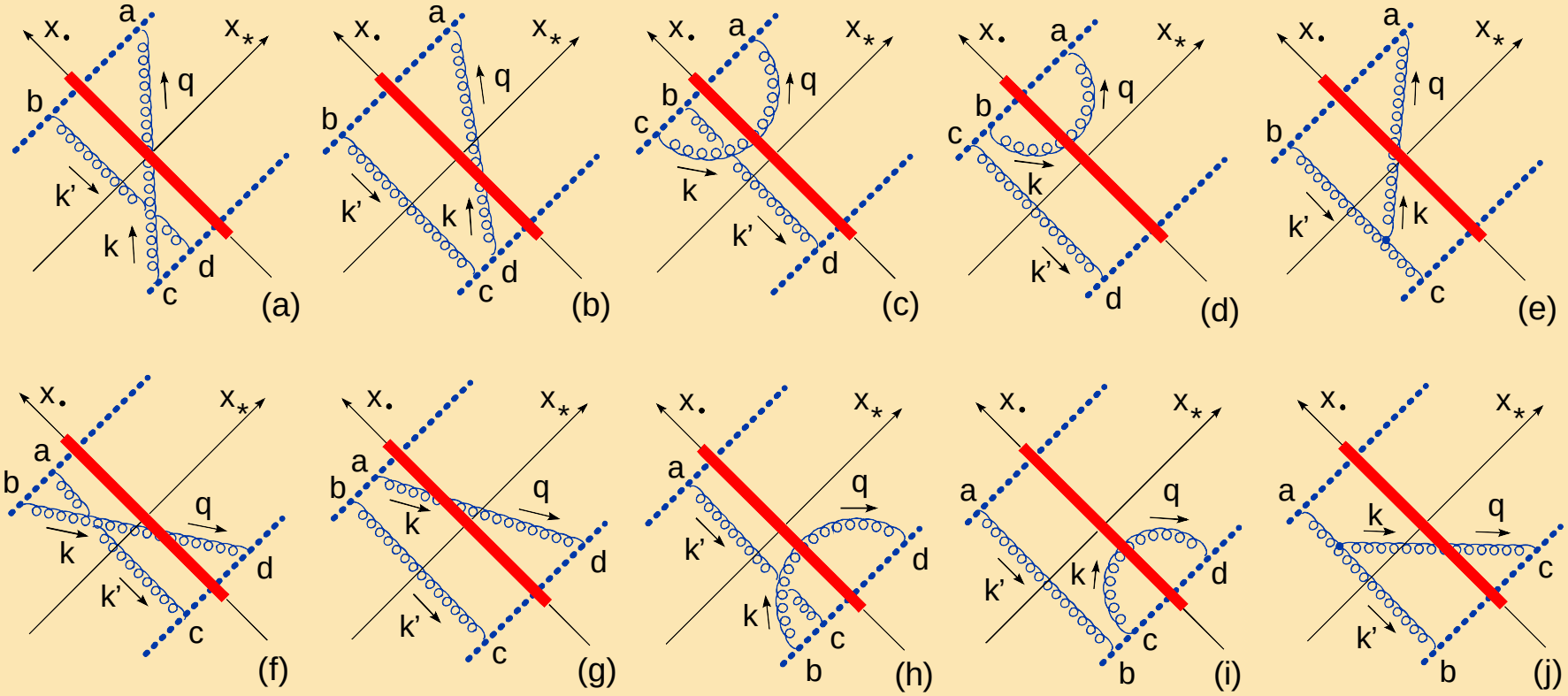


$\partial_Y N/N$

In r behavior caused by the nonconformal double log term

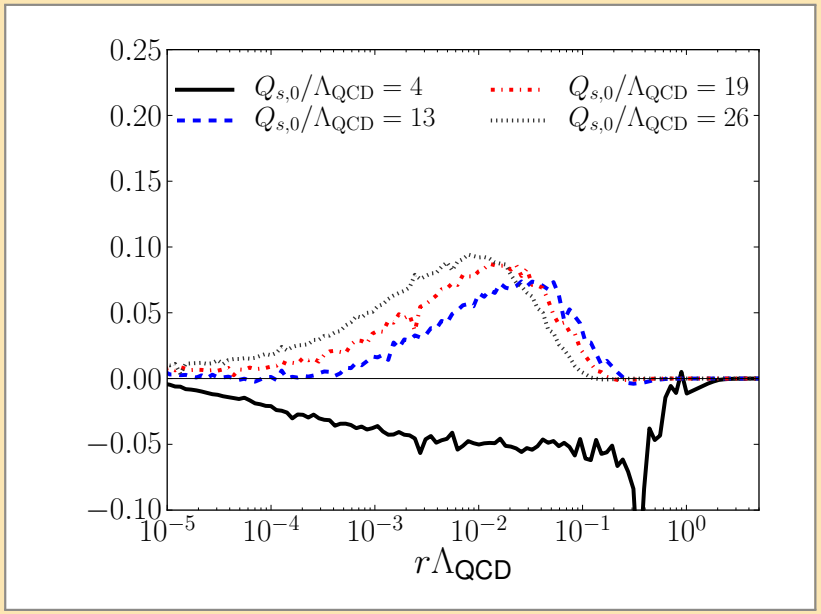
Nonconformal double log in diagrams

Problematic term comes from the following diagrams:

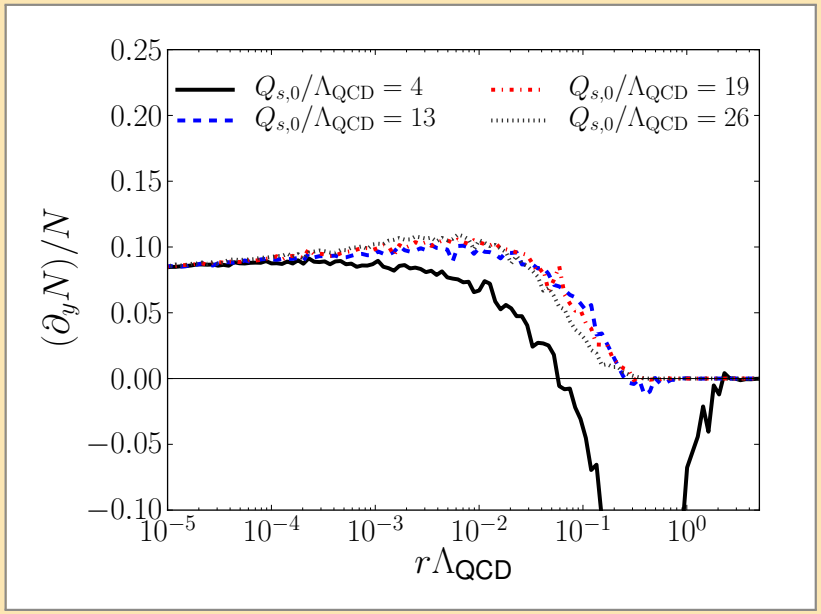


Changing the initial condition

Decrease γ



$\partial_y N/N$



$\partial_y N/N$

$\gamma = 0.8$

- Initally looks ok, if α_s small enough
- But γ gradually increases, blows up by $y \gtrsim 20$

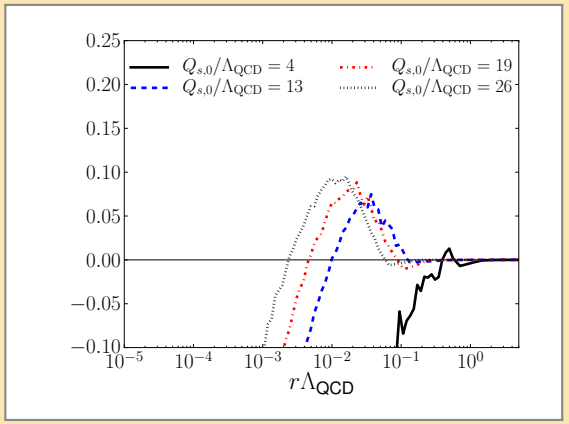
$\gamma = 0.6$

- Requires α_s small enough
- Solution behaves ok at least up to $y = 30$

Interpreting the $\partial_y N/N$ plot

- LO: roughly $N(r) \sim (Q_s r)^{2\gamma}$ & $Q_s^2 \sim e^{\lambda y}$
  $\Longrightarrow \partial_y N/N \approx 2\gamma\lambda > 0$
- If $\partial_y N/N \to -c < 0$ for $r \to 0$,
  $\Longrightarrow N \sim e^{-cy} \Longrightarrow$ still ok

  (But why would $N(y)$ decrease with y?)

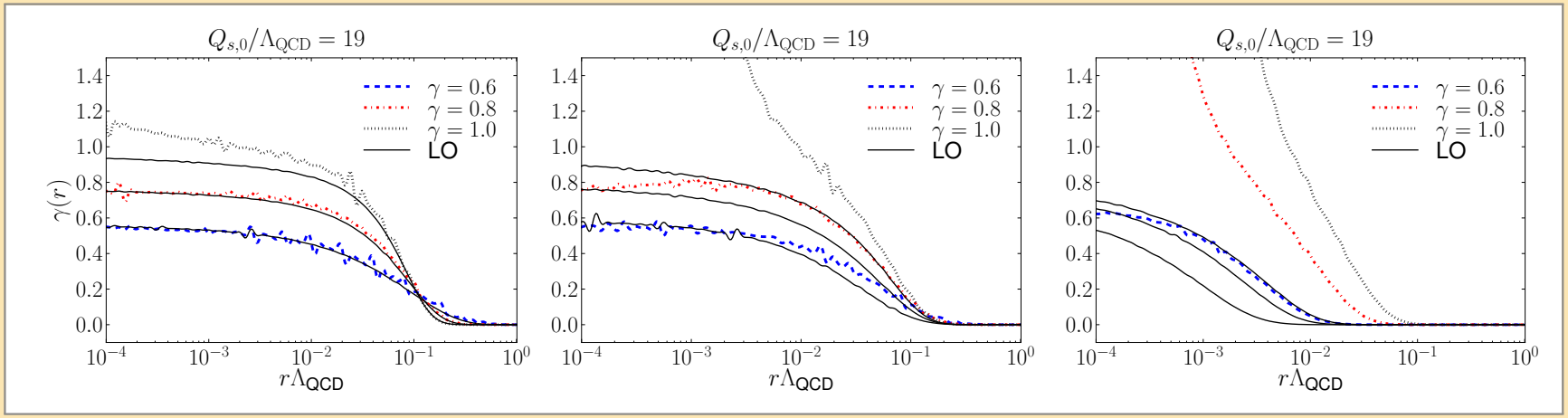


Parametrizing $N(r) \sim (Q_s r)^{2\gamma(y)}$ $\quad \partial_y N/N \sim c \ln r \Longrightarrow \gamma(y) \sim y$

$\Longrightarrow$ front gets steeper $\quad \Longrightarrow$ eventually $N(r) \sim \theta(r - 1/Q_s)$

(This is assuming c =const, but in numerics c grows $\Longrightarrow$ divergence worse)

Plot $\gamma(r) \equiv \mathrm{d}\ln N(r)/\mathrm{d}\ln r^2$:



$y = 1$ $\quad$ $y = 5$ $\quad$ $y = 30$

Conformal composite dipole

Proposal by Balitsky & Chirilli,

Nucl. Phys. B **822** (2009) 45 [arXiv:0903.5326 [hep-ph]] :

absorb logarithm into redefinition of dipole operator:

$$S(r)^{\mathrm{conf}} = S(r) - \frac{\alpha_s N_c}{4\pi^2} \int \mathrm{d}^2 z \frac{r^2}{X^2 Y^2} \ln \frac{a r^2}{X^2 Y^2} [S(X)S(Y) - S(r)].$$

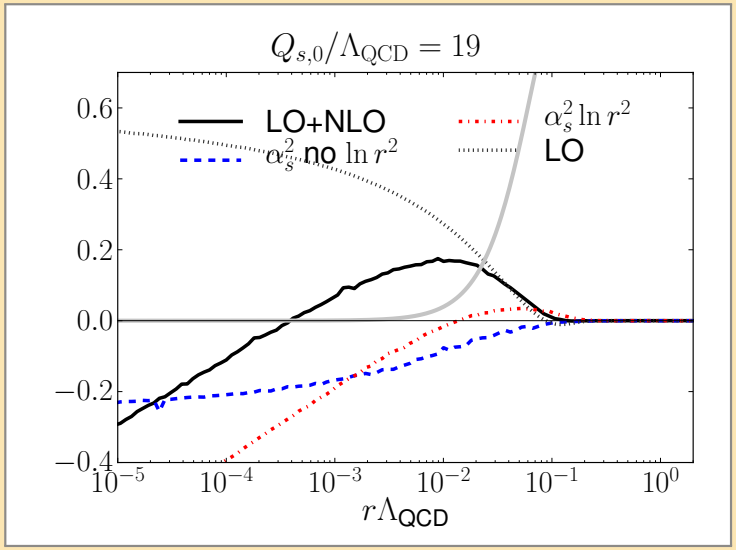
- a: dimensionful constant, cancels out in the end
- Double log $\ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2}$ drops out from K_1
- New term $\frac{2r^2}{X^2 Y'^2 (z-z')^2} \ln \frac{r^2 (z-z')^2}{X'^2 Y^2}$ appears in kernel K_2

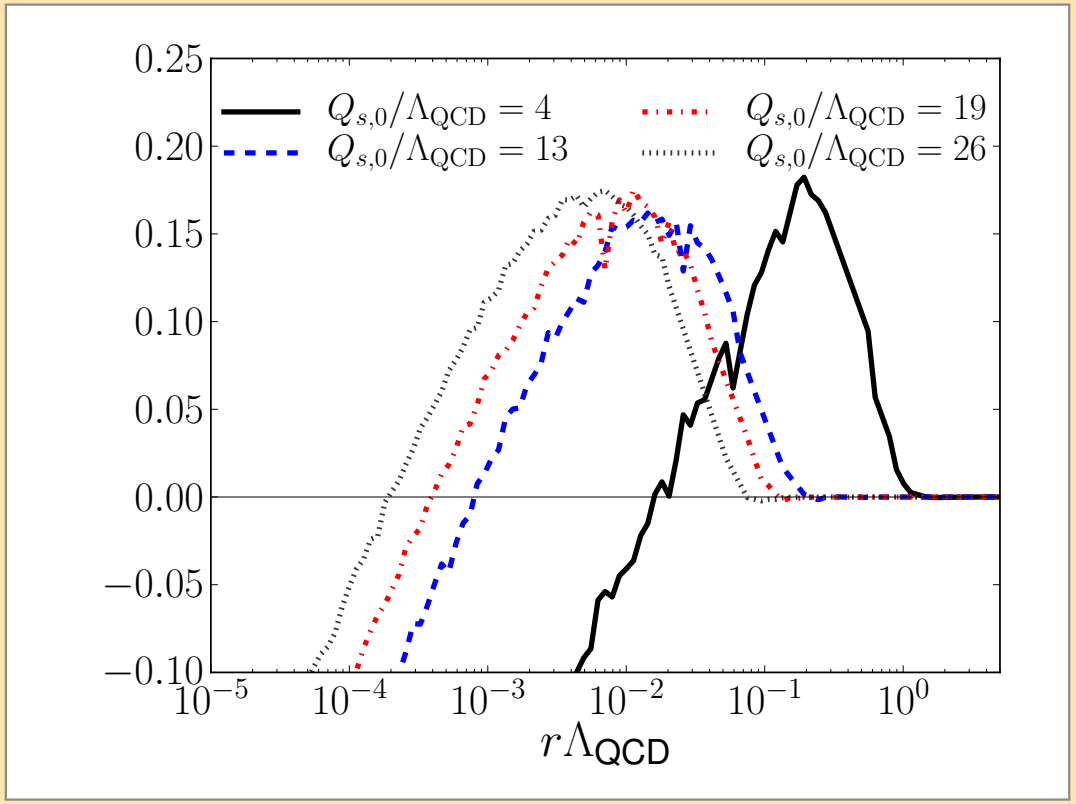
Evolution speed at $y = 0$: conformal dipole

$\gamma = 1$ (MV model)

- $r \sim 1/Q_s$ ok, even for large α_s
- For small dipoles $\partial_y N/N \sim \ln r$ still

As expected: reason is new $\ln r$ term:





$\partial_y N/N$

$\gamma < 1$ similar as for original equation.

Conclusions

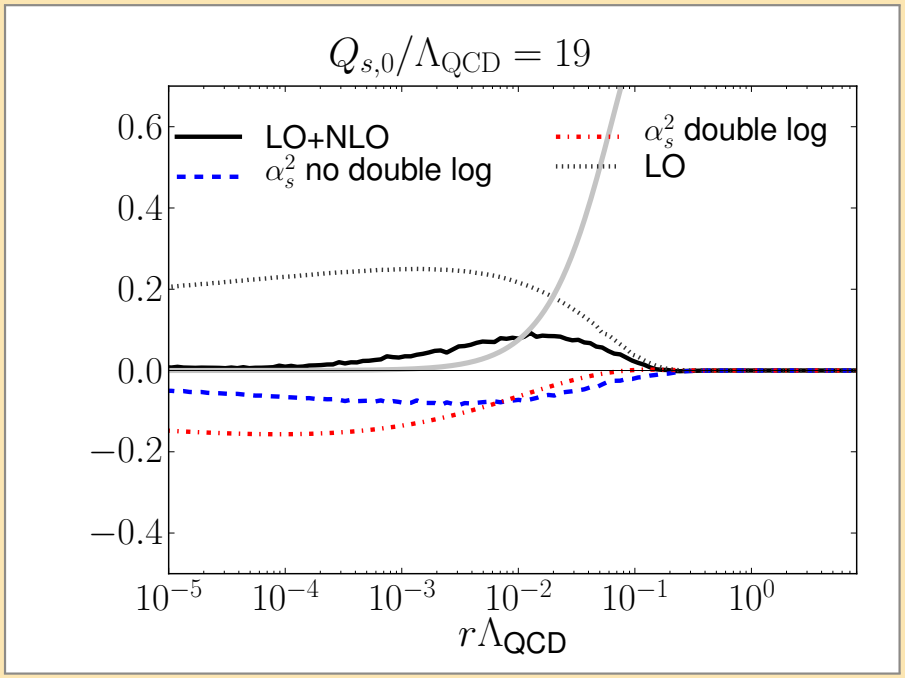
- NLO corrections mostly negative: slow down evolution
  - This is good for phenomenology
- Divergence $\sim \ln r$ for small r problematic
  - Makes solving the equation for $\gamma \sim 1$ impossible
- Problem at small r $\Longrightarrow$ presumably related to large Q^2 logs $\Longrightarrow$ resummation needed?
  - Agrees with Avsar, Stasto, Triantafyllopoulos, Zaslavsky 2011
  - Problem same as for NLO BFKL ?
  - BFKL Resummation in Mellin space: Salam, Ciafaloni, Forte ...
  - No Mellin space resummation for BK (nonlinear)
  - How to resum directly in coordinate space? Recent suggestion Iancu, Madriga, Mueller, Soyez, Triantafyllopoulos 3/2015 :

$$1 - \ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2} \quad \Longrightarrow \quad \frac{J_1\left(\sqrt{\ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2}}\right)}{\sqrt{\ln \frac{X^2}{r^2} \ln \frac{Y^2}{r^2}}}$$

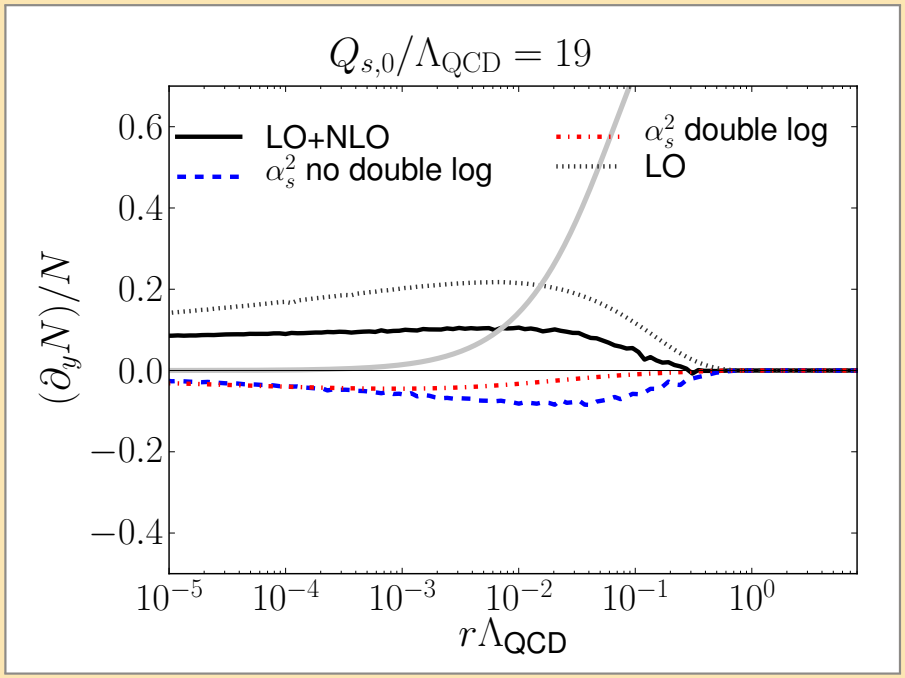
- State of the art for phenomenology is still LO+running α_s

Changing the initial condition

Decrease γ

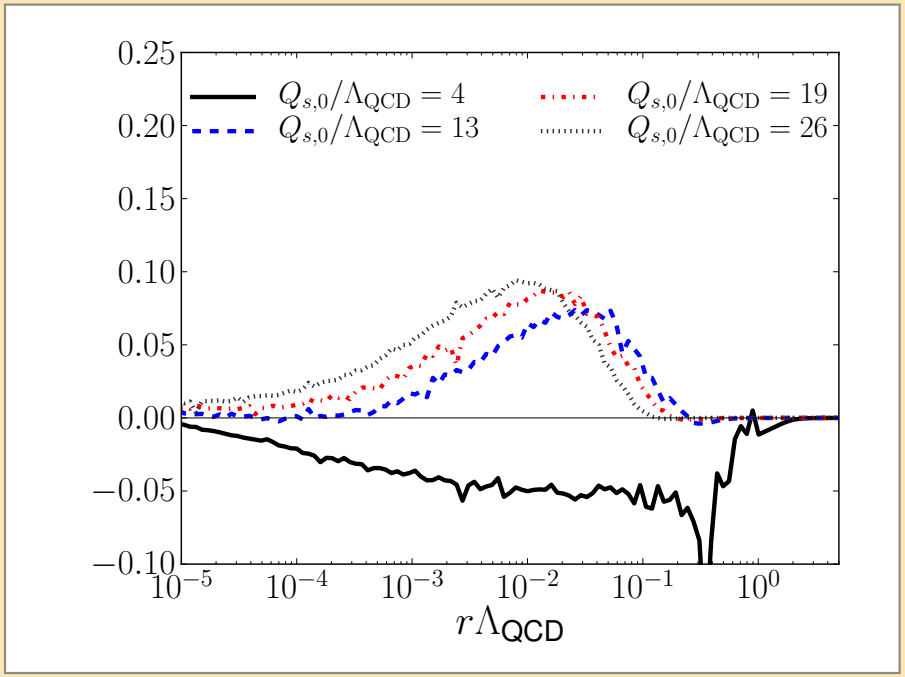


$\gamma = 0.8$

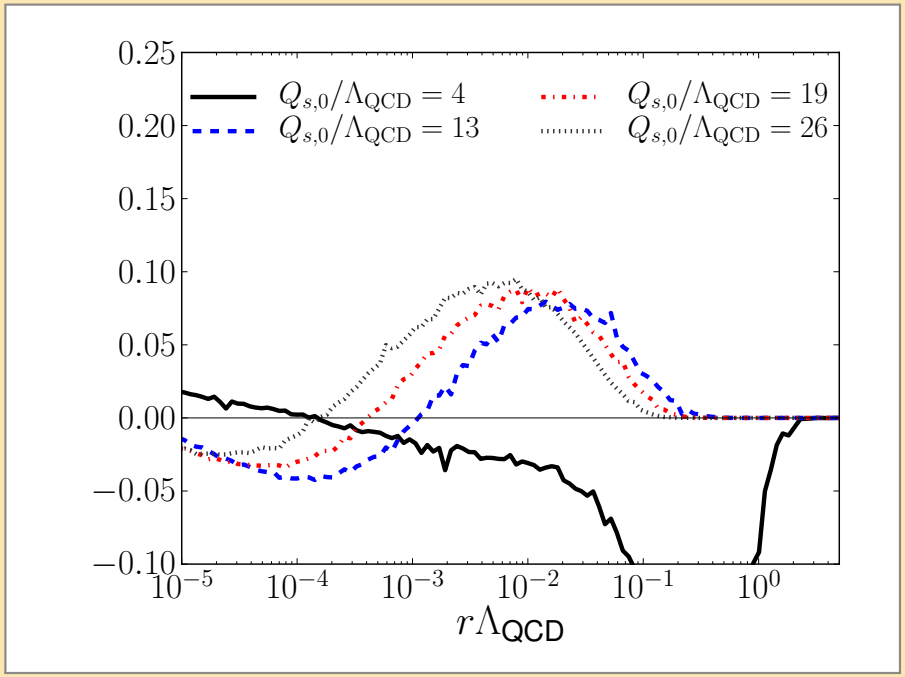


$\gamma = 0.6$

Evolving instability for $\gamma = 0.8$



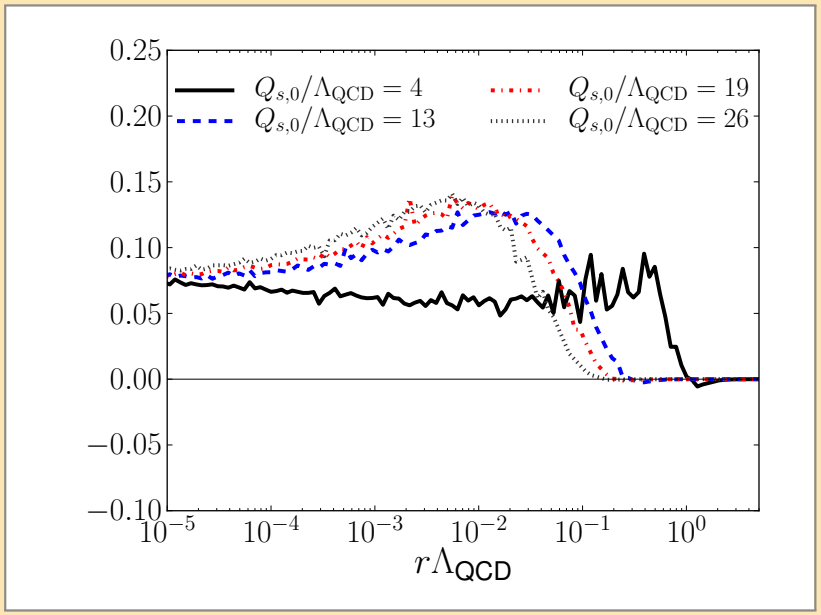
$y = 0$

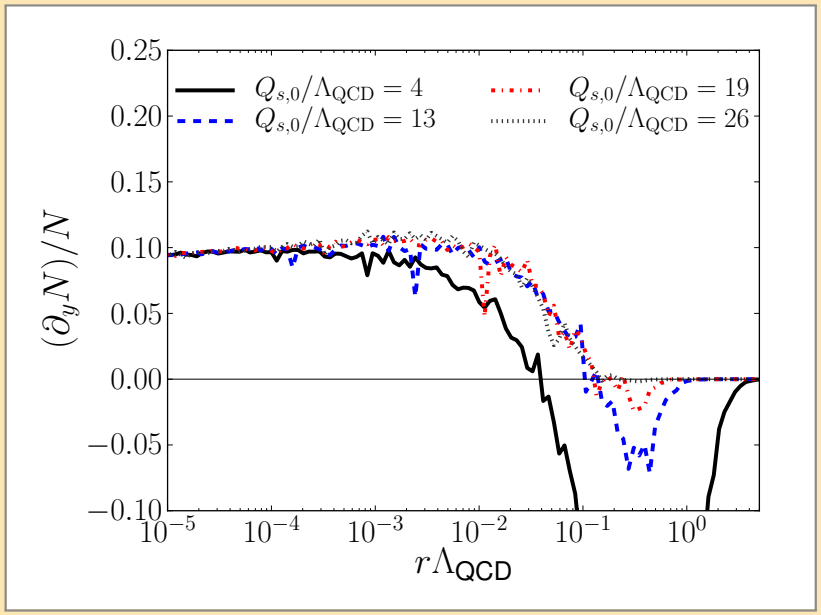


$y = 5$

Changing the initial condition: conformal dipole

Decrease γ





$\gamma = 0.8$

- Kind of ok $y \gtrsim 20$

$\gamma = 0.6$

- Small r ok, large r starts to be erratic