Dynamics of the chiral phase transition
– Universal fluctuations near criticality –

David Mesterházy

University of Illinois at Chicago

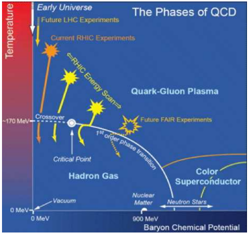
BNL Nuclear Theory Seminar, October 24, 2014

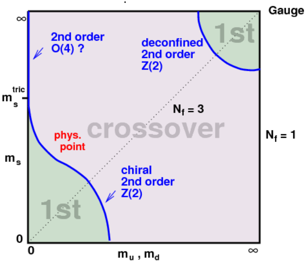
Introduction

The many-body physics of relativistic non-Abelian gauge theories plays an important role in current heavy ion collision experiments and the astrophysics of compact stars. One encounters similar problems in certain dark matter and leptogenesis computations in cosmology, as well as condensed matter and cold atom environments.

The dynamical properties of these many-body systems are encoded in various spectral functions where, in particular, their zero-frequency limits define transport coefficients.

Introduction





Numerical lattice Monte Carlo simulations have been very successful to determine static properties of strongly interacting matter. The reconstruction of real-time quantities based on analytic continuation of Euclidean correlation functions using Bayesian methods such as MEM however is often difficult.

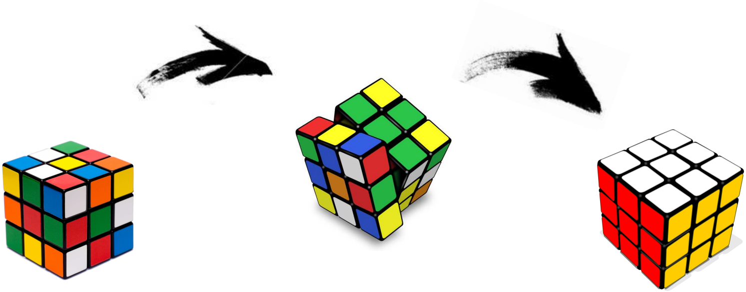
We may employ suitable low-energy effective theories to develop a general understanding of the qualitative structure of spectral functions and associated transport. But what are suitable effective dynamics?

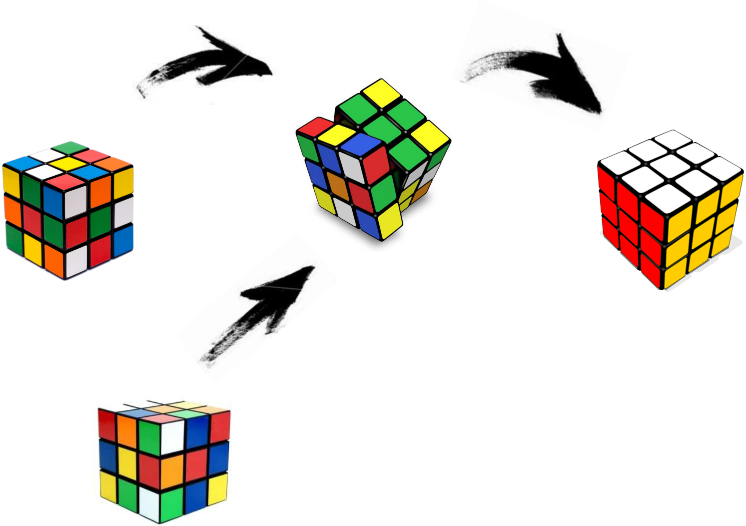
From microscopic to low energy effective dynamics

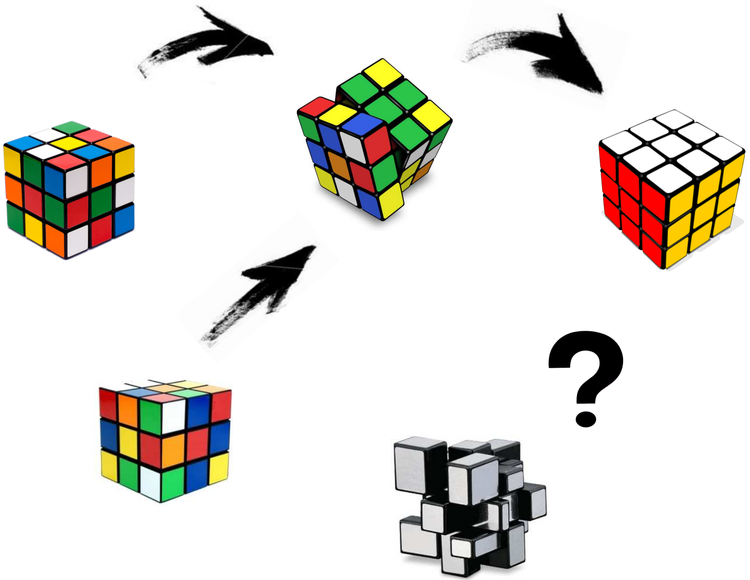
Given a characteristic scale, we typically identify the relevant hydrodynamic degrees of freedom. This provides an appropriate description of the long-wavelength and small-frequency fluctuations in the system away from the critical temperature.

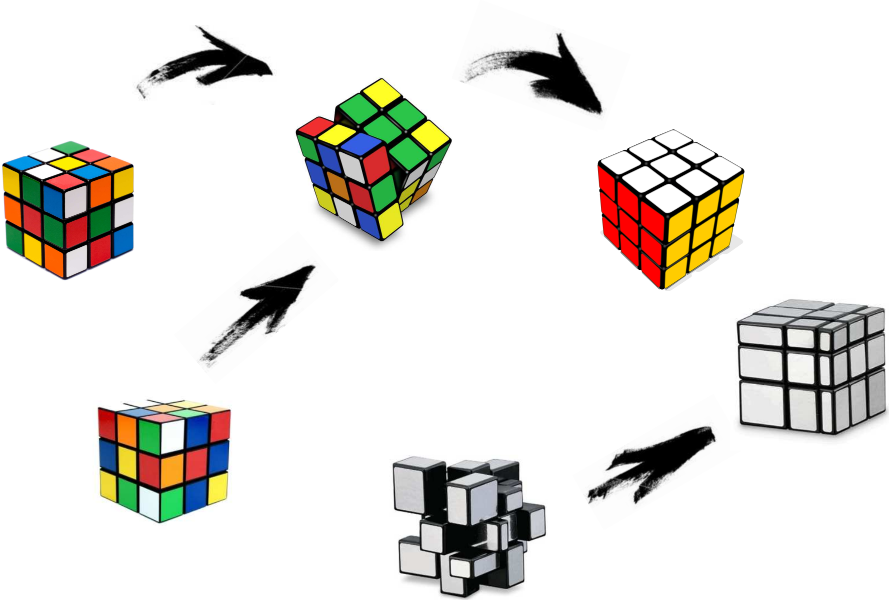
Close to the critical point we need to consider additional slow modes to determine the associated dynamic universality class. In particular, the magnitude of the order parameter will exhibit critical slowing down and might couple to other conserved charges.

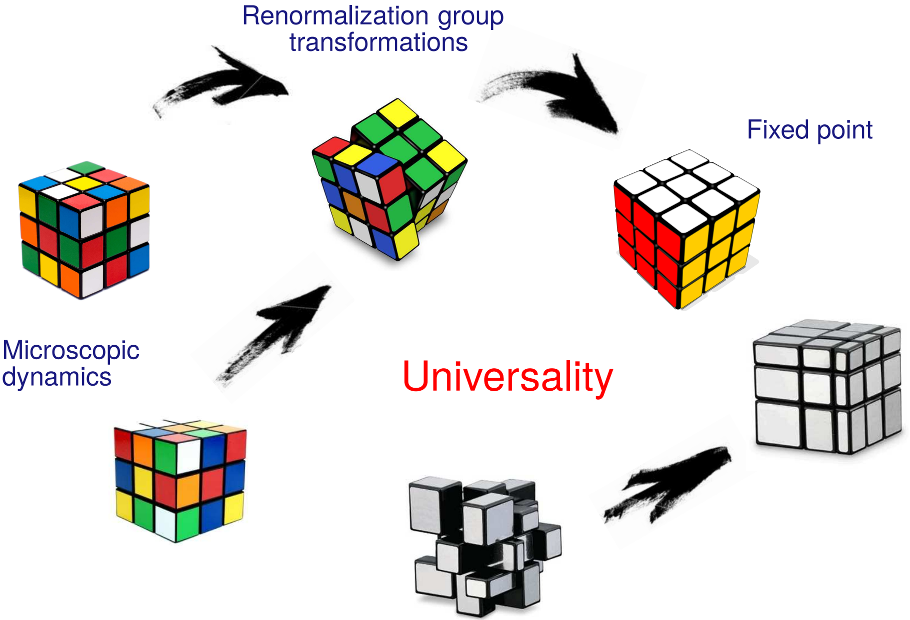
This approach cannot describe the transition from unitary dynamics to dissipative (or possibly dissipationless) transport. In principle, this requires the solution of the full generating functional for correlation functions on a closed time path (Schwinger/Keldysh).










Renormalization group transformations
Fixed point
Microscopic dynamics
Universality

From microscopic to low energy effective dynamics

Low energy effective dynamics

$$\Gamma_{k\to 0} = \Gamma_{1PI}$$

Thermalization?

Long time asymptotics?

$$\frac{\partial \Gamma_k}{\partial s} = \frac{i}{2}\,\mathrm{Tr}\int \frac{d^d q}{(2\pi)^d}\frac{d\omega}{2\pi}\frac{\partial}{\partial s}R_k(q,\omega)\,G_k(q,\omega)$$

$$s = \ln k/\Lambda$$

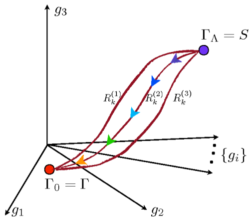
Exact nonperturbative flow equation

Microscopic dynamics

$$\Gamma_{k\to\Lambda} \sim S$$

Unitary time evolution,

sensitive dependence on initial state



The dynamic zoo

At the critical point the system satisfies the scaling assumption. In thermal equilibrium, we can phrase the presence of scaling in terms of the spectral function of the order parameter:

$$\rho(sq, s^z\omega) = s^{-2+\eta}\rho(q,\omega)$$

where z is the dynamic critical exponent, and η is the anomalous dimension.

The coupling to additional fields might be relevant and determine the dynamic unversality class.

Model A	relaxation	Kinetic Ising model, anisotropic magnet	–	$z = 2 + c\eta$
Model B	diffusion	Kinetic Ising model, uniaxial ferromagnet	–	$z = 4 - \eta$
Model C	relaxation	Anisotropic magnet, structural transitions	–	$z = 2 + \frac{\alpha}{\nu}$
Model E		Planar magnet	Spin wave	$z = d/2$
Model F		Planar magnet, superfluid	Second sound	$z = d/2$
Model G		Heisenberg antiferromagnet	Spin wave	$z = d/2$
Model H		Gas-liquid binary fluid	–	$z = d + x_\eta$
Model J		Heisenberg ferromagnet	Spin wave	$z = \frac{1}{2}(d + 2 - \eta)$

Dynamic universality class of the QCD critical point

Near the critical end point of the chiral phase transition in QCD, the modes potentially important for hydrodynamics are given by the fluctuations of:

- the conserved energy and momentum densities, $\varepsilon = T^{00} - \langle T^{00} \rangle$ and $\pi^i = T^{0i}$,
- the conserved baryon number density, $n = \bar{q}\gamma^0 q - \langle \bar{q}\gamma^0 q \rangle$,
- the chiral condensate, $\sigma = \bar{q}q - \langle \bar{q}q \rangle$.

The non-vanishing coupling $[\pi_i(x), \sigma(y)] = \sigma(x)\nabla_i \delta^{(3)}(x-y)$ between the order parameter and momentum density are essential in considerations that lead to Model H.

Son & Stephanov, PRD **70**, 056001 (2004)

In the static case, without a mixing of the order parameter σ and baryon density n, the dynamic universality class might be associated to that of Model C (which asserts the symmetry $\sigma \to -\sigma$).

Berdnikov & Rajagopal, PRD **61**, 105017 (2000)

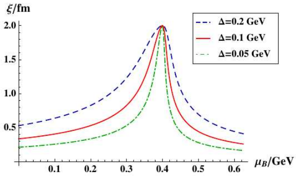
However, $\sigma - n$ mixing (at nonvanishing μ_B) eliminates the nonconserved mode from hydrodynamic theory, leading instead to Model B.

QCD phenomenology

The critical correlations of chiral condensate show up through pion and nucleon coupling in the chiral effective Lagrangian, $\sim g_\pi \sigma \pi^+ \pi^-$ and $\sim g_N \sigma \bar{N} N$.

Cumulants near the critical point:

$$\begin{aligned} \kappa_2 &\sim V n_\pi^2 g_\pi^2 \, \xi^2 \\ \kappa_3 &\sim V n_\pi^3 g_\pi^3 \lambda_3 \, \xi^{9/2} \\ \kappa_4 &\sim V n_\pi^4 g_\pi^4 \lambda_4 \, \xi^7 \end{aligned}$$



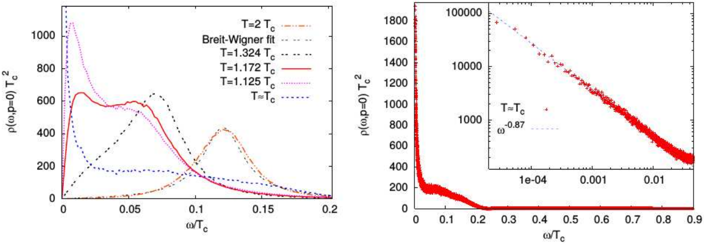
Athanasiou et al., PRD **82**, 074008 (2010)

This leads to strong fluctuations in high order moments, which are limited, however, by finite size and nonequilibrium effects ($\xi \sim \tau^{\frac{1}{z}}$).

Can we determine the dynamic critical exponent z for strongly interacting (relativistic/nonrelativistic) matter?

Model H value of $z \simeq 3$ is larger than the model C value $z \simeq 2.17$, which means that the effect of the time constraint on the correlation length ξ is stronger.

2+1 dimensional relativistic scalar theory



$(2-\eta)/z$
$\simeq 0.87$

Berges et al., NPB **832**, 228 (2010) 228

Dynamic critical scaling over 3 orders of magnitude: $z = 2.0 \pm 0.1$.

Compatible with the relaxational behavior of a single scalar field coupled to an auxiliary field (energy conservation)?

Strong scaling, i.e., $z = 2 + \alpha/\nu$ (Model C) yields: $z = 2$ (according to exact Onsager solution $\alpha = 0, \eta = 1/4, \nu = 1$).

Order parameter coupled to auxiliary field (energy density)

We consider the following dynamics

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_a(x,t) &= -\Omega\,\frac{\delta\mathcal{H}}{\delta\varphi_a(x,t)} + \eta_a(x,t)\\ \frac{\partial}{\partial t}\varepsilon(x,t) &= \Omega_\varepsilon\nabla^2\frac{\delta\mathcal{H}}{\delta\varepsilon(x,t)} + \zeta(x,t)\end{aligned}$$

which defines the kinetic coefficients Ω (dissipation) and Ω_ε (diffusion).

The system is coupled to a thermal bath ($k_B T = 1$)

$$\begin{aligned}\langle\eta_a(x,t)\eta_b(x',t')\rangle &= 2\Omega\delta^{(d)}(x-x')\delta(t-t')\\ \langle\zeta(x,t)\zeta(x',t')\rangle &= -2\Omega_\epsilon\nabla^2\delta^{(d)}(x-x')\delta(t-t')\end{aligned}$$

and the time-dependent GLW functional determines the parameters and couplings of the model:

$$\mathcal{H} = \int d^dx\left\{\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\bar{m}^2\varphi^2 + 3\frac{\bar{\lambda}}{4!}\left(\varphi^2\right)^2 + \frac{1}{2}\varepsilon^2 + \frac{1}{2}\bar{\gamma}\varepsilon\varphi^2\right\}\ .$$

Field-theory formulation

From the classical field equations of motion we may derive the field-theoretic action

$$
\begin{aligned}
S \quad = \quad & \int d^d x \, dt \left\{ \tilde{\varphi}_a \left(\Omega^{-1} \frac{\partial}{\partial t} \varphi_a + \frac{\delta \mathcal{H}}{\delta \varphi_a} \right) - \Omega^{-1} \, \tilde{\varphi}^2 \right. \\
& \left. + \;\; \tilde{\varepsilon} \left(\Omega_\varepsilon^{-1} \frac{\partial}{\partial t} \varepsilon - \nabla^2 \frac{\delta \mathcal{H}}{\delta \varepsilon} \right) + \Omega_\varepsilon^{-1} \, \tilde{\varepsilon} \nabla^2 \tilde{\varepsilon} \right\}
\end{aligned}
$$

$$
Z \sim \int \, [d\tilde{\varphi}] \; [d\varphi] \, [d\tilde{\varepsilon}] \, [d\varepsilon] \, e^{-S}
$$

Scale dependent effective action

We promote the parameters and couplings of the theory to scale-dependent quanitites:

$$
\begin{aligned}
\Gamma_k \quad = \quad & \int d^d x\, dt \left\{ \tilde{\phi}_a \left(\Omega_k^{-1} \frac{\partial}{\partial t} - Z_k \nabla^2 \right) \phi_a + \tilde{\phi}_a \frac{\partial U_k}{\partial \phi_a} - \Omega_k^{-1} \tilde{\phi}^2 \right. \\
& \left. + \tilde{\mathcal{E}} \left(\Omega_{\mathcal{E},k}^{-1} \frac{\partial}{\partial t} - Z_{\mathcal{E},k} \nabla^2 \right) \mathcal{E} - \tilde{\mathcal{E}} \nabla^2 \frac{\partial U_k}{\partial \mathcal{E}} + \Omega_{\mathcal{E},k}^{-1} \tilde{\mathcal{E}} \nabla^2 \tilde{\mathcal{E}} \right\}
\end{aligned}
$$

and determine the components of the inverse propagator:

$$
\begin{aligned}
\left(\Gamma^{(2)}_{k,\tilde{\phi}\phi} \right)_{ab}(q,\omega) \quad &= \quad (- i\Omega_k^{-1}\omega + Z_k q^2)\delta_{ab} + \frac{\partial^2 U_k}{\partial \phi_a \partial \phi_b} \\
\Gamma^{(2)}_{k,\tilde{\mathcal{E}}\mathcal{E}}(q,\omega) \quad &= \quad -i\Omega_{\mathcal{E},k}^{-1}\omega + Z_{\mathcal{E},k} q^2 \\
\left(\Gamma^{(2)}_{k,\phi\tilde{\mathcal{E}}} \right)_{a}(q,\omega) \quad &= \quad q^2 \frac{\partial^2 U_k}{\partial \phi_a \partial \mathcal{E}} \\
\left(\Gamma^{(2)}_{k,\mathcal{E}\tilde{\phi}} \right)_{a}(q,\omega) \quad &= \quad \frac{\partial^2 U_k}{\partial \phi_a \partial \mathcal{E}} \\
\left(\Gamma^{(2)}_{k,\tilde{\phi}\tilde{\phi}} \right)_{ab}(q,\omega) \quad &= \quad -2\Omega_k^{-1}\delta_{ab} \\
\Gamma^{(2)}_{k,\tilde{\mathcal{E}}\tilde{\mathcal{E}}}(q,\omega) \quad &= \quad -2\Omega_{\mathcal{E},k}^{-1} q^2
\end{aligned}
$$

$\Rightarrow$ Conservation of charge via momentum-dependent coupling

Functional renormalization group

The exact nonperturbative functional renormalization group equation

$$k\frac{\partial\Gamma_k}{\partial k} = \frac{1}{2}\int d^d q\, d\omega\, k\frac{\partial R_k(q,\omega)}{\partial k}\left(\Gamma_k^{(2)} + R_k\right)^{-1}(q,\omega)$$

defines and infinite hierarchy of flow equations for irreducible n-point functions:

$$k\frac{\partial}{\partial k}\Gamma_k^{(n)} = \mathrm{Flow}_k^{(n)}\left(\Gamma_k^{(2)},\ldots,\Gamma_k^{(n+2)}\right)$$

It can be closed by truncating the set of operators included in the *ansatz* for the scale dependent effective action. In particular, we choose a gradient expansion to $\mathcal{O}(\partial^2)$ and a finite basis of local field monomials for the potential:

$$U_k(\phi,\mathcal{E}) = \sum_n \bar{g}_{n,k}\mathcal{O}_n(\phi,\mathcal{E})$$

The β functions are defined in terms of the scale derivatives of the couplings

$$\beta_{g_n} = k\frac{\partial g_n}{\partial k} = (-d_{\mathcal{O}_n} + c_n\eta)g_n + \cdots$$

and anomalous dimensions for the fields:

$$\eta(\phi) = -k\frac{\partial Z_k(\phi)}{\partial k}$$

From these quantities we derive fixed points, critical exponents, etc.

Couplings and parameters in Model C

Dimensionless renormalized coupling between the sectors: $\gamma = k^{d/2-2} Z^{-1} Z_{\mathcal{E}}^{-1/2} \bar{\gamma}$

Dynamic properties characterized by parameter:

$$\kappa = 1/(1 + \Omega_{\mathcal{E}} \Omega^{-1} Z^{-1} Z_{\mathcal{E}})$$

which varies in the range $0 \leq \kappa \leq 1$ and captures the asymptotic scaling properties

RG fixed point distinguished by scaling

$$Z \sim k^{-\eta}, \quad Z_{\mathcal{E}} \sim k^{-\eta_{\mathcal{E}}}, \quad \Omega^{-1} \sim k^{-\eta_\Omega}$$

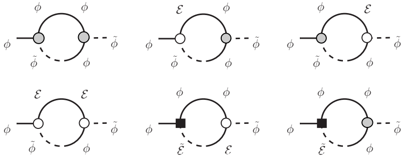
Dynamic critical exponent z is derived by examining the scaling behavior of the spectral function: $\rho = -i \operatorname{Im} G_{\phi\tilde{\phi}} \sim k^{-2+\eta}$, where it is assumed that $q \sim k$ and $\omega \sim k^z$

$$\Omega^{-1} Z^{-1} k^{z-2} \sim k^{z-2+\eta-\eta_\Omega} = \text{const.}$$

$$z = 2 - \eta + \eta_\Omega \,, \qquad z_{\mathcal{E}} = 2 - \eta_{\mathcal{E}} + \eta_{\Omega_{\mathcal{E}}}$$

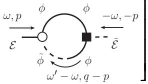
Dynamic critical scaling

Diagrams that contribute to the frequency and momentum-dependent part of the two-point function $\Gamma^{(2)}_{\tilde{\phi}\phi}(p,\omega)$:



Only one diagram contributes to the frequency-dependence in the $\mathcal{E}$-sector:

$$\frac{\partial \Omega_{\mathcal{E}}^{-1}}{\partial s} \sim \mathrm{Im}\left[\lim_{\omega,p\to 0} \frac{\partial}{\partial \omega} \right.$$



$$\left. \vphantom{\frac{\partial}{\partial\omega}} \right] = 0$$

$\Rightarrow$ Absence of dynamic renormalization in $\mathcal{E}$-sector ($\eta_{\Omega_{\mathcal{E}}} = 0$)

$$z = 2 - \eta + \eta_{\Omega}\,, \qquad z_{\mathcal{E}} = 2 - \eta_{\mathcal{E}}$$

Anomalous dimensions

The anomalous dimensions $\eta = -\partial \ln Z/\partial s$, $\eta_{\mathcal{E}} = -\partial \ln Z_{\mathcal{E}}/\partial s$, and $\eta_{\Omega} = -\partial \ln \Omega^{-1}/\partial s$ define the dynamic critical exponent as well as possible fixed points of the model.

$$
\begin{aligned}
\eta &= 16\frac{v_d}{d}\rho_0\lambda^2 m_{2,2}\left(0, 2\rho_0\lambda;\eta\right) , \\
\eta_{\mathcal{E}} &= -2v_d\gamma^2\left\{(N-1)l_2\left(0;\eta\right) + l_2\left(2\rho_0\lambda;\eta\right)\right\} , \\
\eta_{\Omega} &= \frac{1}{\rho_0}2v_d\Big\{l_1(0;\eta) + l_1(2\rho_0\lambda;\eta) - 2\,h_1\Big(\rho_0(\lambda+\gamma^2), \gamma^2\rho_0(1-\kappa)/\kappa, (1-\kappa)/\kappa;\eta\Big)\Big\}
\end{aligned}
$$

Anomalous dimensions (continued)

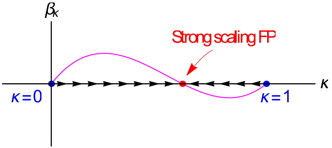
$$
\begin{aligned}
h_1(w_1, w_2, w_3;\eta) \quad = \quad & \frac{1}{(1+w_1)^2}\Bigg\{\frac{(1+w_1)\left(w_2+(1+w_3)^2\right)}{1+w_1-w_2+w_3+w_1w_3} \\
& + \left(\frac{2}{d}-1\right)\,{}_2F_1\left(1,\,\frac{d}{2};\,\frac{d+2}{2};\,\frac{w_2}{w_1+1}-w_3\right) \\
& - \frac{d}{d+2}(w_2+2w_3)\,{}_2F_1\left(1,\,\frac{d+2}{2};\,\frac{d+4}{2};\,\frac{w_2}{w_1+1}-w_3\right) \\
& - \frac{d+2}{d+4}w_3^2\,{}_2F_1\left(1,\,\frac{d+4}{2};\,\frac{d+6}{2};\,\frac{w_2}{w_1+1}-w_3\right) \\
& - \frac{\eta}{2}\Bigg[\left(\frac{2}{d}-1\right)\,{}_2F_1\left(1,\,\frac{d}{2};\,\frac{d+2}{2};\,\frac{w_2}{w_1+1}-w_3\right) \\
& - \frac{d}{d+2}(w_2+2w_3-1)\,{}_2F_1\left(1,\,\frac{d+2}{2};\,\frac{d+4}{2};\,\frac{w_2}{w_1+1}-w_3\right)\Bigg) \\
& - \frac{d+2}{d+4}\left(w_3^2-w_2-2w_3\right)\,{}_2F_1\left(1,\,\frac{d+4}{2};\,\frac{d+6}{2};\,\frac{w_2}{w_1+1}-w_3\right) \\
& + \frac{d+4}{d+6}w_3^2\,{}_2F_1\left(1,\,\frac{d+6}{2};\,\frac{d+8}{2};\,\frac{w_2}{w_1+1}-w_3\right)\Bigg]\Bigg\}
\end{aligned}
$$

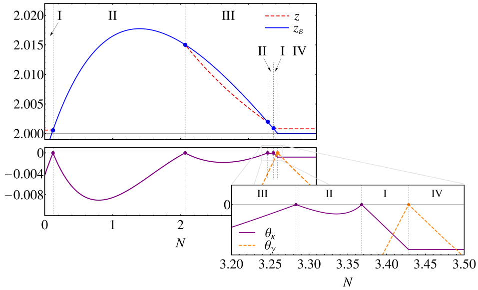
Fixed points and dynamic scaling regimes

Single β-function characterizes the dynamic properties of the non-Gaussian FP associated to the second order phase transition:

$$\beta_\kappa = \kappa(1-\kappa)(\eta_\Omega(\kappa) - \eta + \eta_\mathcal{E})$$

It will depend implicity on the fixed point values of the couplings, the number of field components, and dimensionality of the system. Zeros of β_κ capture the dynamic critical behavior (fixed points) and we find multiple solutions describing to different dynamic universality classes.





Dynamic universality classes for Model C

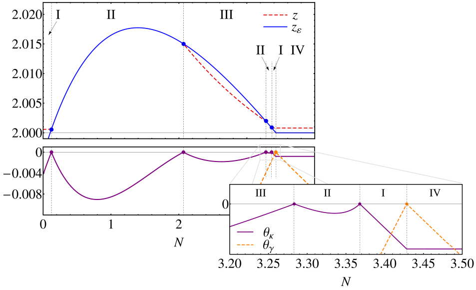
Weak scaling region (I): $\kappa = 0$ and $\gamma \neq 0$

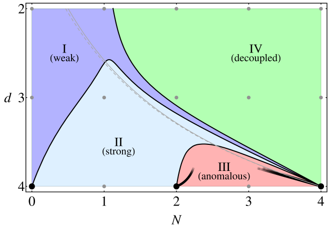
Strong scaling region (II): $0 < \kappa < 1$ and $\gamma \neq 0$

$$z = 2 - \eta_{\mathcal{E}} = z_{\mathcal{E}}$$

Anomalous diffusion region (III): $\kappa = 1$ and $\gamma \neq 0$

Decoupled scaling region (IV): $\kappa = 0$ and $\gamma = 0$





Critical dynamics for relativistic field theories

Coming back to the $2+1$-dimensional relativistic scalar theory, we see that in fact the scaling behavior should match that of Model A: $z = 2.16$ ($d = 2$).

How does that fit with the scaling result $z = 2.0 \pm 0.1$?

Need to revisit classical-statistical simulations which is currently in progress.

In general it is necessary to consider the relativistic scalar theory and determine the appropriate low-energy dynamics. This is work currently in progress.

Questions