Scattering amplitudes and recursion relations on the light front

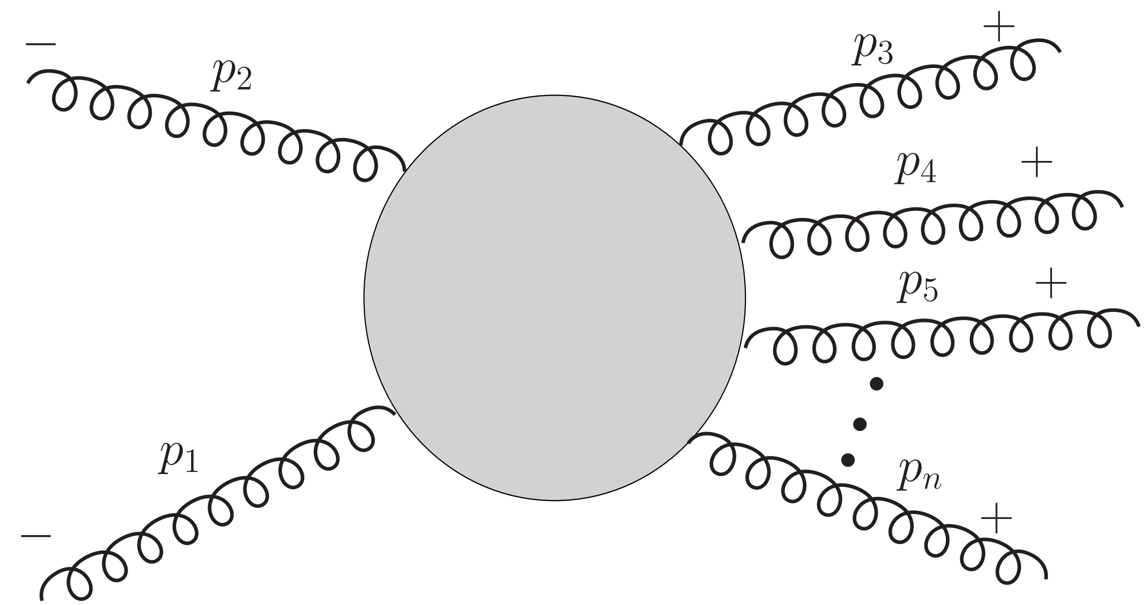
Anna Stasto, Penn State

BNL, November 21st, 2014

Introduction

<u>Helicity amplitudes in QCD</u>

Scattering 2 to n:



Enormous progress in computational techniques in recent decades

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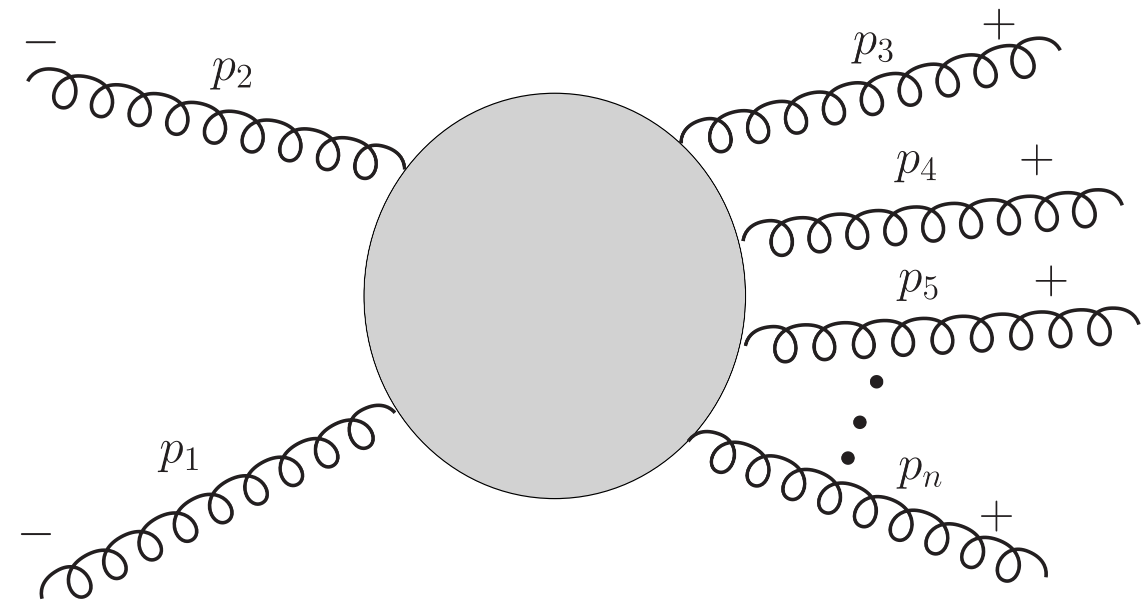

Enormous progress in computational techniques in recent decades

Goals:

Compute amplitudes within framework of light -front perturbation theory.

Construct recursive relations on the light-front.

Obtain insight into patterns and general structure of amplitudes from the light-front theory perspective.

Outline

- Light-front perturbation theory
- Computing off-shell objects: wave functions and fragmentation functions
- MHV amplitudes
- Recursion relations
- Outlook: recursion relations on the light-front and Ward identities

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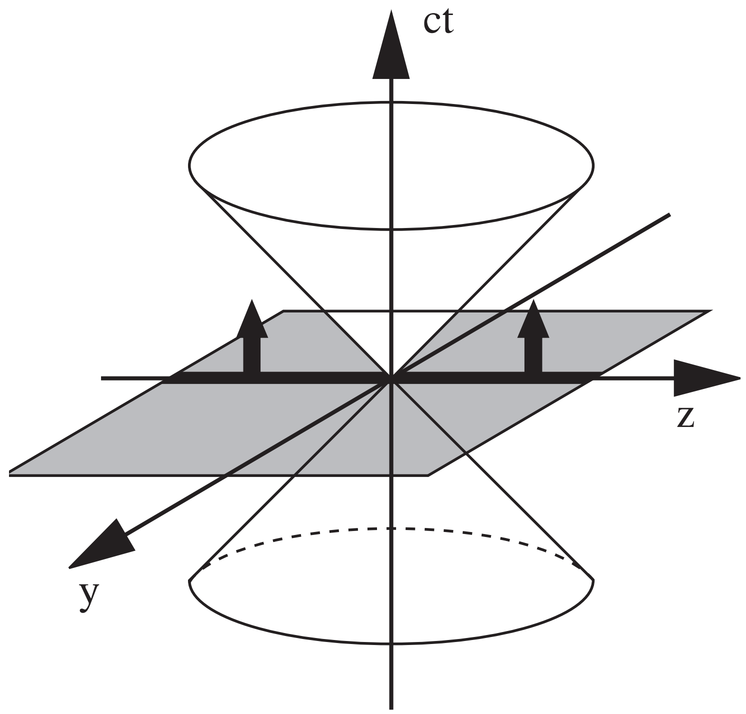
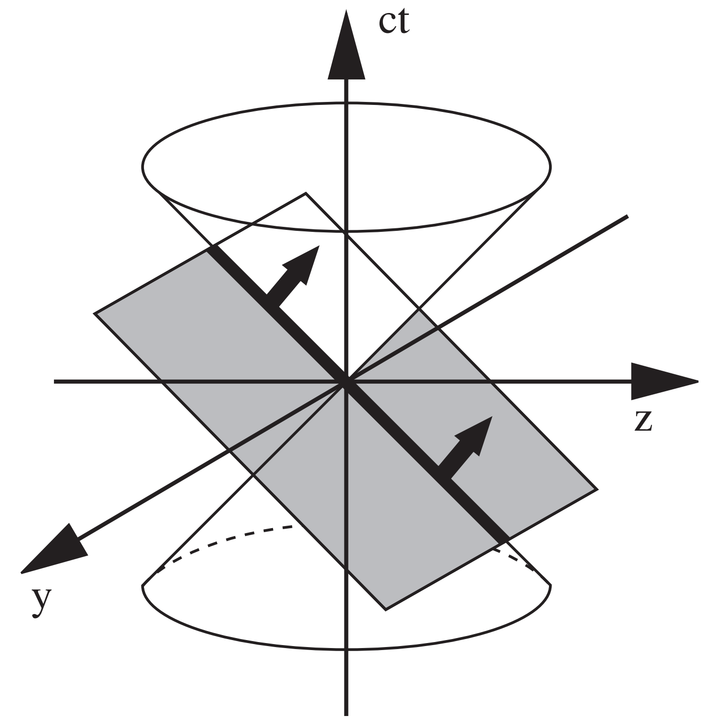
Work in collaboration with L. Motyka, C.Cruz-Santiago and P. Kotko

Light-front formalism

$$(A^0, A^1, A^2, A^3)$$



The instant form



The front form

$$A^+ = \frac{1}{\sqrt{2}}(A^0 + A^3)$$

$$A^- = \frac{1}{\sqrt{2}}(A^0 - A^3)$$

REVIEWS OF MODERN PHYSICS VOLUME 21, NUMBER 3 JULY, 1949

Forms of Relativistic Dynamics

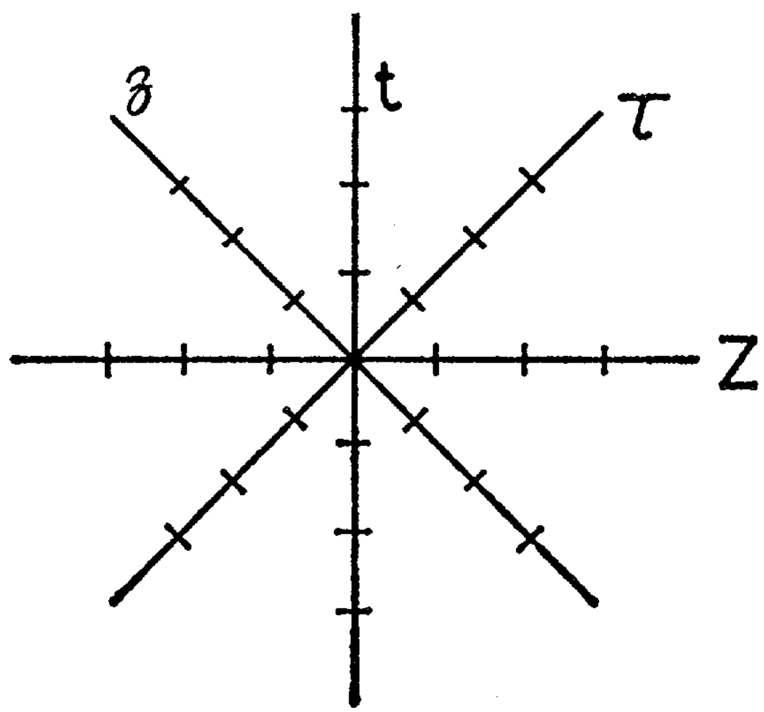
P. A. M. Dirac
St. John's College, Cambridge, England

For the purposes of atomic theory it is necessary to combine the restricted principle of relativity with the Hamiltonian formulation of dynamics. This combination leads to the appearance of ten fundamental quantities for each dynamical system, namely the total energy, the total momentum and the 6-vector which has three components equal to the total angular momentum. The usual form of dynamics expresses everything in terms of dynamical variables at one instant of time, which results in specially simple expressions for six or these ten, namely the components of momentum and of angular momentum. There are other forms for relativistic dynamics in which others of the ten are specially simple, corresponding to various sub-groups of the inhomogeneous Lorentz group. These forms are investigated and applied to a system of particles in interaction and to the electromagnetic field.

The front form has the advantage that it requires only three Hamiltonians, instead of the four of the other forms. This makes it mathematically the most interesting form, and makes any problem of finding Hamiltonians substantially easier. The front form has the further advantage that there is no square root in the Hamiltonians (28), which means that one can avoid negative energies for particles by suitably choosing the values of the dynamical variables in the front, without having to make a special convention about the sign of a square root. It may then be easier to eliminate negative energies from the quantum theory. This

There is no conclusive argument in favor of one or other of the forms. Even if it could be decided that one of them is the most convenient, this would not necessarily be the one chosen by nature, in the event that only one of them is possible for atomic systems. Thus

Light-front formalism



Kogut,Soper

Infinite momentum frame: a limit of a Lorentz frame moving in the -z direction with a (nearly) the speed of light.

$$\tau = \frac{1}{\sqrt{2}}(t+z)$$

Susskind

Isomorphism with the Galilean dynamics in 2 dimensions:

$P^{-} \longrightarrow$ Hamiltonian

$P^{+} \longrightarrow$ Mass

$P_T \longrightarrow$ 2-dim. momentum

Free particle example:

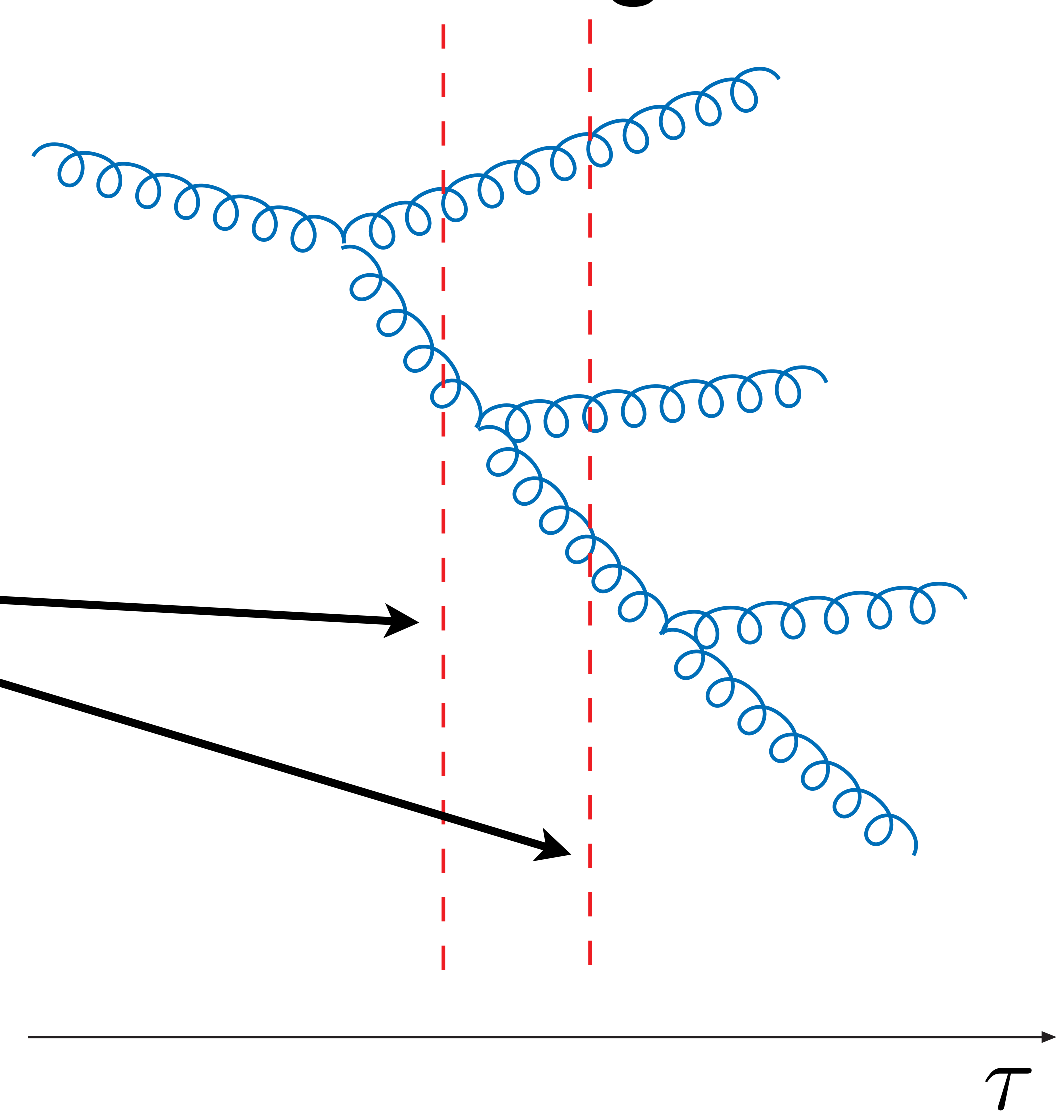
$$m^2 = P^\mu P_\mu = 2P^+ H - P_T^2$$

$$H = \frac{P_T^2}{2P^+} + V_0$$

$$V_0 = \frac{m^2}{2P^+}$$

Light-front rules

Non-covariant (light-front) time ordered diagram



Energy denominators

Difference of light - cone energies:

$$D = \sum_j k^-_{j,f} - \sum_i k^-_i$$

Final state

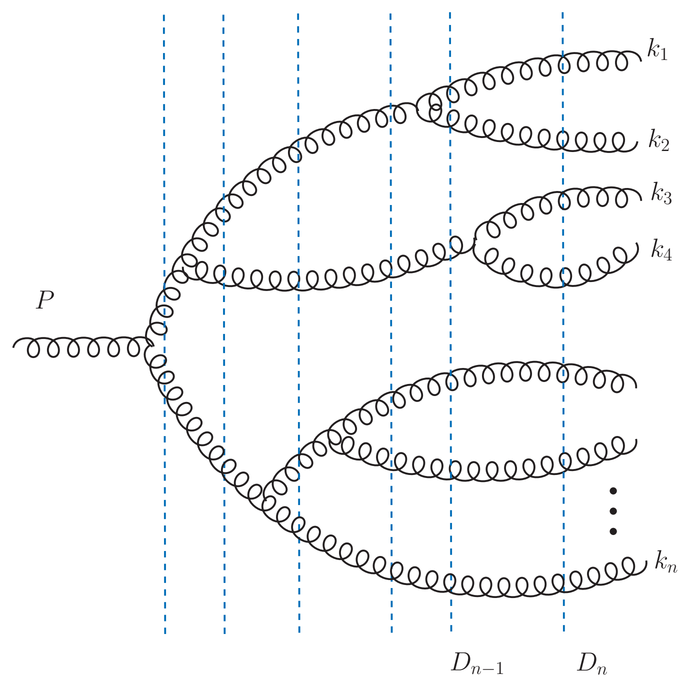
Intermediate state

Particles are on-shell but conservation of 3 momentum components in the intermediate states

Light-front rules

$$\longrightarrow \tau$$

typical graph



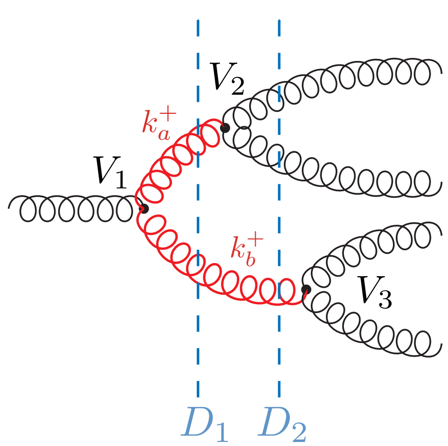
vertices

$$\sim g^n \frac{\Pi_j\, V_j}{\Pi_j\, D_j \;\; \Pi_l\, k_l^+}$$

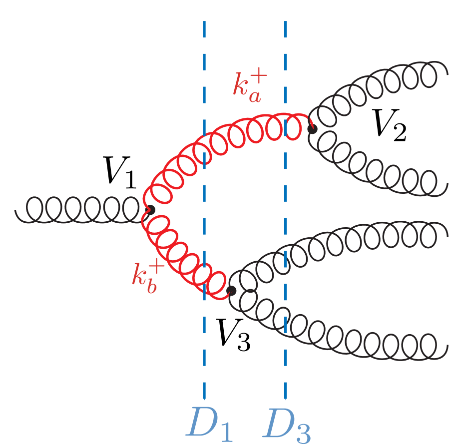
energy denominators

internal lines

Need to sum over all vertex orderings in light - front time:



$+$



$$D_2 \neq D_3$$

$\tau_3 > \tau_2$

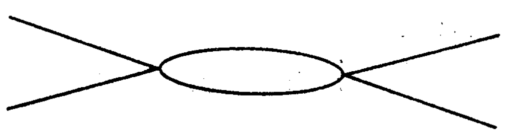
$\tau_2 > \tau_3$

Light-front formalism

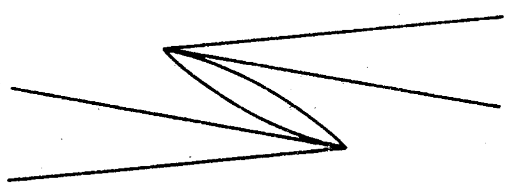
Important simplification on the light front :

Only those diagrams contribute which have positive P^+ for all internal lines

Weinberg:



this diagram does not contribute →



Diagrams in which particles are created or destroyed out of vacuum do not contribute in the light front formalism.

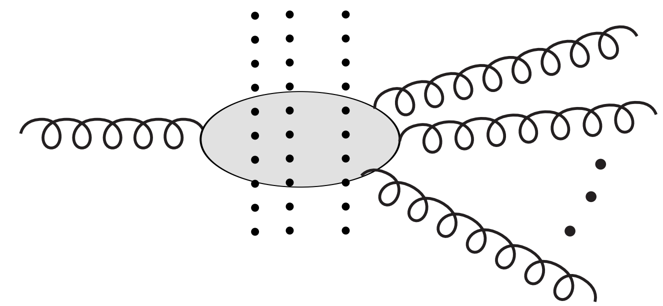
Note that in light front theory individual diagrams not need to be Lorentz invariant. Only the sum of the diagrams is Lorentz invariant.

Amplitudes, wave functions, fragmentation functions

We are interested in computing on-shell amplitude $M(2 \to n-1)$

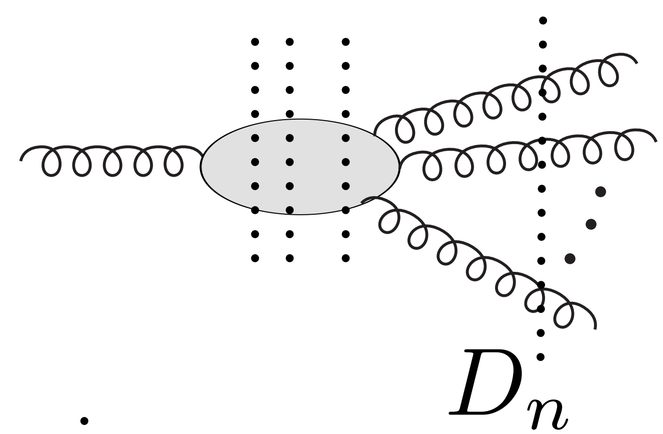
Various objects on the light-front:

<u>transition amplitude</u>



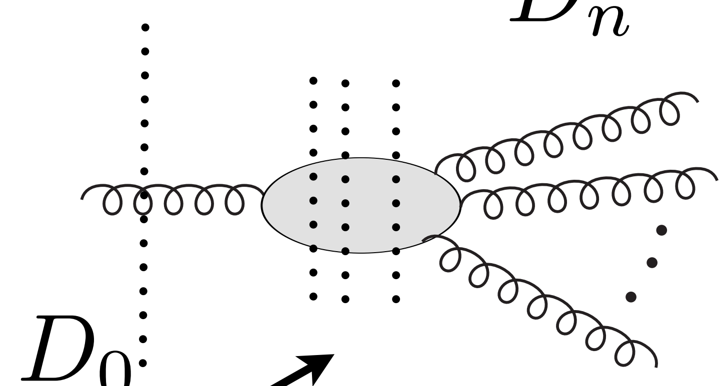
$M(1 \to n)$

<u>wave function:</u> energy denominator in the last state



Ψ_n

<u>fragmentation function:</u> energy denominator in the first state

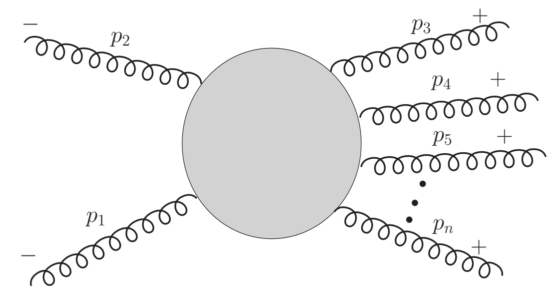


T_n

Here the sum over all light front time orderings has been performed

Amplitudes, wave functions, fragmentation functions

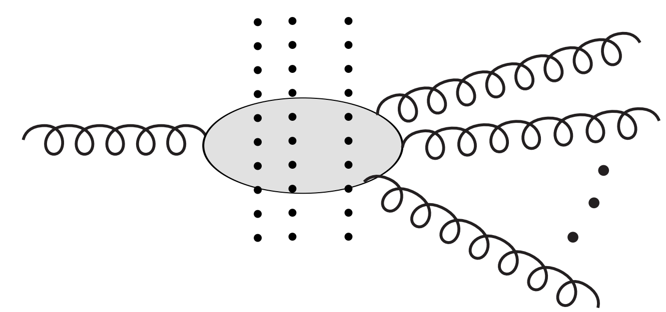
We are interested in computing on-shell amplitude



$M(2 \to n-1)$

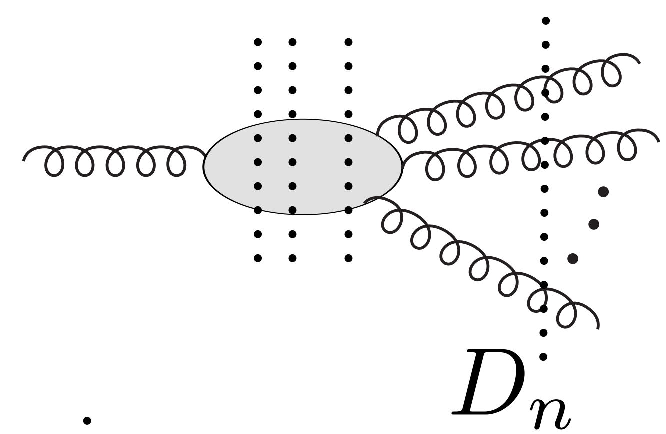
Various objects on the light-front:

<u>transition amplitude</u>



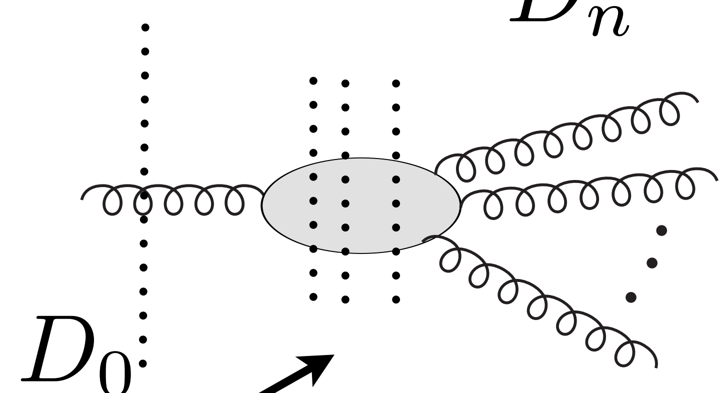
$M(1 \to n)$

<u>wave function:</u> energy denominator in the last state



Ψ_n

<u>fragmentation function:</u> energy denominator in the first state



T_n

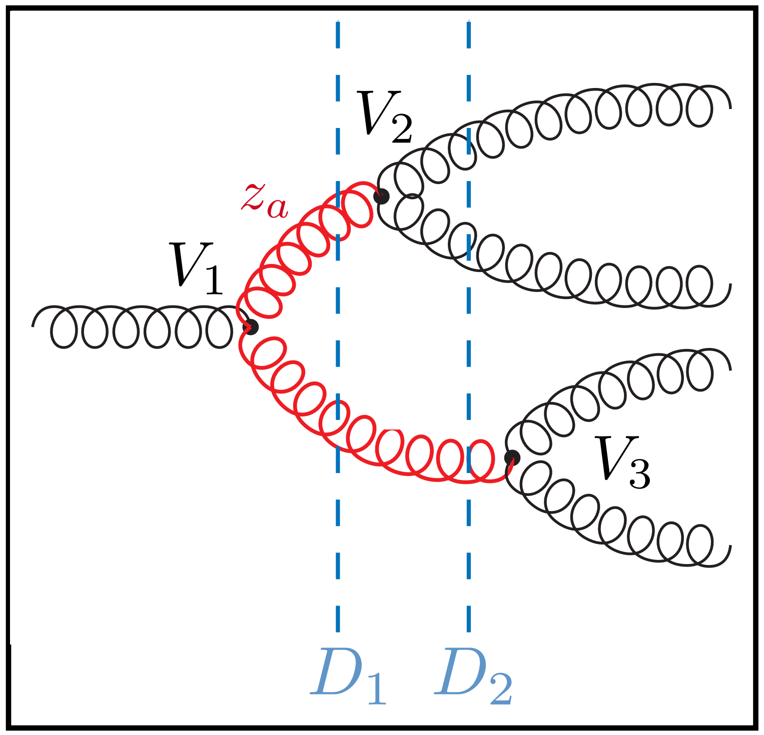
Here the sum over all light front time orderings has been performed

Amplitudes, wave functions, fragmentation functions

$$z = \frac{k^+}{P^+}$$ Fraction of longitudinal momentum

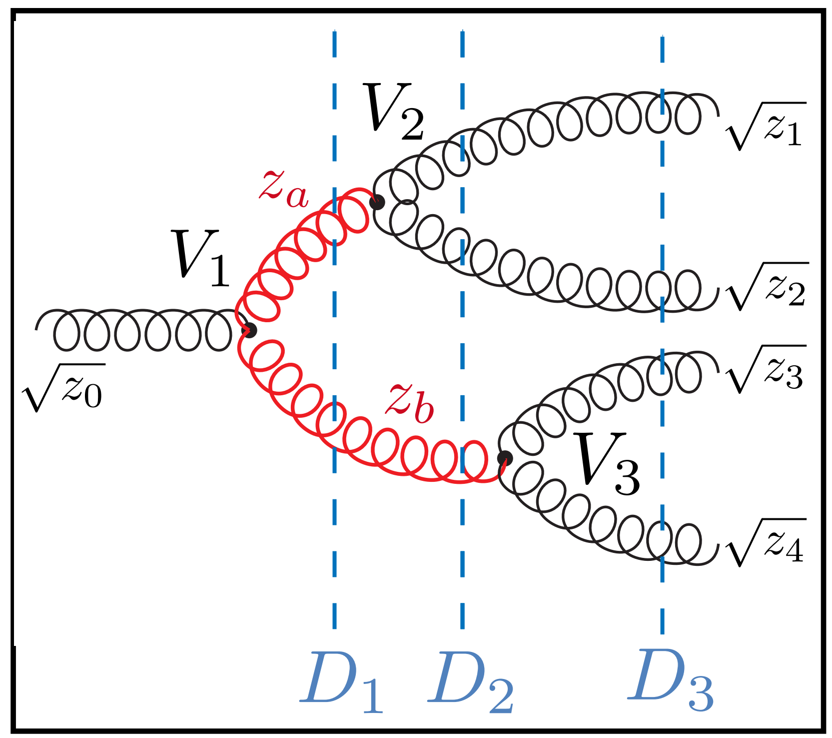
Transition amplitude

M



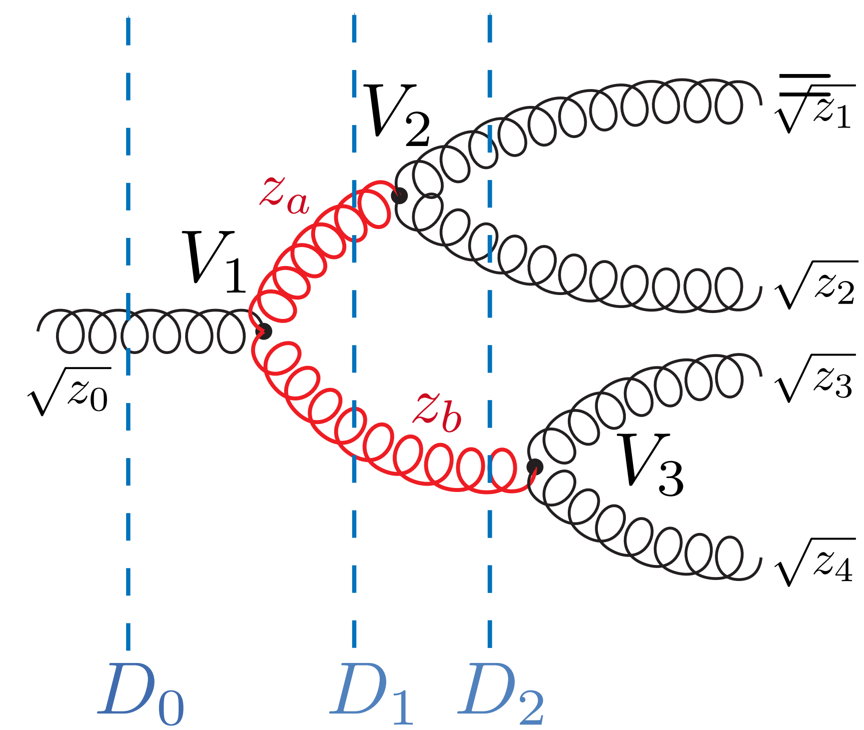
Wavefunction

Ψ



Fragmentation function

T

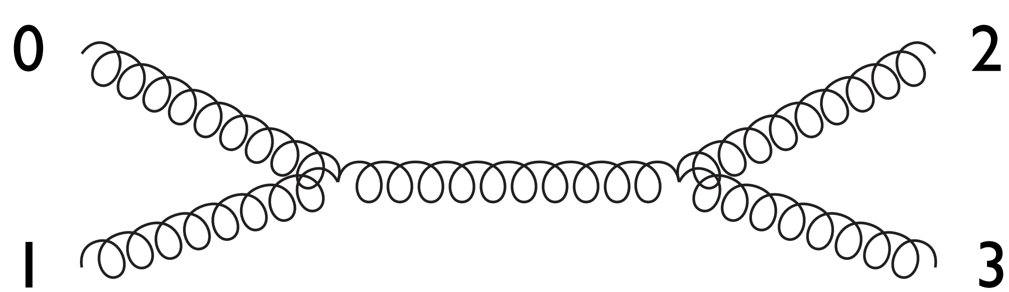


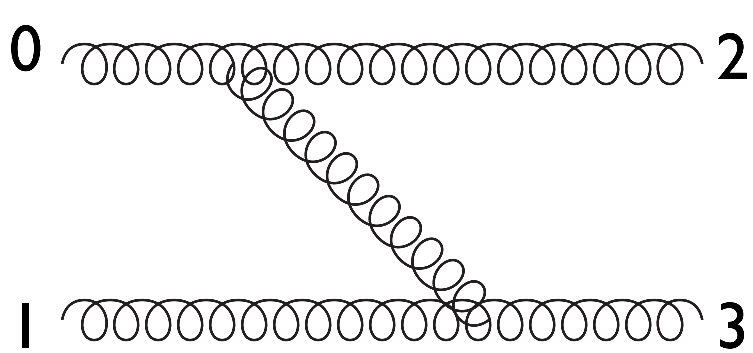
$$M = \sqrt{z_0 z_1 z_2 z_3 z_4} D_3 \Psi|_{D_3=0} = \sqrt{z_0 z_1 z_2 z_3 z_4} D_0 T|_{D_0=0}$$

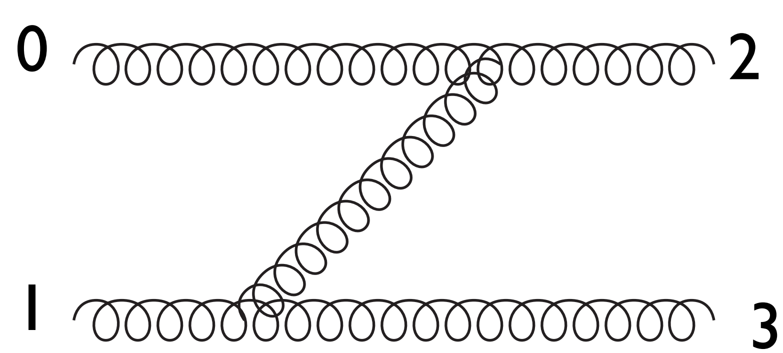
2 to N scattering amplitude

In calculation we want 2 to N amplitude. Since particles move forward and we need to sum over vertex orderings there are different number of graphs depending whether we start from 1 or 2 particles.

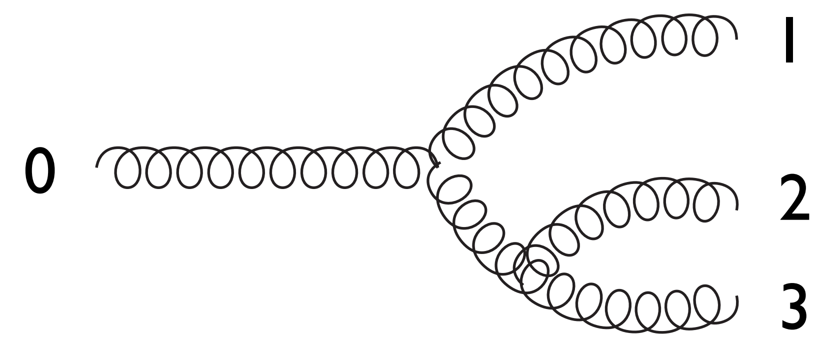
Scattering amplitude

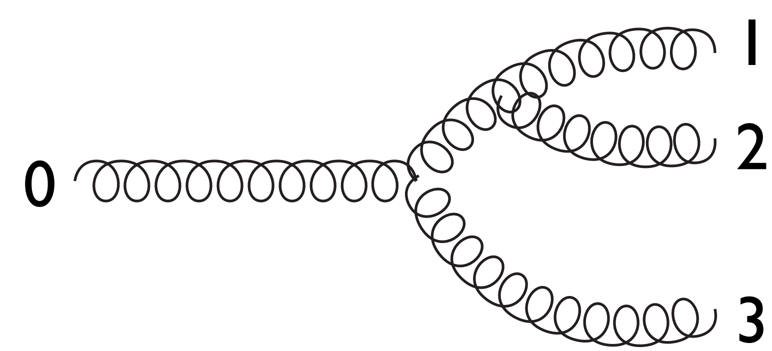






Transition amplitude

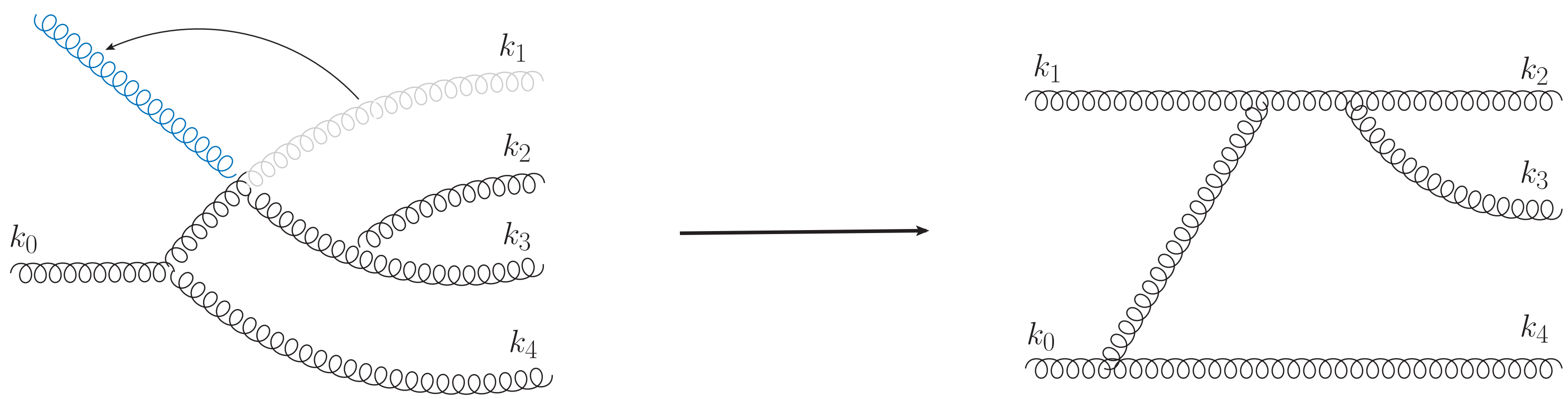




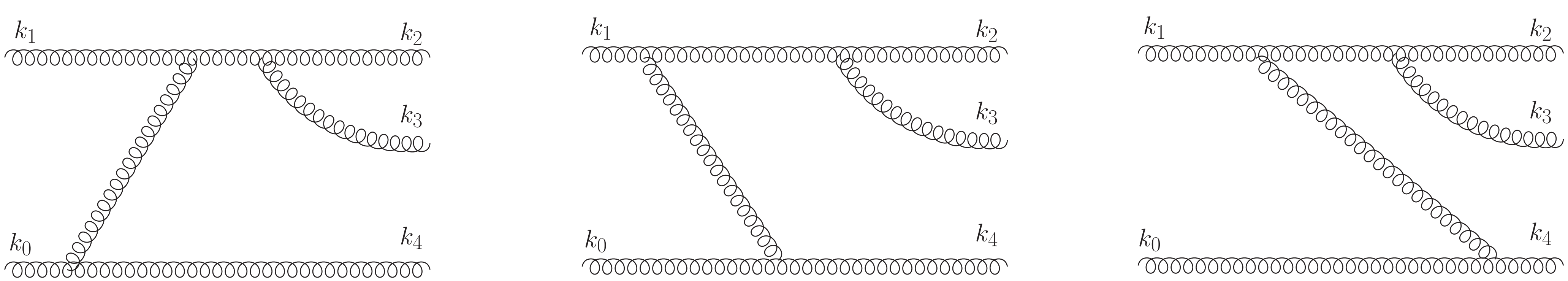
2 to N scattering amplitude

Calculating scattering amplitude: find graphs of given topology

Make the outgoing k_1 gluon into incoming.



Find all graphs with other possible vertex orderings.

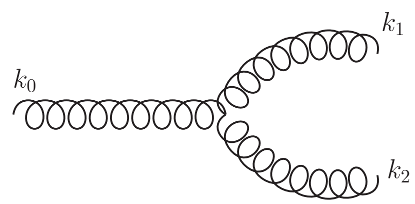


In 2 to N case we have significant increase number of graphs

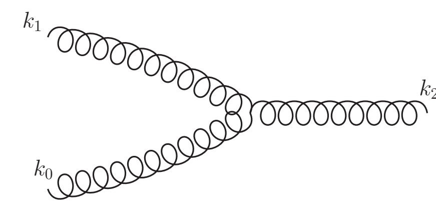
2 to N scattering amplitude

It is however sufficient to compute just 1 to N transition and then reverse the momenta. Even though the number of all topologically equivalent time ordered graphs is different, the result is identical (as it should be since we need to recover Lorentz invariance).

Vertex

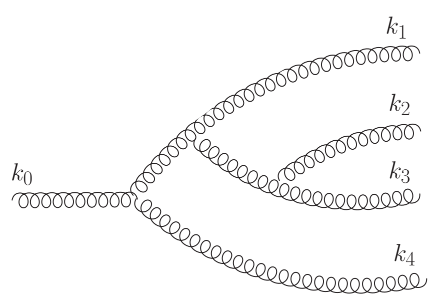


$$v_{21}$$

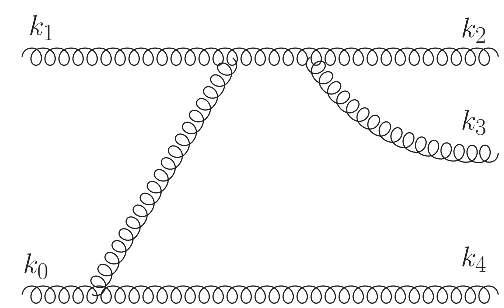


$$-v_{21}$$

Internal line

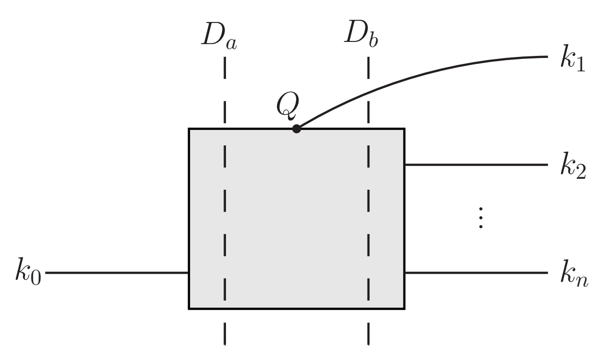


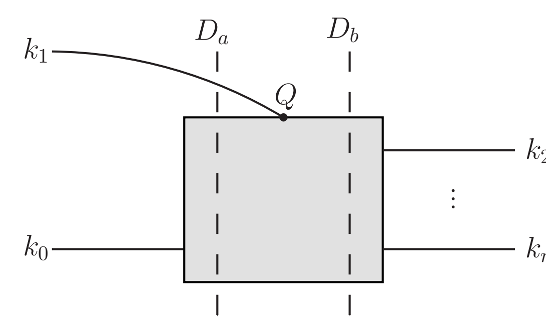
$$z_5 = z_1 + z_2 + z_3$$
$$k_A \equiv k_1$$
$$z_5 = z_A + z_2 + z_3$$



$$z_5 = -z_1 + z_2 + z_3$$
$$k_A \equiv -k_1$$
$$z_5 = z_A + z_2 + z_3$$

Energy denominator





$$D_a = \frac{k_0^2}{z_0 P^+} - E_a = \sum_{i=2}^{n} \frac{k_i^2}{z_i P^+} + \frac{k_1^2}{z_1 P^+} - E_a = \frac{k_A^2}{z_A P^+} + \sum_{i=2}^{n} \frac{k_i^2}{z_i P^+} - E_a$$

$$D_b = \frac{k_0^2}{z_0 P^+} - E_b - \frac{k_1^2}{z_1 P^+} = \frac{k_0^2}{z_0 P^+} - E_b - \frac{k_A^2}{z_A P^+} = \sum_{i=2}^{n} \frac{k_i^2}{z_i P^+} - E_b$$

$$D_a = \frac{k_0^2}{z_0 P^+} - E_a = \sum_{i=2}^{n} \frac{k_i^2}{z_i P^+} - \frac{k_1^2}{z_1 P^+} - E_a = \frac{k_A^2}{z_A P^+} + \sum_{i=2}^{n} \frac{k_i^2}{z_i P^+} - E_a$$

$$D_b = \frac{k_0^2}{z_0 P^+} - E_b + \frac{k_1^2}{z_1 P^+} = \frac{k_0^2}{z_0 P^+} - E_b - \frac{k_A^2}{z_A P^+} = \sum_{i=2}^{n} \frac{k_i^2}{z_i P^+} - E_b$$

2 to N scattering amplitude

Relation on the light - front:

$$M_{2\to n}(\{\underline{k}_0, z_0; \underline{k}_1, z_1\}; \{\underline{k}_2, z_2; \ldots; \underline{k}_n, z_n\}) =$$

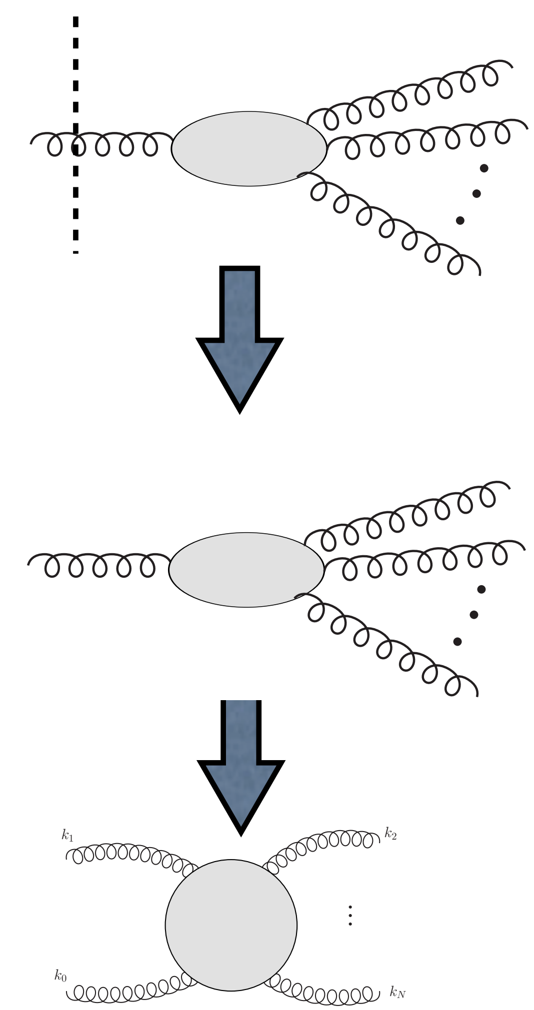
$$-M_{1\to n+1}(\{\underline{k}_0, z_0\}; \{\underline{k}_A, z_A; \underline{k}_2, z_2; \ldots; \underline{k}_n, z_n\})|_{\underline{k}_A \to -\underline{k}_1,\, z_A \to -z_1}$$

Practical setup for computing off-shell/on-shell amplitudes on the light front:

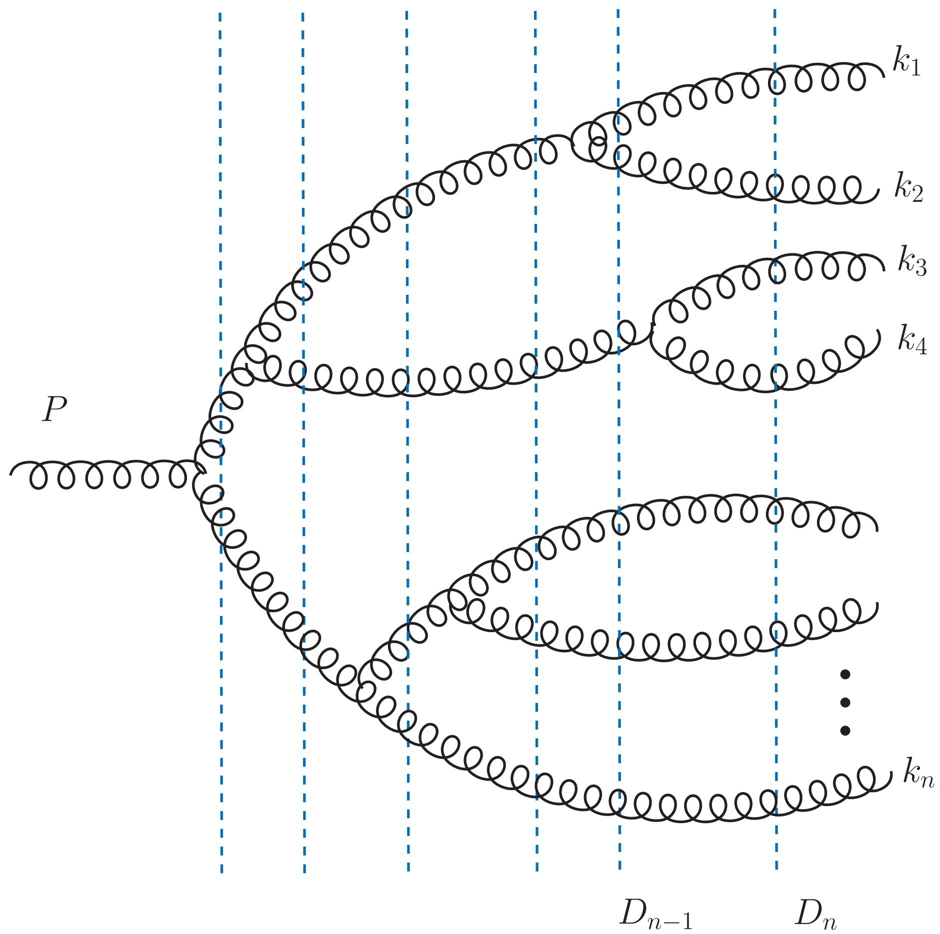
Compute, off-shell wave functions and/or fragmentation functions.

This gives off shell amplitudes as in above relation.

Set , $D = 0 = \sum k^-_{\text{final}} - \sum k^-_{\text{initial}}$ which gives transition amplitude and on-shell 2 to N amplitude.



Wave-functions



First state is on-shell (no energy denominator), last state off-shell energy denominator present.
To simplify, consider only the case with the same helicities:

$$+ \rightarrow + + \cdots +$$

The initial particle is incoming, all the other particles are outgoing.

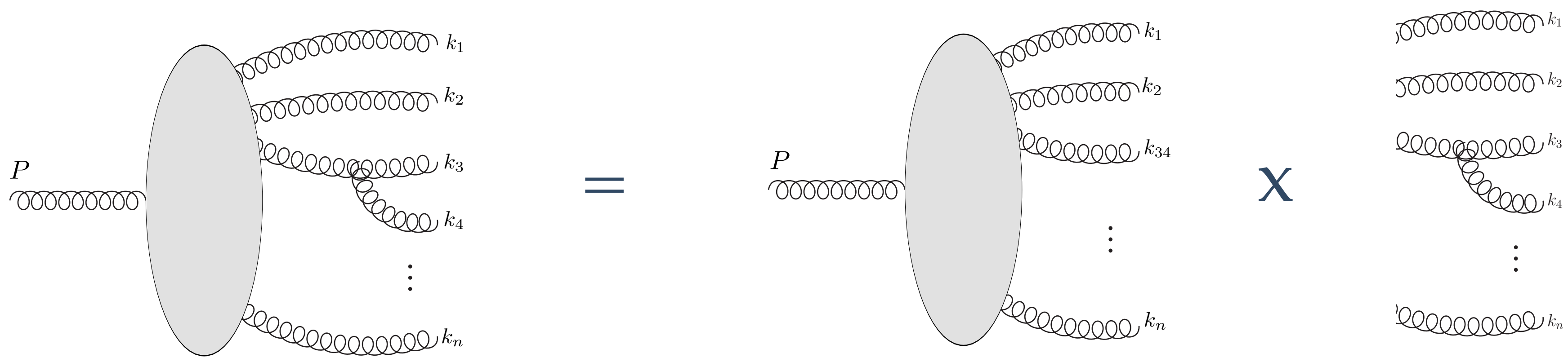
Note that this corresponds to the 2 to n amplitude

$$- + + \cdots +$$

all the particles outgoing, which for the on-shell case should be identically zero. However, we are computing off-shell object which does not have to be equal to zero.

Note: all the calculations done in light - cone gauge.

Solving recurrence relation



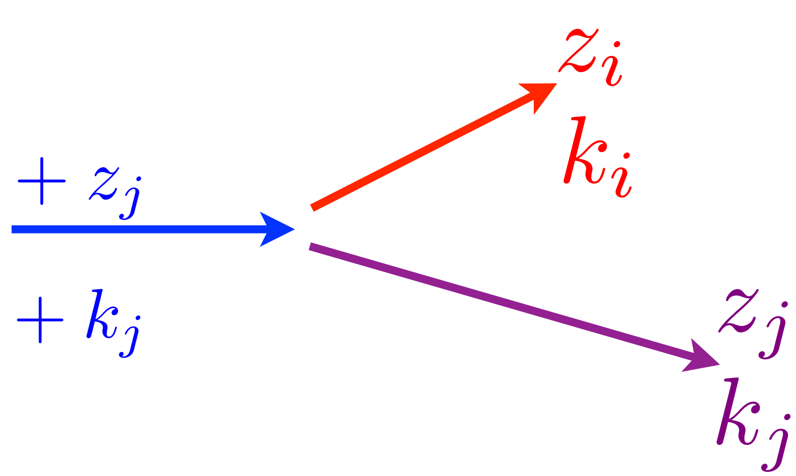
$$\Psi_n = \Psi_{n-1} \times \frac{g}{\sqrt{\xi_{34}}} \frac{v_{34}}{D_n}$$

z_i fraction of longitudinal momentum of i'th particle

$\underline{k}_i$ transverse momentum of the i'th particle

$$z_{ij} = z_i + z_j$$

$$\underline{k}_{ij} = \underline{k}_i + \underline{k}_j$$



reduced mass:

$$\boxed{\xi_{ij} = \frac{z_i z_j}{z_i + z_j}}$$

relative velocity:

$$\boxed{\underline{v}_{ij} = \frac{\underline{k}_i}{z_i} - \frac{\underline{k}_j}{z_j}}$$

$\underline{\epsilon}^{(-)}$ transverse polarization vector

$$v_{ij} = \underline{\epsilon}^{(-)} \cdot \underline{v}_{ij}$$

Light cone wave function

Case of the on-shell incoming gluon.
Can resum the wave function completely. Need to sum over all possible splittings.

$$-D_n \Psi_n(1,2,\ldots,n) \;=\; g \sum_{k=1}^{n-1} \frac{v^*_{(k,k+1)}}{\sqrt{\xi_{(k,k+1)}}} \Psi_{n-1}(1,2,\ldots,(k\,k+1),\ldots,n) \qquad n-1 \to n$$

$$-\,D_{n+1}\,\Psi_{n+1}(1,2,\ldots,n+1) \;=\; g \sum_{i=1}^{n} \frac{v^*_{(i,i+1)}}{\sqrt{\xi_{(i,i+1)}}} \Psi_n(1,2,\ldots,(i\,i+1),\ldots,n+1) \qquad n \to n+1$$

...

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...

Tree-level gluon wave function with exact kinematics

$$\Psi_n(1,2,\ldots,n) \;=\; (-1)^{n-1} g^{n-1}\,\Delta^{(n)}\,\frac{1}{\sqrt{z_1 z_2\ldots z_n}}\,\frac{1}{\xi_{(12\ldots n-1)n}\,\xi_{(12\ldots n-2)(n-1\,n)}\;\cdots\;\xi_{1(2\ldots n)}}$$

$$\times\;\frac{1}{v_{(12\ldots n-1)n}\,v_{(12\ldots n-2)(n-1\,n)}\;\cdots\;v_{1(2\ldots n)}}\;.$$

$$v_{(i_1 i_2\ldots i_p)(j_1 j_2\ldots j_q)} = \frac{k_{i_1}+k_{i_2}+\ldots+k_{i_p}}{z_{i_1}+z_{i_2}+\ldots+z_{i_p}} - \frac{k_{j_1}+k_{j_2}+\ldots+k_{j_q}}{z_{j_1}+z_{j_2}+\ldots+z_{j_q}},$$

$$\xi_{(i_1 i_2\ldots i_p)(j_1 j_2\ldots j_q)} = \frac{(z_{i_1}+z_{i_2}+\ldots+z_{i_p})(z_{j_1}+z_{j_2}+\ldots+z_{j_q})}{z_{i_1}+z_{i_2}+\ldots+z_{i_p}+z_{j_1}+z_{j_2}+\ldots+z_{j_q}}\;,$$

Light cone wave function

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Can resum the wave function completely. Need to sum over all possible splittings.

$$-D_n \Psi_n(1,2,\ldots,n) \;=\; g \sum_{k=1}^{n-1} \frac{v^*_{(k,k+1)}}{\sqrt{\xi_{(k,k+1)}}} \Psi_{n-1}(1,2,\ldots,(k\,k+1),\ldots,n) \qquad n-1 \to n$$

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...

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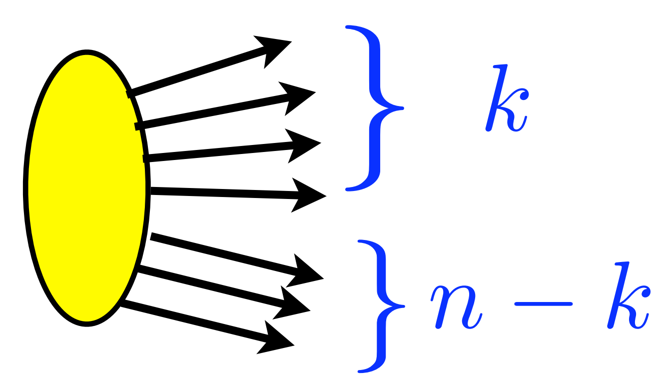
$$\Psi_n(1,2,\ldots,n) \;=\; (-1)^{n-1} g^{n-1}\,\Delta^{(n)}\, \frac{1}{\sqrt{z_1 z_2 \ldots z_n}}\, \frac{1}{\xi_{(12\ldots n-1)n}\,\xi_{(12\ldots n-2)(n-1\,n)}\,\cdots\,\xi_{1(2\ldots n)}}$$

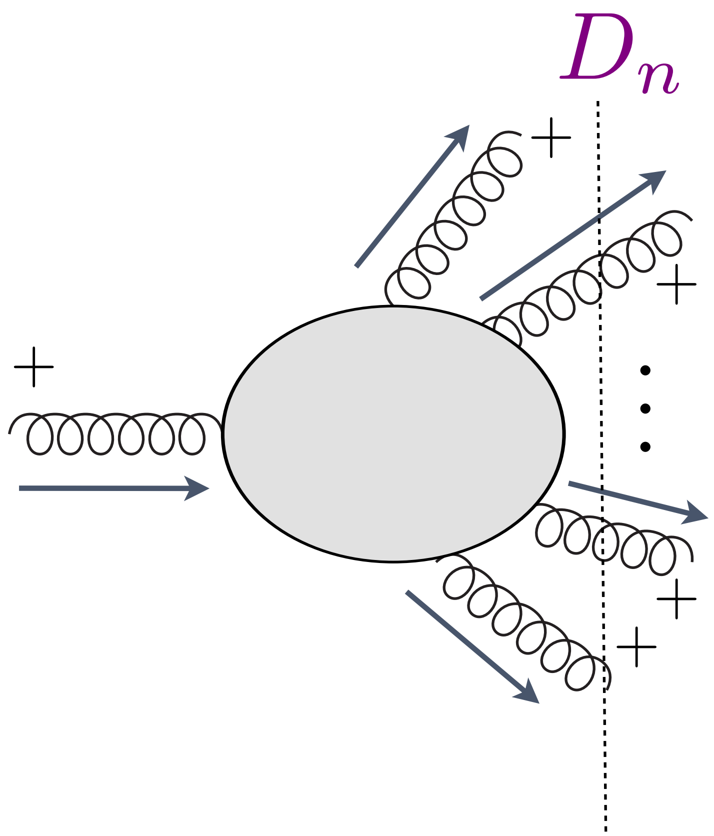
$$\times\, \frac{1}{v_{(12\ldots n-1)n}\, v_{(12\ldots n-2)(n-1\,n)}\,\cdots\, v_{1(2\ldots n)}}\,.$$

$$v_{(i_1 i_2\ldots i_p)(j_1 j_2\ldots j_q)} = \frac{k_{i_1}+k_{i_2}+\ldots+k_{i_p}}{z_{i_1}+z_{i_2}+\ldots+z_{i_p}} - \frac{k_{j_1}+k_{j_2}+\ldots+k_{j_q}}{z_{j_1}+z_{j_2}+\ldots+z_{j_q}},$$

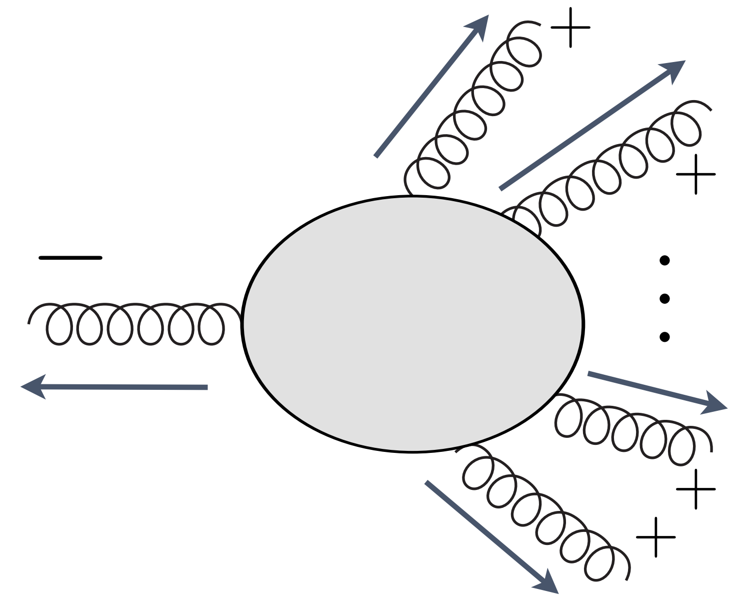
$$\xi_{(i_1 i_2\ldots i_p)(j_1 j_2\ldots j_q)} = \frac{(z_{i_1}+z_{i_2}+\ldots+z_{i_p})(z_{j_1}+z_{j_2}+\ldots+z_{j_q})}{z_{i_1}+z_{i_2}+\ldots+z_{i_p}+z_{j_1}+z_{j_2}+\ldots+z_{j_q}},$$



Light cone wave function and the Parke - Taylor amplitude



Wave function in the light-front formalism.



$$= 0$$

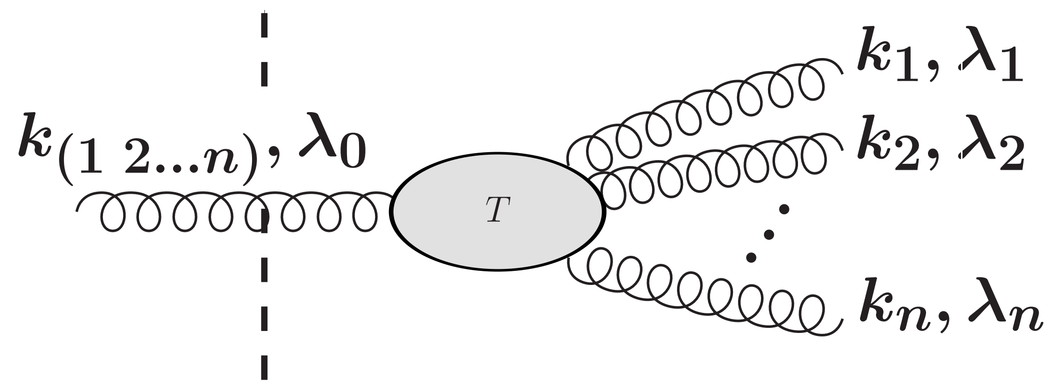
Standard result vanishes for this choice of helicities.

$$M(- + \cdots +) \sim \left(D_n \Psi_n(+ \to + \cdots +)\right)|_{D_n=0} = 0$$

$\Psi_n(+ \to + \cdots +)$ this wave function is non-singular when $D_n \to 0$

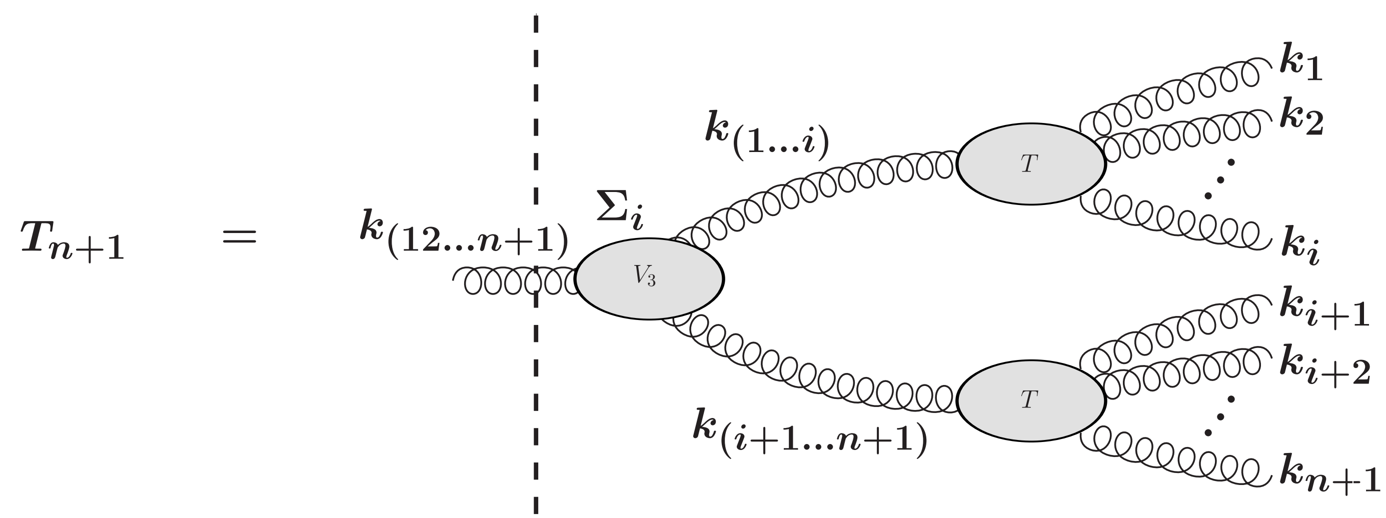
Consistent with the standard result, the on-shell amplitude vanishes as it is proportional to the denominator between initial and final states which vanishes due to the energy conservation. However the off-shell wave function is non zero here.

Fragmentation functions



Energy denominator included the first state

Factorization property (see also cluster expansion Brodsky et al)



Fragmentation function is factorized into sum over products of two fragmentation functions. This property holds when summed over all the light-front time orderings of the intermediate states. It does not hold at the level of the individual light-front diagrams since the energy denominators couple all the parts of the graph.

Fragmentation functions

Factorization property (see also cluster expansion Brodsky et al)

for the special case of only '+' helicities:

$$T_{n+1}[(12\ldots n+1) \to 1,2,\ldots,n+1] \;=\; -\frac{2ig}{D_{n+1}} \sum_{i=1}^{n} \left\{ \frac{v^*_{(1\ldots i)(i+1\ldots n+1)}}{\sqrt{\xi_{(1\ldots i)(i+1\ldots n+1)}}} \times T_i[(1\ldots i) \to 1,\ldots,i\,]\; T_{n+1-i}[(i+1\ldots n+1) \to i+1,\ldots,n+1] \right\}$$

Special case of light-front Berends-Giele recursion relation

Explicit solution in this case

$$T_n[(12\ldots n)^+ \to 1^+,2^+,\ldots,n^+] = (-ig)^{n-1}\left(\frac{z_{1\ldots n}}{z_1\,\ldots z_n}\right)^{3/2} \frac{1}{v_{n\,n-1}v_{n-1\,n-2}\ldots v_{21}}$$

$$T_{n+1} \to \mathcal{A}_{2\to n}$$

Fragmentation functions

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$$\left. \times\, T_i[(1\ldots i) \to 1,\ldots,i]\; T_{n+1-i}[(i+1\ldots n+1) \to i+1,\ldots,n+1] \right\}$$

Special case of light-front Berends-Giele recursion relation

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$$T_n[(12\ldots n)^+ \to 1^+,2^+,\ldots,n^+] = (-ig)^{n-1}\left(\frac{z_{1\ldots n}}{z_1\ldots z_n}\right)^{3/2} \frac{1}{v_{n\,n-1}v_{n-1\,n-2}\ldots v_{21}}$$

From this (off- shell) object one can obtain on-shell amplitude

$$T_{n+1} \to \mathcal{A}_{2\to n}$$

$$k_1 \to -k_1 \qquad \lambda_1 \to -\lambda_1$$

and remove the first energy denominator

$$\mathcal{A} \sim T_n D_n$$

$$D_n \to 0 \qquad \mathcal{A}(+- \;\to\; ++\cdots+) = 0$$

as a result of energy conservation

Relation to Parke-Taylor amplitudes

$$\mathcal{M}_n = \sum_{\{1,\ldots,n\}} \mathrm{tr}(t^{a_1} t^{a_2} \ldots t^{a_n})\, m(p_1, \epsilon_1; p_2, \epsilon_2; \ldots; p_n, \epsilon_n)\ ,$$

Color part

Kinematical part

Relation to Parke-Taylor amplitudes

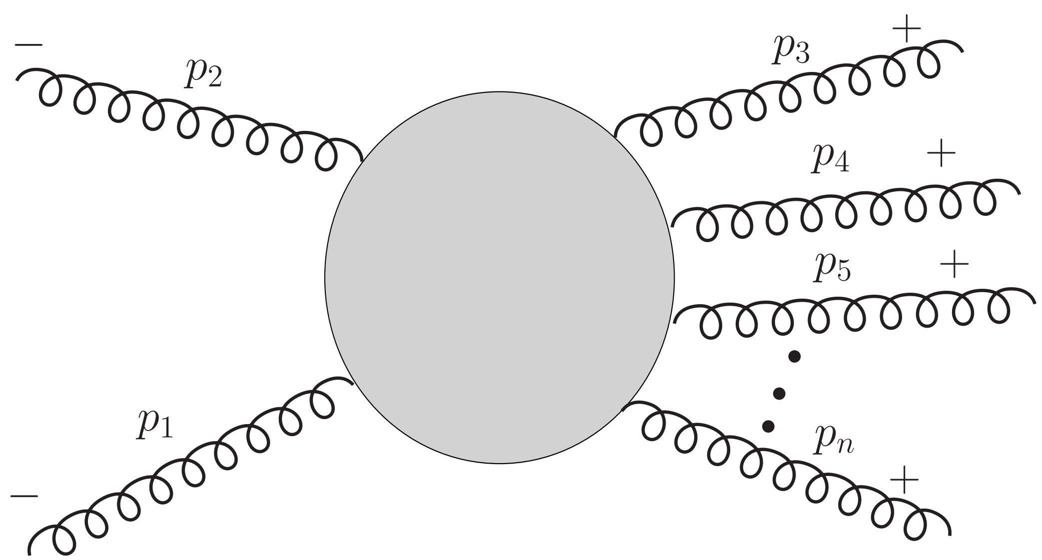
$$\mathcal{M}_n = \sum_{\{1,\dots,n\}} \mathrm{tr}(t^{a_1} t^{a_2} \dots t^{a_n})\, m(p_1, \epsilon_1; p_2, \epsilon_2; \dots; p_n, \epsilon_n)\ ,$$

Color part

Kinematical part

Maximally Helicity Violating amplitude for gluons: 2 to n

Here: all gluons are outgoing



$$\langle ij \rangle = \sqrt{z_i z_j}\, \underline{\epsilon}^{(+)} \cdot \left(\frac{\underline{k}_i}{z_i} - \frac{\underline{k}_j}{z_j} \right)$$

$$\langle\, ij\, \rangle = \sqrt{z_i z_j}\, \underline{\epsilon}^{(+)} \cdot \underline{v}_{ij},$$

Relation to Parke-Taylor amplitudes

$$\mathcal{M}_n = \sum_{\{1,\ldots,n\}} \mathrm{tr}(t^{a_1} t^{a_2} \ldots t^{a_n})\, m(p_1,\epsilon_1; p_2,\epsilon_2; \ldots; p_n,\epsilon_n)\ ,$$

Color part

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Maximally Helicity Violating amplitude for gluons: 2 to n

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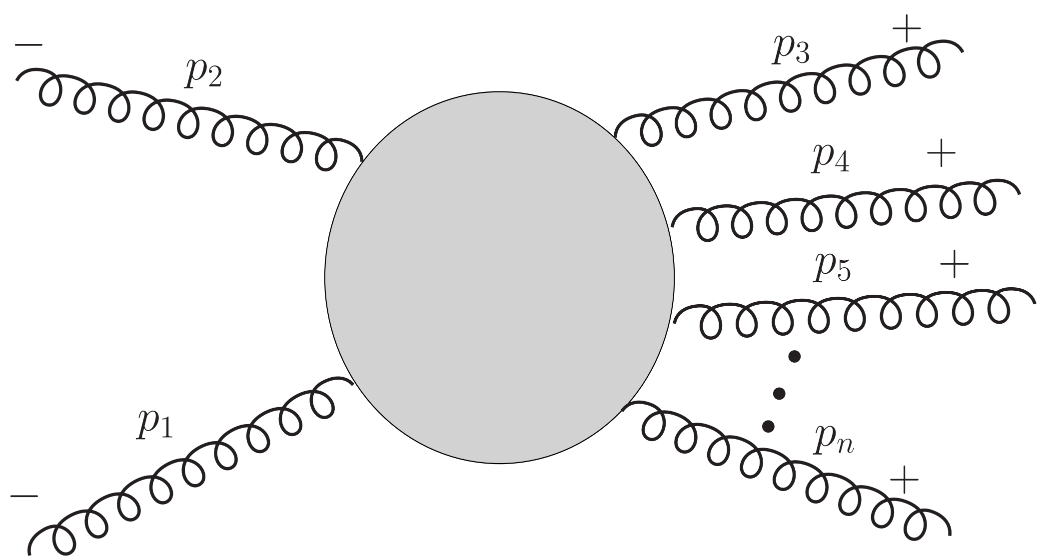


$$\langle ij \rangle = \sqrt{z_i z_j}\, \underline{\epsilon}^{(+)} \cdot \left(\frac{\underline{k}_i}{z_i} - \frac{\underline{k}_j}{z_j} \right)$$

$$\langle\, ij\, \rangle = \sqrt{z_i z_j}\, \underline{\epsilon}^{(+)} \cdot \underline{v}_{ij},$$

Tree level, Parke-Taylor formula

$$m(1^-, 2^-, 3^+, \ldots, n^+) = i g^{n-2} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n{-}2\; n{-}1 \rangle \langle n{-}1\; n \rangle \langle n1 \rangle}\ ,$$

Light-front to MHV dictionary...

$$|i\pm\rangle \;=\; \psi_\pm(k_i) \;=\; \frac{1}{2}(1\pm\gamma_5)\psi(k_i)\;\;, \quad \langle\pm i| \;=\; \overline{\psi_\pm(k_i)}\,,$$

$$\langle i|j\rangle = \langle i-|j+\rangle\,, \quad [ij] = \langle i+|j-\rangle$$

spinor products

$$\langle ij\rangle = \sqrt{2z_iz_j}\,\underline{\epsilon}^{(+)}\cdot\left(\frac{\underline{k}_i}{z_i}-\frac{\underline{k}_j}{z_j}\right)$$

$$[ij] = \sqrt{2z_iz_j}\,\underline{\epsilon}^{(-)}\cdot\left(\frac{\underline{k}_i}{z_i}-\frac{\underline{k}_j}{z_j}\right)$$

$$\boxed{\langle ij\rangle = \sqrt{2z_iz_j}\,\underline{\epsilon}^{(+)}\cdot\underline{v}_{ij}}$$

$$\boxed{[ij] = \sqrt{2z_iz_j}\,\underline{\epsilon}^{(-)}\cdot\underline{v}_{ij}}$$

Duality: wave function vs fragmentation

Wave function
initial state

Fragmentation
final state

$$\Psi_n \sim \frac{1}{v_{(12\ldots n-1)n}\, v_{(12\ldots n-2)(n-1\, n)} \;\cdots\; v_{1(2\ldots n)}}$$

$$T_n \sim \frac{1}{v_{12} v_{23} \ldots v_{n-1\, n}}$$

Nearly identical expressions (the same topology of graphs): different combinations of momenta

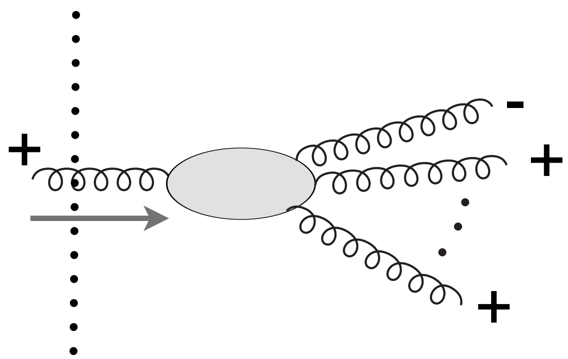
$$v_{(i_1 i_2 \ldots i_p)(j_1 j_2 \ldots j_q)} = \frac{k_{i_1} + k_{i_2} + \ldots + k_{i_p}}{z_{i_1} + z_{i_2} + \ldots + z_{i_p}} - \frac{k_{j_1} + k_{j_2} + \ldots + k_{j_q}}{z_{j_1} + z_{j_2} + \ldots + z_{j_q}},$$

$$\langle ij \rangle = \sqrt{2 z_i z_j}\, \underline{\epsilon}^{(+)} \cdot \underline{v}_{ij}$$

Similar form as in MHV

More general recursion

MHV amplitude $\quad (- - + \cdots +) \quad$ (here all particles outgoing)



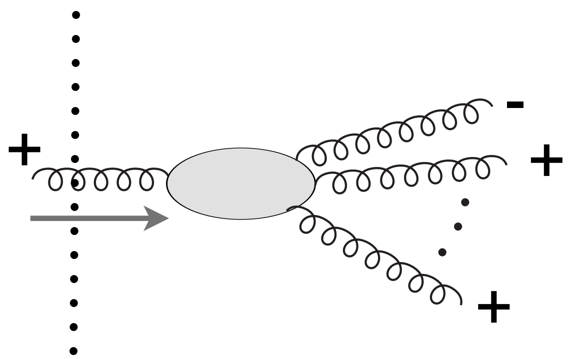
$T_n \quad (+ \to - + \cdots +)$ (1st is incoming particle)

$$T_n[(12\ldots n)^+ \to 1^-, 2^+, \ldots, n^+] = \frac{1}{\sqrt{z_{1\ldots n} z_1 \ldots z_n}} \frac{i}{D_n} \overline{M}_{1\to n}$$

Here $\overline{M}_{1\to n}$ is the off-shell amplitude with the denominator amputated.

More general recursion

MHV amplitude $\quad (- - + \cdots +)$ $\quad$ (here all particles outgoing)



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$$T_n[(12\ldots n)^+ \to 1^-, 2^+, \ldots, n^+] = \frac{1}{\sqrt{z_{1\ldots n} z_1 \ldots z_n}} \frac{i}{D_n} \overline{M}_{1\to n}$$

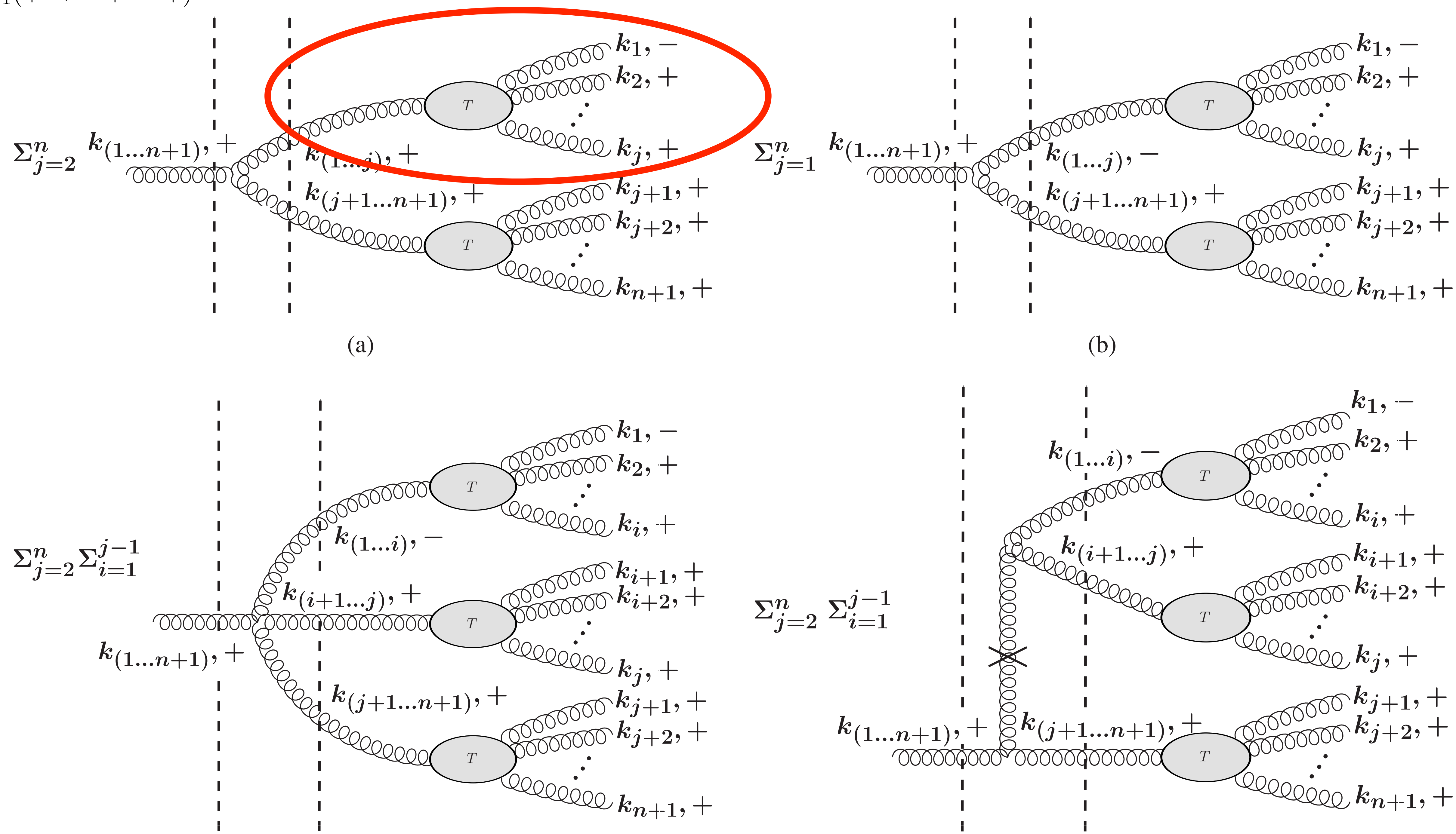
Here $\overline{M}_{1\to n}$ is the off-shell amplitude with the denominator amputated.

$$\begin{aligned}
\overline{M}_{1\to n+1} = &\sum_{j=2}^{n} V_+ \sqrt{\frac{z_1 z_2 \ldots z_{n+1}}{z_{1\ldots j} z_{j+1\ldots n+1}}}\; T_j[(1\ldots j)^+ \to 1^-, 2^+, \ldots, j^+]\; T_{n+1-j}[(j+1\ldots n+1)^+ \to (j+1)^+, \ldots, (n+1)^+] \\
&+ \sum_{j=1}^{n} V_- \sqrt{\frac{z_1 z_2 \ldots z_{n+1}}{z_{1\ldots j} z_{j+1\ldots n+1}}}\; T_j[(1\ldots j)^- \to 1^-, 2^+, \ldots, j^+]\; T_{n+1-j}[(j+1\ldots n+1)^+ \to (j+1)^+, \ldots, (n+1)^+] \\
&+ \sum_{j=2}^{n} \sum_{i=1}^{j-1} (V_4 + V_{\mathrm{Coul}}) \sqrt{\frac{z_1 z_2 \ldots z_{n+1}}{z_{1\ldots i} z_{i+1\ldots j} z_{j+1\ldots n+1}}}\, T_i[(1\ldots i)^- \to 1^-, 2^+, \ldots, i^+] \\
&\qquad \times\; T_{j-i}[(i+1\ldots j)^+ \to (i+1)^+, \ldots, j^+]\; T_{n+1-j}[(j+1\ldots n+1)^+ \to (j+1)^+, \ldots, (n+1)^+]\,.
\end{aligned}$$

light-front analog of Berends-Giele recursion relation. It is written on the level of off-shell fragmentation functions (or off-shell currents).

Recursion in terms of diagrams

$T_{n+1}(+ \to - + \cdots +) =$



Coulomb term

All MHV on the light front

Lowest order amplitudes:

$$\overline{M}_{1\to 2} = 2i^2 g^1 M_{1\to 2}, \qquad \overline{M}_{1\to 3} = 2i^3 g^2 \left\{ M_{1\to 3} - \frac{D_3}{D_2}\frac{z_{123}}{z_{12}}\frac{1}{z_3}\frac{M_{1\to 2}}{v_{3(123)}} \right\},$$

$$\overline{M}_{1\to 4} = 2i^4 g^3 \left\{ M_{1\to 4} - \frac{D_4}{D_3}\frac{z_{1234}^2}{z_{123}z_{1234}}\frac{1}{z_4}\frac{1}{v_{4(1234)}} M_{1\to 3} - \frac{D_4}{D_2}\frac{z_{1234}^2}{z_{12}z_{123}}\frac{1}{z_3 z_4}\frac{1}{v_{34}}\frac{M_{1\to 2}}{v_{3(123)}} \right\}$$

off-shell

MHV amplitudes

All MHV on the light front

Lowest order amplitudes:

$$\overline{M}_{1\to 2} = 2i^2 g^1 M_{1\to 2}, \qquad \overline{M}_{1\to 3} = 2i^3 g^2 \left\{ M_{1\to 3} - \frac{D_3}{D_2}\frac{z_{123}}{z_{12}}\frac{1}{z_3}\frac{M_{1\to 2}}{v_{3(123)}} \right\},$$

$$\overline{M}_{1\to 4} = 2i^4 g^3 \left\{ M_{1\to 4} - \frac{D_4}{D_3}\frac{z_{1234}^2}{z_{123}z_{1234}}\frac{1}{z_4}\frac{1}{v_{4(1234)}} M_{1\to 3} - \frac{D_4}{D_2}\frac{z_{1234}^2}{z_{12}z_{123}}\frac{1}{z_3 z_4}\frac{1}{v_{34}}\frac{M_{1\to 2}}{v_{3(123)}} \right\}$$

off-shell

MHV amplitudes

Solution by induction:

$$\overline{M}_{1\to n} = 2i^n g^{n-1} \left\{ M_{1\to n} - z_{1\ldots n}^2 D_n \sum_{i=2}^{n-1} \frac{1}{z_{1\ldots i} z_{1\ldots i+1}} \frac{1}{z_{i+1}\ldots z_n} \frac{1}{v_{i+1\, i+2}\ldots v_{n-1\, n}} \frac{M_{1\to i}}{v_{i+1(1\ldots i+1)} D_i} \right\}$$

$$\boxed{M_{1\to n} \equiv \frac{z_{1\ldots n}\, z_1}{z_2 z_3 \ldots z_n} \frac{v_{(1\ldots n)1}^3}{v_{12}v_{23}\ldots v_{n-1\, n} v_{n(1\ldots n)}}}$$

$$\boxed{i g^{n-2} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n-2\; n-1\rangle\langle n-1\; n\rangle\langle n1\rangle}}$$

Recursion formula

$\sim$

It is on-shell amplitude but evaluated for off-shell momenta

In the on-shell case the expected MHV result is recovered.

Recursion formula

$$\overline{M}_{1\to n} = 2i^n g^{n-1} \left\{ M_{1\to n} - z_{1\ldots n}^2 D_n \sum_{i=2}^{n-1} \frac{1}{z_{1\ldots i} z_{1\ldots i+1}} \frac{1}{z_{i+1} \ldots z_n} \frac{1}{v_{i+1\ i+2} \ldots v_{n-1\ n}} \frac{M_{1\to i}}{v_{i+1(1\ldots i+1)} D_i} \right\}$$

$$M_{1\to n} \equiv \frac{z_{1\ldots n}\ z_1}{z_2 z_3 \ldots z_n} \frac{v_{(1\ldots n)1}^3}{v_{12} v_{23} \ldots v_{n-1\ n} v_{n(1\ldots n)}} \quad \sim \quad i g^{n-2} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n-2\ n-1 \rangle \langle n-1\ n \rangle \langle n1 \rangle}$$

It is on-shell amplitude but evaluated for off-shell momenta

$$\overline{M}_{1\to n} \xrightarrow{D_n \to 0} 2i^n g^{n-1} M_{1\to n}$$

In the on-shell case the expected MHV result is recovered.

Recursion formula

What is the meaning of the derived recursion formula?

$$\overline{M}^{(+\to -+\cdots+)}_{1\to n} = 2i^n g^{n-1}\Big\{ M^{(+\to -+\cdots+)}_{1\to n} -$$

$$z^2_{1\ldots n} D_n \sum_{i=2}^{n-1} \frac{1}{z_{1\ldots i} z_{1\ldots i+1}} \frac{1}{z_{i+1}\ldots z_n} \frac{1}{v_{i+1\,i+2}\ldots v_{n-1\,n}} \frac{M^{(+\to -+\cdots+)}_{1\to i}}{v_{i+1(1\ldots i+1)} D_i}\Big\}\,.$$

$$D_i = \frac{1}{P^+}\left(\sum_{j=1}^{i} E_j^- - E_{1\ldots i}^-\right) = \sum_{j=1}^{i} \frac{k_j^2}{z_j} - \frac{k_{1\ldots i}^2}{z_{1\ldots i}}$$

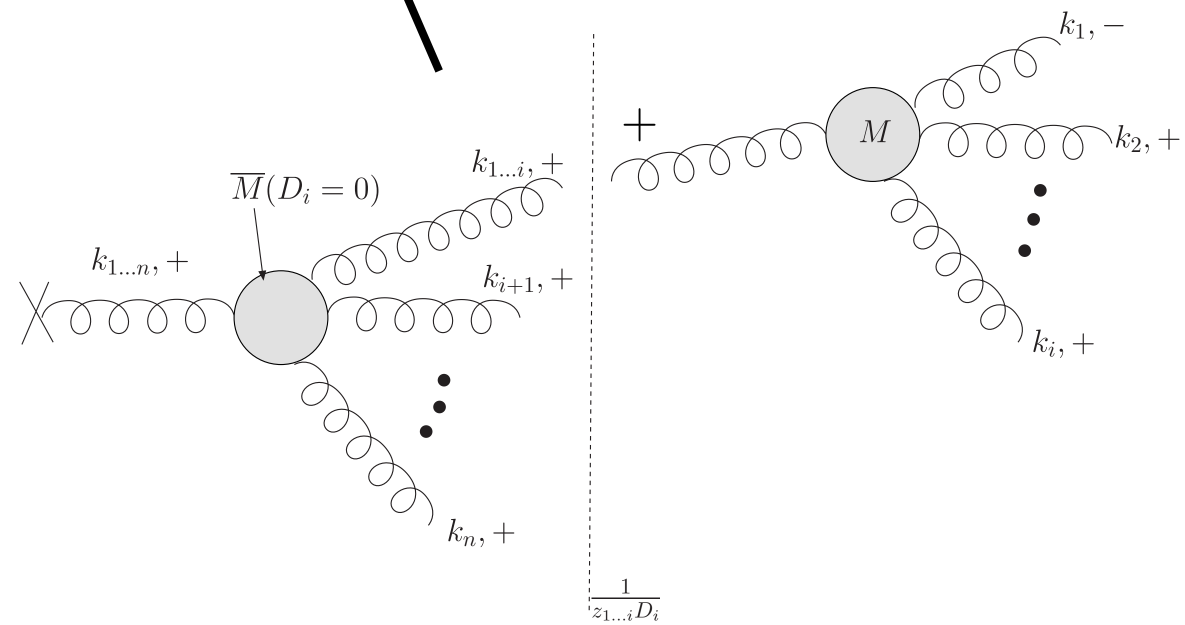
$$D_n = \frac{1}{P^+}\left(\sum_{j=1}^{n} E_j^- - E_{1\ldots n}^-\right) = \sum_{j=1}^{n} \frac{k_j^2}{z_j} - \frac{k_{1\ldots n}^2}{z_{1\ldots n}}$$

Can be recast into more compact form:

$$\overline{M}^{(+\to -+\cdots+)}_{1\to n} = 2i^n g^{n-1} M^{(+\to -+\cdots+)}_{1\to n} + i\sum_{i=2}^{n-1} \frac{D_n}{\tilde{D}_{n-i+1}} \overline{M}^{(+\to ++\cdots+)}_{1\to n-i+1} \frac{1}{z_{1\ldots i} D_i} (2i^i g^{i-1} M^{(+\to -+\cdots+)}_{1\to i})$$

$$\tilde{D}_{n-i+1} = D_n - D_i$$

$$\frac{D_n}{\tilde{D}_{n-i+1}} \overline{M}_{1\to n-i+1} = (\overline{M}_{1\to n-i+1})|_{D_i=0}$$



Recursion formula

What is the meaning of the derived recursion formula?

$$\overline{M}^{(+\to -+\cdots+)}_{1\to n} = 2i^n g^{n-1}\left\{ M^{(+\to -+\cdots+)}_{1\to n} - z^2_{1\ldots n} D_n \sum_{i=2}^{n-1} \frac{1}{z_{1\ldots i} z_{1\ldots i+1}} \frac{1}{z_{i+1}\ldots z_n} \frac{1}{v_{i+1\, i+2}\ldots v_{n-1\, n}} \frac{M^{(+\to -+\cdots+)}_{1\to i}}{v_{i+1(1\ldots i+1)} D_i} \right\} .$$

$$D_i = \frac{1}{P^+}\left(\sum_{j=1}^{i} E_j^- - E^-_{1\ldots i}\right) = \sum_{j=1}^{i} \frac{k_j^2}{z_j} - \frac{k^2_{1\ldots i}}{z_{1\ldots i}}$$

$$D_n = \frac{1}{P^+}\left(\sum_{j=1}^{n} E_j^- - E^-_{1\ldots n}\right) = \sum_{j=1}^{n} \frac{k_j^2}{z_j} - \frac{k^2_{1\ldots n}}{z_{1\ldots n}}$$

Can be recast into more compact form:

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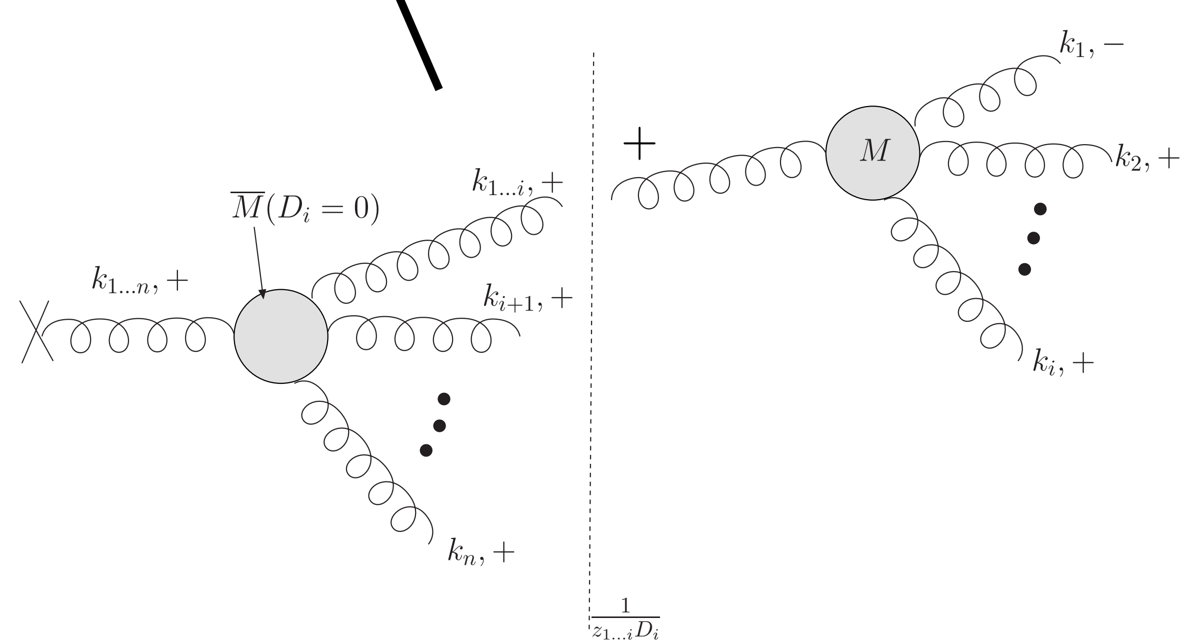
$$\tilde{D}_{n-i+1} = D_n - D_i$$

$$\frac{D_n}{\tilde{D}_{n-i+1}} \overline{M}_{1\to n-i+1} = (\overline{M}_{1\to n-i+1})|_{D_i=0}$$



Recursion formula

The second term is reminiscent of the BCFW formula but in the case of the off-shell amplitude.

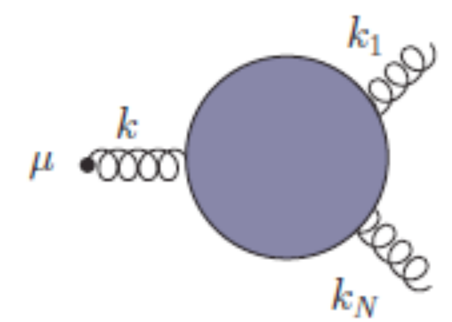
However, it is not obvious how the first term arises in the context of BCFW (could it be included in the sum as well). Also, shifts of the momenta into complex plane, which are the basis of BCFW are not entirely visible in this context. More analysis is needed here.

Interesting connection with gauge invariant off-shell amplitudes and Wilson lines

P. Kotko

Current with one off-shell leg

$$\mathcal{J}^{\mu}\left(\varepsilon_{k_1}^{\lambda_1}(q_1),\ldots,\varepsilon_{k_N}^{\lambda_N}(q_N)\right) =$$



Amplitude with one off-shell leg:

$$\mathcal{M}(\varepsilon;\varepsilon_1,\ldots,\varepsilon_N) = k^2\varepsilon_\mu\mathcal{J}^{\mu}(\varepsilon_1,\ldots,\varepsilon_N)$$

$$k\cdot\varepsilon_k^{\pm}(q) = q\cdot\varepsilon_k^{\pm}(q) = 0.$$

q reference momentum

Is not gauge invariant, thus does not satisfy Ward identity

$$\mathcal{M}(\varepsilon;\varepsilon_1,\ldots,k_i,\ldots,\varepsilon_N) \neq 0 \text{ for } i = 1,\ldots,N$$

Gauge invariant off-shell amplitudes

Matrix element

$$\mathfrak{M}^{aa_1\ldots a_N}\left(\varepsilon;\varepsilon_1,\ldots,\varepsilon_N\right)=\frac{1}{g}\int d^4x\, e^{ik\cdot x}$$
$$\left\langle 0\left|\mathcal{T}\left\{\mathrm{Tr}\left[t^a\mathcal{P}\exp\left(ig\int_{-\infty}^{+\infty}ds\, A_\mu^b\left(x+s\varepsilon\right)\varepsilon^\mu t^b\right)\right]e^{iS_{\mathrm{Y-M}}}\right\}\right|k_1,\varepsilon_1,a_1;\ldots;k_N,\varepsilon_N,a_N\right\rangle_c$$

Color ordered amplitude

$$\mathfrak{M}\left(\varepsilon;\varepsilon_1,\ldots,\varepsilon_N\right)=\delta^4\left(k-k_1-\ldots-k_N\right)\delta\left(\varepsilon\cdot k\right)\tilde{\mathcal{M}}\left(\varepsilon;\varepsilon_1,\ldots,\varepsilon_N\right)$$

Wilson line direction and the momentum need to be mutually transverse

Ward identity is satisfied with respect to the on-shell legs.

$$\tilde{\mathcal{M}}\left(\varepsilon;\varepsilon_1,\ldots,k_i,\ldots,\varepsilon_N\right)=0\quad\text{for}\quad i=1,\ldots,N.$$

The relation between different objects is that:

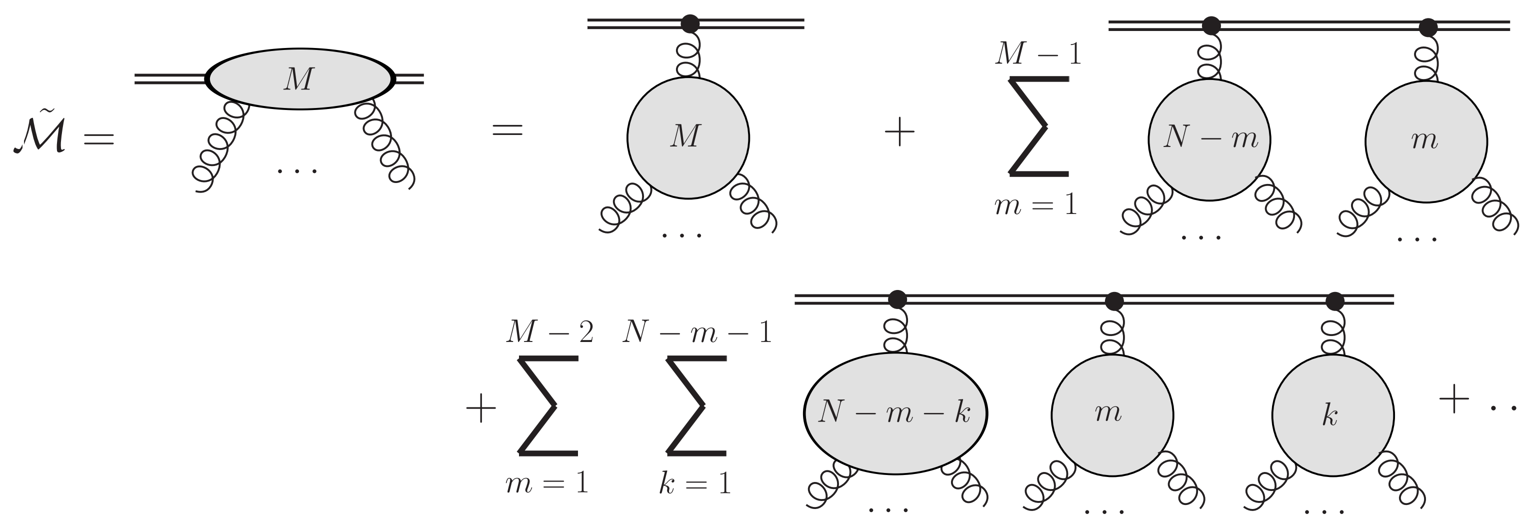
$$\tilde{\mathcal{M}}=\overline{\mathcal{M}}+\text{terms required for gauge invariance.}$$

Ward identity ok

Ward identity not-ok

Gauge invariant off-shell amplitudes

Relation between the different objects: P. Kotko



It can be shown that (at least for the sample choices of helicities (-+...+) and (--...+))
the results are identical to the ones derived in the light-cone perturbation theory

$$\frac{1}{k^2}\mathcal{M}(-;1^+,\dots,N^+) = \quad T_n[(12\dots n)^+ \to 1^+,2^+,\dots,n^+] = (-ig)^{n-1}\left(\frac{z_{1\dots n}}{z_1\dots z_n}\right)^{3/2}\frac{1}{v_{n\,n-1}v_{n-1\,n-2}\dots v_{21}}$$

For the (--+...+) case the above relation is identical to the light-front recursion

$$\overline{M}_{1\to n}^{(+\to -+\cdots+)} = 2i^n g^{n-1} M_{1\to n}^{(+\to -+\cdots+)} + i\sum_{i=2}^{n-1}\frac{D_n}{\tilde{D}_{n-i+1}}\overline{M}_{1\to n-i+1}^{(+\to ++\cdots+)}\frac{1}{z_{1\dots i}D_i}\left(2i^i g^{i-1}M_{1\to i}^{(+\to -+\cdots+)}\right)$$

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- Can the last recursion (i.e. between off-shell and on-shell amplitudes) be used for higher order computation?