Renormalized entanglement entropy, and RG flows

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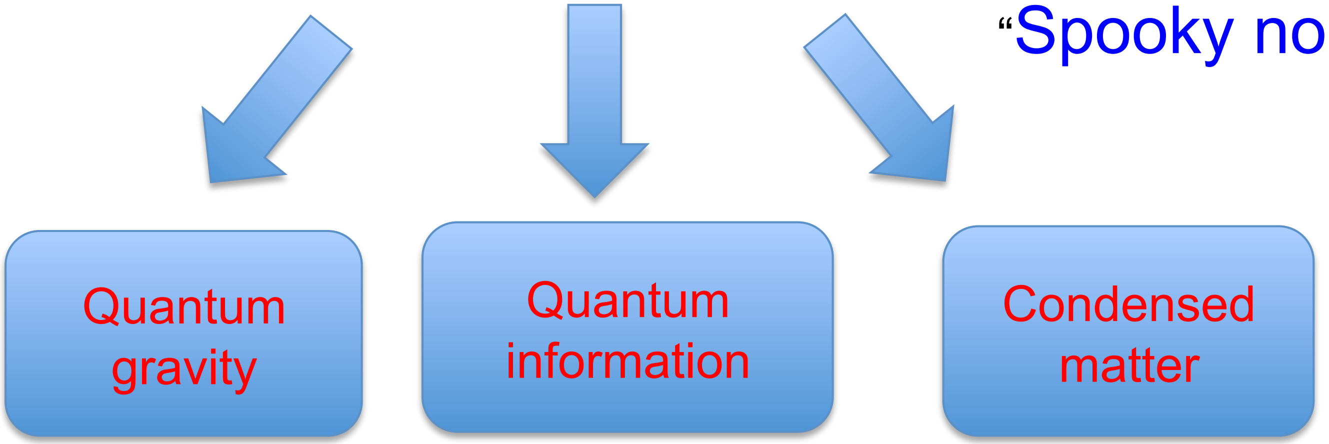
Based on Mark Mezei and HL, **arXiv:1202.2070**
Mark Mezei and HL, **arXiv:1309.6935**

Quantum entanglement

Quantum entanglement

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

"Spooky non-locality"

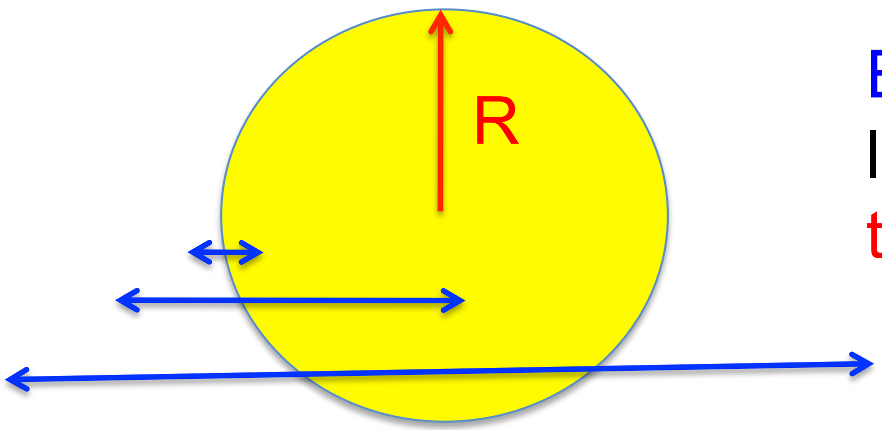


Bi-partite entanglement: entanglement entropy

$$S^{(\Sigma)} = -\mathrm{Tr}\rho_A \log \rho_A$$

(will focus on vacuum)

Entanglement entropy



Expect it to depend on physics at length scales ranging from size R all the way to short-distance cutoff δ .

$$S^{(\Sigma)} \propto \frac{A_\Sigma}{\delta^{d-2}} + \cdots$$

dominated by short-distance physics

δ : Short-distance cutoff

(Bombelli et al, Srednicki)

"unpleasant" features:

- ill-defined in the continuum limit: Divergent for a renormalizable QFT
- Long range correlations hard to extract.

Even in the large R limit, still sensitive to all the shorter-distance d.o.f., not clear it will reduce to the behavior of the IR fixed point.

Note: entanglement entropy cannot be defined in terms of operators, standard QFT techniques of renormalization do not apply.

Common practice:

subtract the UV divergent parts by hand, often ambiguous (e,g. typically not invariant under reparametrizations of the cutoff)

Even after the subtraction, could still depend on physics at scales much smaller than the size of the entangled region.

Free massive fields

Consider a free massive scalar field for a spherical region in the regime mR >> 1 in d=3:

Herzberg and Wilczek, Huerta

$$S_{\text{scalar}}(mR) = \# \cancel{\frac{R}{\delta}} - \frac{\pi}{6} mR - \frac{\pi}{240} \frac{1}{mR} + \cdots$$

The finite part diverges linearly in R and does not have a well defined limit in the large R limit.

At long distances, the system contains nothing.

Ideally, we would have liked to have the EE to go to zero.

- Some short-distance physics (1/m) remaining
- ambiguous: can be changed by $\delta \to \delta(1 + m\delta + \cdots)$

Entanglement entropy (even after subtracting the divergences) contains too much short-distance “junk”

Would like to be able to directly probe entanglement relations at a given scale.

Would like to understand how entanglement correlations change with scale: RG flow of entanglement.

It turns out an almost trivial procedure goes some distance toward these goals.

Plan

- Introduce Renormalized entanglement entropy
  - UV finite, well-defined in the continuum limit
  - characterize quantum entanglement at a given scale.
  - "track" the number of degrees of freedom of the system at a given scale.
  - reveal new features of renormalization group flows: holographic examples

"Renormalized entanglement entropy"

For any entangling (smooth) surface Σ with a scalable size R:

$$\mathcal{S}_d^{(\Sigma)}(R) = \begin{cases} \frac{1}{(d-2)!!}\left(R\frac{d}{dR}-1\right)\left(R\frac{d}{dR}-3\right)\cdots\left(R\frac{d}{dR}-(d-2)\right) S^{(\Sigma)}(R) & \text{d odd} \\ \frac{1}{(d-2)!!}R\frac{d}{dR}\left(R\frac{d}{dR}-2\right)\cdots\left(R\frac{d}{dR}-(d-2)\right) S^{(\Sigma)}(R) & \text{d even} \end{cases}.$$

d=2: $\mathcal{S}_2(R) = R\dfrac{dS}{dR}$

d=3: $\mathcal{S}_3^{(\Sigma)}(R) = R\dfrac{\partial S^{(\Sigma)}}{\partial R} - S^{(\Sigma)}$

d=4:

$$\mathcal{S}_4^{(\Sigma)}(R) = \frac{1}{2}R\partial_R(R\partial_R S^{(\Sigma)} - 2S^{(\Sigma)}) = \frac{1}{2}\left(R^2\frac{\partial^2 S^{(\Sigma)}}{\partial R^2} - R\frac{\partial S^{(\Sigma)}}{\partial R}\right)$$

Divergence structure of entanglement entropy

Grover, Turner, Vishwanath; HL and Mezei

The divergent part of EE should only depend on local physics at the cutoff scale near the entangling surface,

$$S^{(\Sigma)}_{\rm div} = \int_\Sigma d^{d-2}\sigma \sqrt{h} F(K_{ab}, h_{ab})$$

h: induced metric,
K: extrinsic curvature

F: sum of all possible geometric invariants

$$S^{(\Sigma)}_{\rm div} = S^{(\bar\Sigma)}_{\rm div}$$

$$\Rightarrow \quad S^{(\Sigma)}_{\rm div} = a_1 R^{d-2} + a_2 R^{d-4} + \cdots$$ (scalable surface)

a_1,a_2 : divergent coefficients, in general complicated functions of dimensional parameters of a system.

Renormalizability: No negative powers of R.

UV finiteness

Given: $$S_{\text{div}}^{(\Sigma)} = a_1 R^{d-2} + a_2 R^{d-4} + \cdots$$

$$\mathcal{S}_d^{(\Sigma)}(R) = \begin{cases} \frac{1}{(d-2)!!}\left(R\frac{d}{dR}-1\right)\left(R\frac{d}{dR}-3\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d odd} \\ \frac{1}{(d-2)!!}R\frac{d}{dR}\left(R\frac{d}{dR}-2\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d even} \end{cases}.$$

will then get rid of all UV divergent terms for any QFT.

The differential operator also gets rid of finite terms of the same R-dependence.

Such terms can be modified by redefining the cutoff, thus not well defined in the continuum limit (“contaminated”).

$\mathcal{S}_d^{(\Sigma)}(R)$ is thus UV finite, and unambiguous (independent of reparametrizations of the cutoff).

CFT

For a scale invariant system, we must have:

$$\mathcal{S}_d^{(\Sigma)}(R) = s_d^{(\Sigma)}$$

Converting it back to the EE itself, we then have

$$S^{(\Sigma)} = \begin{cases} \frac{R^{d-2}}{\delta_0^{d-2}} + \cdots + \frac{R}{\delta_0} + (-1)^{\frac{d-1}{2}} s_d^{(\Sigma)} & \text{odd d} \\ \frac{R^{d-2}}{\delta_0^{d-2}} + \cdots + \frac{R^2}{\delta_0^2} + (-1)^{\frac{d-2}{2}} s_d^{(\Sigma)} \log \frac{R}{\delta_0} + \text{const} & \text{even d} \end{cases}$$

This agrees with what was previously found from holographic calculations. (Ryu, Takayanagi)

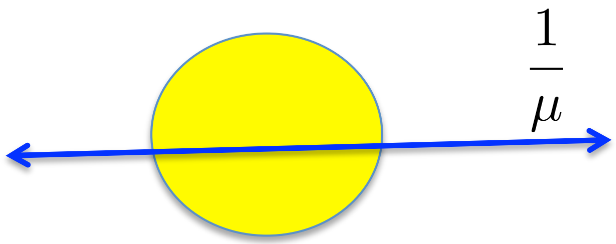
$s_d^{(\Sigma)}$ is the “universal” part of the entanglement entropy.

General QFTs

Contains mass parameters: $\mu_1, \mu_2, \cdots$

In the small R limit:

$$S^{(\Sigma)}(R) \to S^{(\Sigma,\mathrm{UV})}, \quad R \to 0$$



$$\mathcal{S}_d^{(\Sigma)}(R) \to s_d^{(\Sigma,\mathrm{UV})}, \quad R \to 0$$

In the large R limit: $\left(R \gg \frac{1}{\mu_1}, \frac{1}{\mu_2}, \cdots\right)$

$S^{(\Sigma)}(R)$ depends on physics at all scales from δ_0 to R including $\mu_1, \mu_2, \cdots$

Introducing a floating cutoff δ : $\frac{1}{\mu_1}, \frac{1}{\mu_2}, \cdots \ll \delta \ll R$

$$\mathcal{S}_d^{(\Sigma)}(R) \to s_d^{(\Sigma,\mathrm{IR})}, \quad R \to \infty$$

It is most sensitive to degrees of freedom at scale R.

Summary

$$\mathcal{S}_d^{(\Sigma)}(R) = \begin{cases} \frac{1}{(d-2)!!}\left(R\frac{d}{dR}-1\right)\left(R\frac{d}{dR}-3\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d odd} \\ \frac{1}{(d-2)!!}R\frac{d}{dR}\left(R\frac{d}{dR}-2\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d even} \end{cases}.$$

- UV finite, well-defined in the continuum limit
- R-independent for a scale invariant system $\mathcal{S}_d^{(\Sigma)}(R) = s_d^{(\Sigma)}$
- For a general quantum field theory $\mathcal{S}_d^{(\Sigma)}(R) \to \begin{cases} s_d^{(\Sigma,\mathrm{UV})} & R \to 0 \\ s_d^{(\Sigma,\mathrm{IR})} & R \to \infty \end{cases}.$

⇒ $\mathcal{S}_d^{(\Sigma)}(R)$ can be interpreted as characterizing entanglement at scale R.

The R-dependence can be interpreted as describing the “RG” flow of entanglement entropy with distance scale.

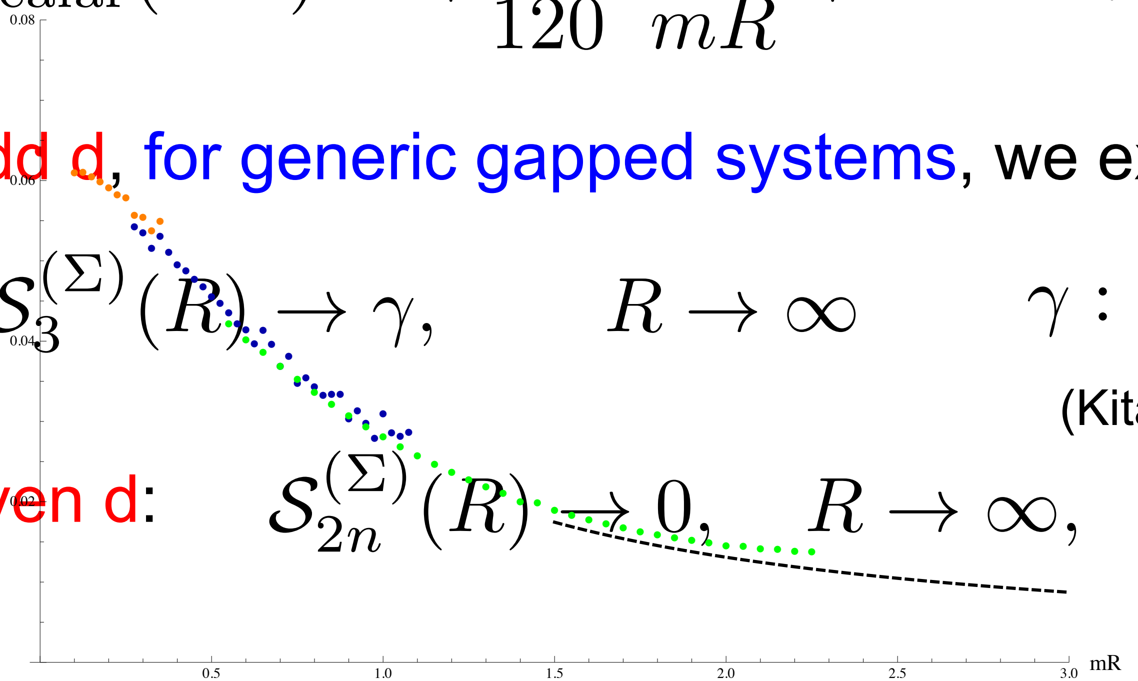
Note: definition not unique, simplest

Gapped systems

For a free massive scalar field for a spherical region in the regime mR >> 1 in d=3:

$$S_{\rm scalar}(mR) = \# \cancel{\frac{R}{\delta}} - \cancel{\frac{\pi}{6} mR} - \frac{\pi}{240}\frac{1}{mR} + \cdots$$

$$\mathcal{S}_{\rm scalar}(mR) = +\frac{\pi}{120}\frac{1}{mR} + \cdots \quad \to 0$$



In odd d, for generic gapped systems, we expect: (e.g. d=3)

$$\mathcal{S}_3^{(\Sigma)}(R) \to \gamma, \qquad R \to \infty \qquad \gamma :$$ Topological entanglement entropy

(Kitaev, Preskill; Levin, Wen)

In even d: $\mathcal{S}_{2n}^{(\Sigma)}(R) \to 0, \quad R \to \infty, \qquad n = 1, 2, \cdots$

An application: non-Fermi liquids

For a system with a Fermi surface, expect at large R:

$$\mathcal{S}_d^{(\Sigma)}(R) \propto A_{FS} \propto k_F^{d-2}$$

(independent whether the system has quasiparticles or not)

$$\mathcal{S}_d^{(\Sigma)}(R) \propto k_F^{d-2} R^{d-2} \propto A_{FS} A_{\Sigma}, \qquad R \to \infty$$

$$S^{\Sigma}(R) \sim k_F^{d-2} R^{d-2} \log(k_F R) \sim A_{FS} A_{\Sigma} \log(A_{FS} A_{\Sigma})$$

Wolf; Gioev, Klich
Swingle,

Similarly for higher co-dimensional Fermi surfaces:

$$\mathcal{S}_d^{(\Sigma)}(R) \propto (k_F R)^{d-n}$$

$$S^{\Sigma}(R) \propto \begin{cases} (k_F R)^{d-n} \log(k_F R) & \text{n even} \\ (k_F R)^{d-n} & \text{n odd} \end{cases}$$

Renyi entropy

$$S_n(A) = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n$$

One can similarly define "renormalized Renyi entropies."

All the earlier discussions also apply to the renormalized Renyi entropies including log-enhancement for a non-Fermi liquid.

Entropic function in theory space

From REE, one can introduce an "entropic function" defined in the space of couplings:

$$\mathcal{C}^{(\Sigma)}(g^a(\Lambda)) \equiv \mathcal{S}^{(\Sigma)}\left(R\Lambda, g^a(\Lambda)\right)\Big|_{R=\frac{1}{\Lambda}} = \mathcal{S}^{(\Sigma)}\left(1, g^a(\Lambda)\right)$$

$$\Lambda: \quad \mathrm{RG\ scale}$$

$$\Lambda \frac{d\mathcal{C}^{(\Sigma)}(g^a(\Lambda))}{d\Lambda} = -R\frac{d\mathcal{S}^{(\Sigma)}\left(R\Lambda, g^a(\Lambda)\right)}{dR}\Bigg|_{R=\frac{1}{\Lambda}}$$

Application: EE and the number of d.o.f.

RG: integrating out degrees of freedom

Could $\mathcal{S}_d^{(\Sigma)}(R)$ track the loss of d.o.f.?

Expect the tendency is for it to decrease with R.

If $$R\frac{d\mathcal{S}_d^{(\Sigma)}(R)}{dR} < 0$$

Given $$\mathcal{S}_d^{(\Sigma)}(R) \to \begin{cases} s_d^{(\Sigma,\mathrm{UV})} & R \to 0 \\ s_d^{(\Sigma,\mathrm{IR})} & R \to \infty \end{cases} .$$

$\Rightarrow$ $s_d^{(\Sigma,\mathrm{UV})} > s_d^{(\Sigma,\mathrm{IR})}$ i.e. a c-theorem.

$s_d^{(\Sigma)}$: central charge

$\mathcal{S}_d^{(\Sigma)}(R)$: central function

d=2

$$\mathcal{S}_2(R) = R\frac{dS}{dR}$$

For a CFT $\quad \mathcal{S}_2 = \frac{c}{3}$ Holzhey, Larsen, Wilczek

For Lorentz-invariant, unitary QFTs Casini and Huerta

$$\mathcal{S}_2(R) \text{ monotonic}$$

alternative proof of
Zamolodchikov's c-theorem

using strong subadditivity condition.

Higher dimensions

$\mathcal{S}_d^{(\Sigma)}(R)$ now depends on the shape of Σ.

Will all shapes work?

d=4: for a CFT

Solodukhin

$$s_4^{(\Sigma)} = 2a_4 \int_\Sigma d^2\sigma \sqrt{h}\, E_2 + c_4 \int_\Sigma d^2\sigma \sqrt{h}\, I_2$$

a_4,c_4 : coefficients of trace anomaly

I_2 vanishes for sphere, $s_4^{(\mathrm{sphere})} = 4a_4$

For a general shape, will be a combination of a and c.

Thus only for a sphere, do we always have

$$s_4^{(\Sigma,\mathrm{UV})} > s_4^{(\Sigma,\mathrm{IR})}$$

Higher dimensions

$\mathcal{S}_d^{(\Sigma)}(R)$ now depends on the shape of Σ.

Sphere has the best chance

For all even spacetime dimensions:

Myers, Sinha
Casini, Myers, Heurta

$$s_{2n}^{(\mathrm{sphere})} = 4a_{2n}$$

Casini, Myers, Heurta

For all odd dimension: $s_d^{(\mathrm{sphere})} = (\log Z)_{\mathrm{finite}}$

$(\log Z)_{\mathrm{finite}}$: finite part of the Euclidean partition for the CFT on S^d

$\mathcal{S}_d(R)$ (for a sphere) if monotonic,

Lead to the conjectured a-theorem or F-theorem in all dimensions

Cardy,
Myers, Sinha
Jefferis, Klebanov, Pufu and Safdi

d=3

$$\mathcal{S}_3(R) = R\frac{dS}{dR} - S$$

Free massive scalar and various holographic examples:

Conjecture: $\mathcal{S}_3(R)$ monotonically decreasing with R and non-negative

for all Lorentz invariant, unitary QFTs

Monotonicity $\Longleftrightarrow$ $S''(R) < 0$

Casini and Huerta have given a proof shortly after (1202.5650).

But the conjecture of non-negativeness apperas open.

d=4

$$\mathcal{S}_4(R) = \frac{1}{2}\left(R^2 \frac{d^2 S}{dR^2} - R\frac{dS}{dR}\right)$$

Various holographic examples: $\mathcal{S}_4(R)$ neither monotonic nor non-negative

Nevertheless $\mathcal{S}_4(R \to 0) > \mathcal{S}_4(R \to \infty)$ from a-theorem

- the function form should be modified
- Monotonicity of $\mathcal{S}_4$ or its improvement would imply an inequality for S with least three derivatives.

$$R^3\partial_R^3 S + R^2\partial_R^2 S < R\partial_R S$$

not clear it could arise from the strong subadditivity condition.

Application: new perspectives on renormalization group flows

Asymptotic behavior near a UV fixed point (small R behavior) ?

Asymptotic behavior near an IR fixed point (large R behavior) ?

Any interesting behavior along the flow ?

Strategy: use Ryu-Takayanagi prescription to compute renormalized entanglement entropy (for a sphere) in QFTs with a gravity dual.

Zoo of holographic systems

The RG flow of a Lorentz-invariant holographic system in the vacuum:

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right) \qquad f(z) \to 1, \; z \to 0$$

1. IR fixed point:

$$f(z) \to \frac{L^2}{L_{IR}^2} \equiv f_\infty > 1, \quad z \to \infty$$

2. Singular geometries:

$$f(z) = a z^n + \cdots, \quad a > 0, \quad n > 0$$

Behavior near a UV fixed point

In all holographic theories:

For small R $\mathcal{S}_d(R) = s_d^{(\mathrm{UV})} - A(\alpha)(\mu R)^{2\alpha} + \cdots \; A(\alpha) > 0$

$\alpha = d - \Delta$ $\quad\Delta$: dimension of least relevant operator

$\mathcal{C}_d(g) = s_d^{(\mathrm{UV})} - A(\Delta) g_{eff}^2(\Lambda)$ g: least relevant coupling

See also Klebanov, Nishioka, Pufu, Safdi

However, in d=2:

Casini, Huerta

$$\text{free scalar :} \qquad \mathcal{S}_2(R) = \frac{1}{3} + \frac{1}{\log\left(m^2 R^2\right)} + \cdots$$

$$\text{Dirac fermion :} \qquad \mathcal{S}_2(R) = \frac{1}{3} - 4 m^2 R^2 \, \log^2\left(m^2 R^2\right) + \cdots$$

Non-analytic behavior for free scalar in d=3

Klebanov, Nishioka, Pufu, Safdi

Behavior near an IR fixed point

HL, Mezei

Δ Dimension of leading irrelevant operator $\qquad \tilde{\alpha} = \Delta - d$

$$\text{For} \quad \tilde{\alpha} < \begin{cases} \frac{1}{2} & \text{odd d} \\ 1 & \text{even d} \end{cases}$$

For large R

$$\mathcal{S}_d(R) = s_d^{(\mathrm{IR})} + \frac{B(\tilde{\alpha})}{(\tilde{\mu}R)^{2\tilde{\alpha}}} + \cdots \quad B(\tilde{\alpha}) > 0$$

$$\sim s_d^{(IR)} + O(g^2)$$

$$B(\tilde{\alpha}) = \frac{1}{(1-2\tilde{\alpha}) f_\infty^{3/2}} \qquad \text{(d=3)}$$

Behavior near an IR fixed point

HL, Mezei

Δ Dimension of leading irrelevant operator $\qquad \tilde{\alpha} = \Delta - d$

$$\tilde{\alpha} > \begin{cases} \frac{1}{2} & \text{odd d} \\ 1 & \text{even d} \end{cases} \qquad \text{i.e.} \qquad \Delta > \begin{cases} 3.5 & d = 3 \\ 5 & d = 4 \end{cases}$$

$$\mathcal{S}_d(R) = s_d^{(\mathrm{IR})} + \begin{cases} \frac{\#}{R} + \cdots & \text{odd d} \\ \frac{\#}{R^2} + \cdots & \text{even d} \end{cases}$$

$$\mathcal{C}_d(g) = s_d^{(\mathrm{IR})} + C g^{\gamma} \qquad \gamma = \begin{cases} \tilde{\alpha}^{-1} & \text{odd d} \\ 2\tilde{\alpha}^{-1} & \text{even d} \end{cases} < 2$$

$$C = \int_0^\infty dz \left(\frac{z^2}{\sqrt{f(z)}} \left[\int_z^\infty dv \, \frac{1}{v^2 \sqrt{f(v)}} \right]^2 - \frac{1}{f_\infty^{3/2}} \right) \quad \text{(d=3)}$$

Physical reason: remnant UV sensitivity

$$\Delta > \begin{cases} 3.5 & d = 3 \\ 5 & d = 4 \end{cases}$$

Entanglement entropy provides a sensitive probe of aspects of RG flow.

Understanding singular geometries

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right)$$

$$f(z) = az^n + \cdots, \quad a > 0, \quad n > 0$$

n>2: $$\mathcal{S}_d(R) = \begin{cases} \frac{1}{R} + \cdots & \text{d odd} \\ \frac{1}{R^2} + \cdots & \text{d even} \end{cases}$$

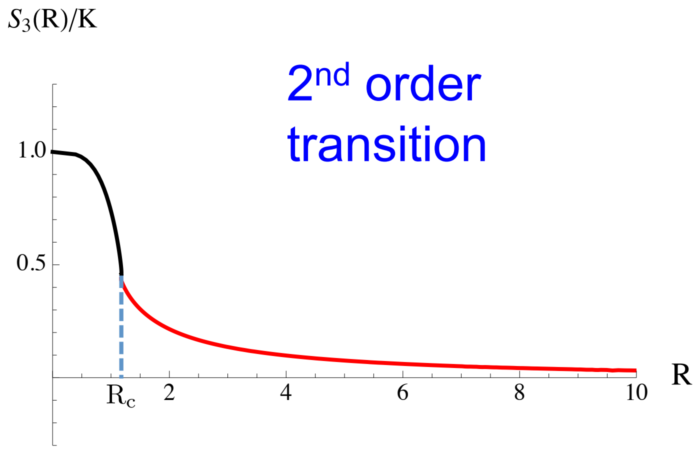
gapped systems

n<2: Additional terms non-analytic powers in R

contain gapless modes

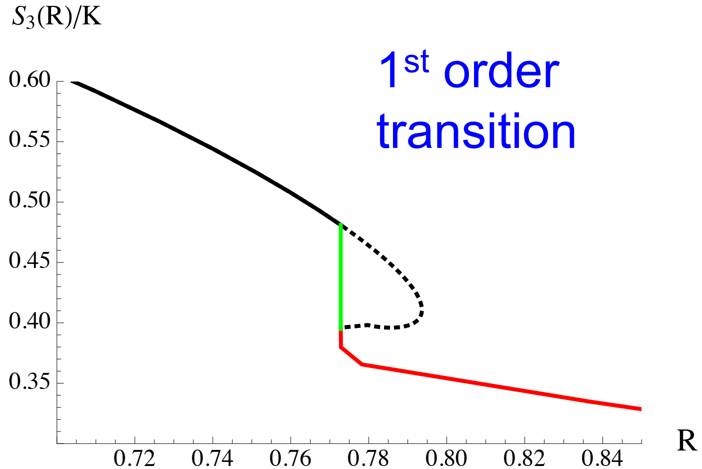
n=2: Gap plus a continuum

“Phase transitions”

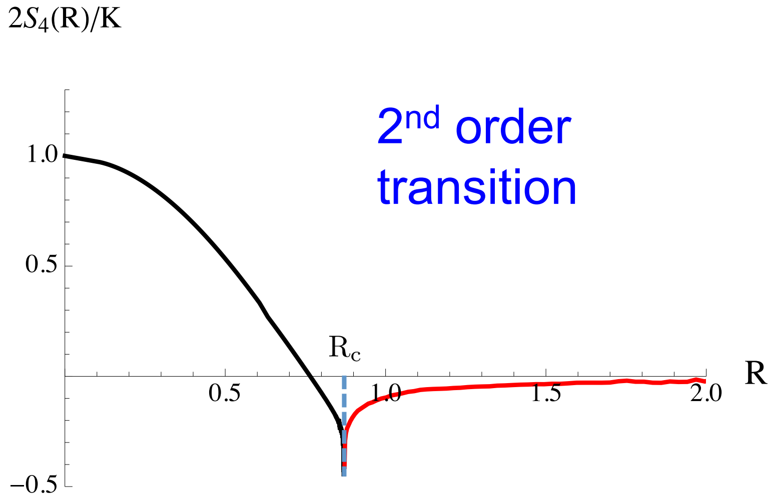


2nd order transition

d=3



1st order transition



2nd order transition

d=4

See also Klebanov, Nishioka, Pufu,Safdi

GPPZ flow

(Girardello, Petrini, Porrati,Zaffaroni)

“Phase transitions”

These are non-analytic behavior in entanglement entropy as a function of size for the vacuum of a Lorentz invariant QFT.

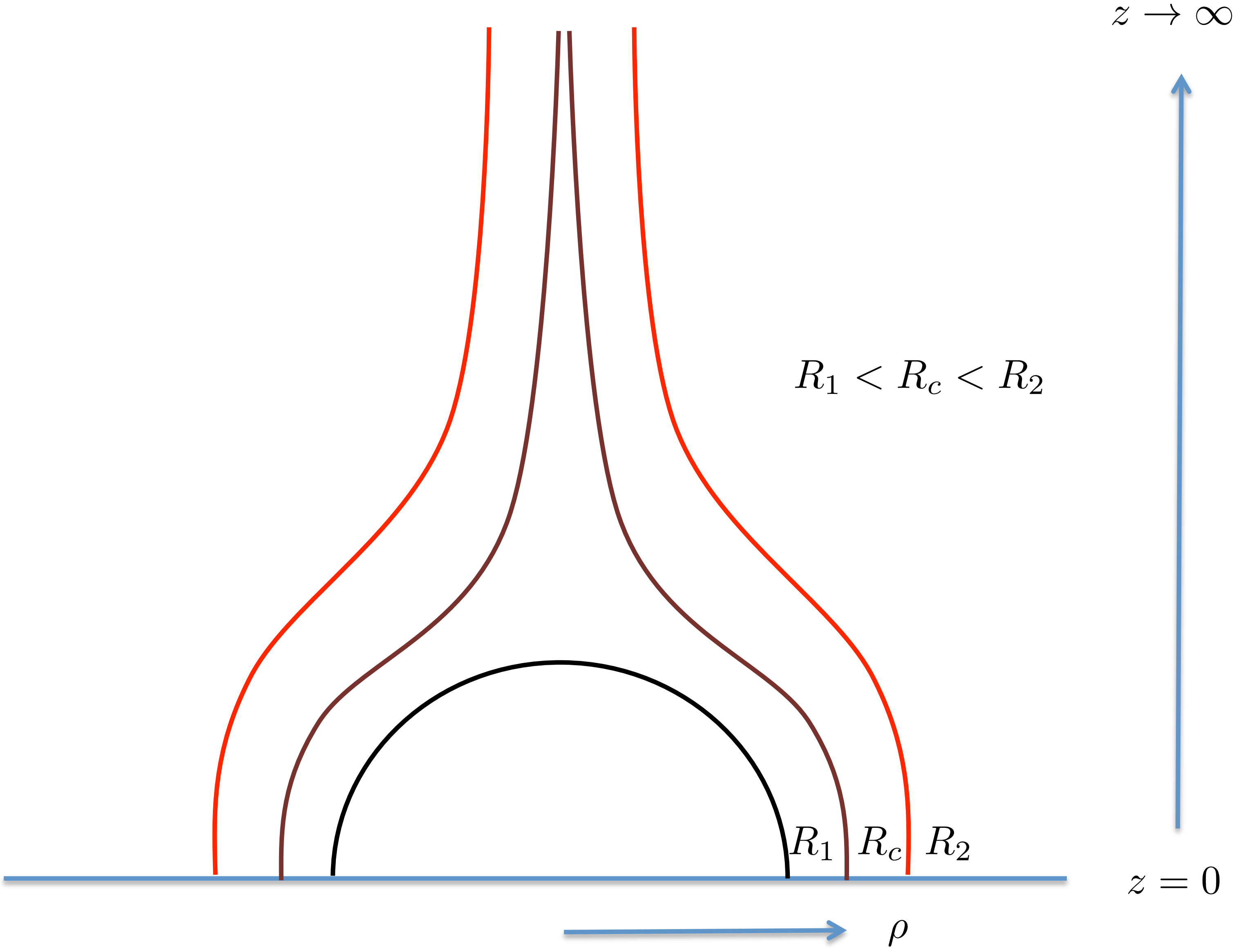
Large N artifact?

These phase transitions should tell us “something” about RG flow of a system.

It appears to happens when a flow is “fast”

Nishioka and Takayanagi
Klebanov, Kutasov, Murugan
Pakman, Parnachev
Headrick
Albash and Johnson….

2^{nd} order phase transitions involving topology change



Summary

For any renormalizable quantum field theory (not necessarily Lorentz-invariant), "renormalized entanglement entropy:"

- characterize quantum entanglement at a given scale.
- for d=2,3, c-function, gives a measure of the number of degrees of freedom of the system at a given scale (with Lorentz symmetry)
- Reveal some surprising features of RG flow

Thank You !