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FINITE SIZE SCALING AND LOW MASS GLUEBALLS*

Richard C. BROWER

Physics Department, University of California, Santa Cruz, CA 95064, USA

Michael CREUTZ

Brookhaven National Lab., Upton, Long Island, NY 11973, USA

Michael NAUENBERG[1]

Institute for Theoretical Physics, University of California, Santa Barbara, CA 92106, USA

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We propose finite lattice effects as a probe of the glueball mass spectrum, and give an analysis of the recent SU(2) Monte Carlo data of Brower, Nauenberg and Schalk in terms of a gas of free glueballs. For L^4 lattices with $L=4$, 5, 6 fits are made to $\xi(m=1/a\xi)$ which indicate a rather large effective number of degrees of freedom (i.e. statistical degeneracy where a spin J counts as $2J+1$) from 5 to 15 states. As the degeneracy is increased, the central glueball mass increases from $m=(1.3\pm0.2)\sqrt{\kappa}$ at degeneracy 5 to about $m=(1.9\pm0.2)\sqrt{\kappa}$ at degeneracy 15, relative to the SU(2) string tension $\sqrt{\kappa}$.

Recent Monte Carlo simulations for lattice quantum chromodynamics have come close to experimental confrontation for predicted mass ratios in the continuum limit. For SU(2) and SU(3) quantum chromodynamics without quarks, calculations of the ratio of the square root of the string tension ($\sqrt{\kappa}$) relative to the minimal subtracted $\Lambda_{\overline{\text{ms}}}$ parameter of logarithmic scaling violations [1, 2] have given

$$\begin{aligned} \Lambda_{\overline{\text{ms}}}/\sqrt{\kappa} &= (1.3\pm0.2)(0.199)\ \text{for SU(2)}\,, \\ \Lambda_{\overline{\text{ms}}}/\sqrt{\kappa} &= (0.5\pm0.15)(0.289)\ \text{for SU(3)}\,. \end{aligned} \tag{1}$$

In SU(3), with $\sqrt{\kappa}=420$ MeV, this is a suitable Λ parameter ($\Lambda_{\overline{\text{ms}}}\simeq 60$ MeV); however, the experimental uncertainty for $\Lambda_{\overline{\text{ms}}}$ does not encourage much more streneous computations of this ratio.

Within quarkless QCD, a more definitive test should be the prediction of the lowest mass glueball and its degeneracy (i.e. statistical weight $2J+1$ for spin J). After all, the glueball is a salient non-perturbative effect of QCD, absent in earlier quark models. Even a 10% bound on its mass could in principle prove fatal to the standard QCD theory.

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[1] Permanent address: University of California, Santa Cruz, CA 95064.

However, even for quarkless SU(2) Monte Carlo, the computation of the glueball mass has appeared extremely difficult. By measuring the plaquette–plaquette correlation function at separation L, and assuming

$$\langle \mathrm{Tr}\,(U_{P_1})\,\mathrm{Tr}\,(U_{P_2})\rangle - \langle \mathrm{Tr}\,(U_P)\rangle^2 = C_L\, e^{-L/\xi_{GB}} \tag{2}$$

Bhanot and Rebbi [2] estimated the SU(2) glueball mass $m \equiv 1/a\xi$ relative to their string tension* (κ) to be

$$m/\sqrt{\kappa} = 3 \pm 1\,, \tag{3}$$

and Berg [3] by a similar method obtained

$$m/\sqrt{\kappa} = 3.7 \pm 1.2\,. \tag{4}$$

A major difficulty is the need to have good statistical accuracy so that the small exponential *difference* in the connected part can be measured.

Here we attempt a new approach to measuring the glueball mass or inverse correlation length. We look at the finite lattice behavior of the one-point function,

$$E_P(L, \beta) = \tfrac{1}{2}\langle \mathrm{Tr}(U_P)\rangle_L \tag{5}$$

in a periodic box of volume L^4, and bare coupling $\beta = 4/g_0^2$. Subsequently, we will show that the leading exponential corrections are due to glueball propagation to a plaquette's periodic image:

$$E_P(L, \beta) - E_P(\infty, \beta) \sim C_L\, e^{-L/\xi}\,. \tag{6}$$

Moreover, Brower, Nauenberg and Schalk [4] have demonstrated that this difference obeys the scaling behavior dictated by finite lattice scaling theory and asymptotic freedom for $L \geqslant 4$ and $\beta = 4/g_0^2 \geqslant 2.05$,

$$E_P(L, \beta) - E_P(\infty, \beta) = \frac{1}{L^4}\,\varepsilon\left(\frac{L}{\xi}\right), \tag{7}$$

where the correlation length obeys the scaling law

$$\xi = C\left(\frac{6\pi^2}{11}\beta\right)^{51/121} e^{-(3\pi^2/11)\beta}\,. \tag{8}$$

We now turn to a model of these finite scaling effects based on a free gas of glueballs.

A GLUEBALL GAS

Formulated in a periodic box of finite temporal extent, a path integral represents the partition function for a finite temperature quantum field theory. The period in

* Bhanot and Rebbi's estimate for $\sqrt{\kappa}$ for SU(2) is 1.2 times bigger, so that their glueball mass is actually $3.6\sqrt{\kappa}$ in terms of the string tension of ref. [1], which we use throughout.

time corresponds to the inverse physical temperature. For a cool system, a dilute gas of the lightest particles in the theory will dominate the finite size effects. In this way, we can obtain information on the light particle spectrum.

These ideas can be formalized with the transfer matrix expression for the path integral

$$Z = \int \mathrm{d}U\, \mathrm{e}^{-S(U)} = \mathrm{Tr}\,(\hat{T}^{L_0})\,. \tag{9}$$

Here L_0 is the number of discrete time intervals in our lattice and the trace is over the quantum mechanical Hilbert space. The hamiltonian $\hat{H}$ is defined by

$$\hat{T} = \mathrm{e}^{-a\hat{H}}\,, \tag{10}$$

where a in the lattice spacing. In a continuous time limit, this establishes the connection between a path integral and a canonical formalism.

Working on a hyper-rectangular lattice of dimension $L_\mu = (L_0, \boldsymbol{L})$, we assume that the lowest eigenstate of the hamiltonian is an isolated vacuum state with energy per unit volume

$$\mathscr{E}(a, \boldsymbol{L}) = \frac{1}{V_3}\langle 0|\hat{H}|0\rangle\,, \tag{11}$$

where V_3 is the spatial volume

$$V_3 = \prod_{\mu=1}^{3} L_\mu\,. \tag{12}$$

Using dimensional transmutation, we place all coupling constant dependence in the lattice spacing a. We adopt a renormalization scheme whereby the lowest glueball mass is held fixed.

We now assume that the first excited states of the system are single glueballs of momentum $\boldsymbol{q}$ and energy $E(\boldsymbol{q})$. In a finite box the momentum assumes only discrete values, however for the leading behavior we can replace sums over $\boldsymbol{q}$ with continuous integrals. Putting the vacuum and one glueball state into the trace in eq. (9), we find

$$Z = \exp\,(-V_4\mathscr{E}(a, \boldsymbol{L}))\left\{1 + rV_3 \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, \mathrm{e}^{-L_0 aE(\boldsymbol{q})} + \text{higher states}\right\}. \tag{13}$$

Here we define the four-dimensional volume of our lattice

$$V_4 = \prod_{\mu=1}^{4} L_\mu\,. \tag{14}$$

The factor r represents the degeneracy of the first state; for a spin J state we have

$$r = 2J + 1\,. \tag{15}$$

The glueball energy above the vacuum, $E(\boldsymbol{q})$, should become relativistic for small lattice spacing; consequently, we assume the form

$$E(\boldsymbol{q}) = m + \frac{\boldsymbol{q}^2}{2m} + \mathrm{O}(\boldsymbol{q}^4)\,. \tag{16}$$

At strong coupling the rest and kinetic masses need not be equal; we assume that the coupling is small enough that such deviations are negligible. Inserting the spectrum (16) into eq. (13), we obtain

$$Z = \exp\left(-V_4 \mathscr{E}(a, \boldsymbol{L})\right)\left\{1 + rV_3\left(\frac{ma}{2\pi L_0}\right)^{3/2} \mathrm{e}^{-maL_0}\left(1 + \mathrm{O}\left(\frac{1}{maL_0}\right)\right)\right\}. \tag{17}$$

We now must take account of the finite size dependence of the vacuum energy density $\mathscr{E}$. Such corrections should also be exponential in the glueball mass

$$\mathscr{E}(a, \boldsymbol{L}) = \mathscr{E}(a, \infty) + \mathrm{O}(\mathrm{e}^{-maL_i})\,. \tag{18}$$

In eq. (17) we found the finite size correction exponential in L_0. A transfer matrix in the ith direction will give the contribution exponential in L_i. We now consider all components of L_μ to be equal, i.e. we work on an L^4 lattice. Each direction contributes equally to the finite size effect, and we conclude

$$Z(a, L^4) = Z(a, \infty)\left\{1 + 4r\left(\frac{maL}{2\pi}\right)^{3/2} \mathrm{e}^{-maL}\left(1 + \mathrm{O}\left(\frac{1}{maL}\right)\right)\right\}. \tag{19}$$

We now use the connection between Z and the internal energy per plaquette

$$E_{\mathrm{P}} = \frac{1}{6L^4}\frac{\mathrm{d}}{\mathrm{d}\beta}\log Z\,, \tag{20}$$

and use the renormalization group equation to convert the derivative with respect to β into a derivative with respect to a:

$$\frac{\mathrm{d}}{\mathrm{d}\beta} = \frac{g_0^3}{8\gamma(g_0)} a \frac{\mathrm{d}}{\mathrm{d}a}\,, \tag{21}$$

where $\gamma(g_0)$ in the Gell-Mann–Low function

$$a\frac{\mathrm{d}}{\mathrm{d}a} g_0(a) = \gamma(g_0)\,. \tag{22}$$

Combining these equations gives the result

$$E_{\mathrm{P}}(L) - E_{\mathrm{P}}(\infty) = \frac{2^{1/2} r (ma)^{5/2}}{48\pi^{3/2} L^{3/2}} \frac{g_0^3}{\gamma(g_0)} \mathrm{e}^{-maL}\left(1 + \mathrm{O}\left(\frac{1}{maL}\right)\right). \tag{23}$$

To further study the corrections to this formula, we model the glueball gas using the partition function for a free scalar field of mass $m = 1/a\xi$:

$$Z = \int \mathrm{d}\phi_n \exp\left(-\tfrac{1}{2}\sum_n \{(\Delta_\mu \phi_n)^2 + (ma)^2\phi_n^2\}\right). \tag{24}$$

Again from scaling (22), we compute

$$E_{\mathrm{P}} = \frac{1}{6L^4}\frac{\partial}{\partial\beta}\log Z = \frac{g_0^3 (ma)^2}{16\gamma(g_0)}\langle \phi_n^2\rangle_L, \tag{25}$$

where the finite lattice propagator is

$$\langle \phi_n^2\rangle_L = \frac{1}{L^4}\sum_{q_\mu}\frac{1}{(ma)^2 + 4\sum_\mu \sin^2(aq_\mu/2)}. \tag{26}$$

The sum extends over $aq_\mu = 2\pi l_\mu/L$ for $l_\mu = 1, 2, \cdots, L$. This may be recognized as simply the infinite lattice propagator for the glueball from $x = 0$ to all the periodic images at $x_n = aL(n_1, n_2, n_3, n_4)$ by using the identity,

$$\frac{1}{(aL)^4}\sum_{q_\mu} = \int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}^4 q}{(2\pi)^4}\sum_{n_\mu} \mathrm{e}^{iq\cdot x_n}. \tag{27}$$

The leading scaling contribution to $\varepsilon(L/\xi) = L^4(E_{\mathrm{P}}(L) - E_{\mathrm{P}}(\infty))$ for large $aLm = L/\xi$ is easily evaluated from the nearest image,

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{e}^{iaLq_0}}{m^2 + q_0^2 + \boldsymbol{q}} \simeq \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{\mathrm{e}^{-aLE(\boldsymbol{q})}}{2E(\boldsymbol{q})}, \tag{28}$$

and the approximation $aE(\boldsymbol{q}) \simeq 1/\xi + \frac{1}{2}\xi a^2 \boldsymbol{q}$. The result for $r = 1$ and $\gamma(g_0^2)/g_0^3 \simeq 11/24\pi^2$ is

$$\varepsilon_0(L/\xi) \simeq \tfrac{1}{2}\sqrt{\pi}\tfrac{1}{11}(L/\xi)^{5/2}\,\mathrm{e}^{-L/\xi}, \tag{29}$$

in accord with our transfer matrix result (23). However, as we will see shortly, this approximation is inadequate for the values of L/ξ we have in our Monte Carlo data. Most importantly, it misses the next nearest images at $\sqrt{2}L$ and $\sqrt{3}L$. But even for rather large values of L/ξ the polynomial factor $(L/\xi)^{5/2}$ is an important effect. A similar factor of $(L/\zeta)^{3/2}$ has been neglected in earlier work [2, 4].

Fortunately it is not difficult to include exactly all scaling contributions of our glueball gas. After several manipulations, we obtain

$$\varepsilon(x) = \tfrac{1}{11}r \cdot \tfrac{1}{16}x^3\int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-x/2\lambda}\left\{\left(\sum_{n=-\infty}^{\infty}\mathrm{e}^{-n^2 x\lambda/2}\right)^4 - 1\right\}, \tag{30}$$

where $x = L/\xi = amL$ and r is the statistical weight. The contribution to the first three nearest images at L, $\sqrt{2}L$ and $\sqrt{3}L$ is

$$\varepsilon_3(x) = \tfrac{1}{11}r \cdot \tfrac{1}{16}x^3\int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-x/2\lambda}[\mathrm{e}^{-\lambda x/2} + 3\,\mathrm{e}^{-\lambda x} + 4\,\mathrm{e}^{-3\lambda x/2}]. \tag{31}$$

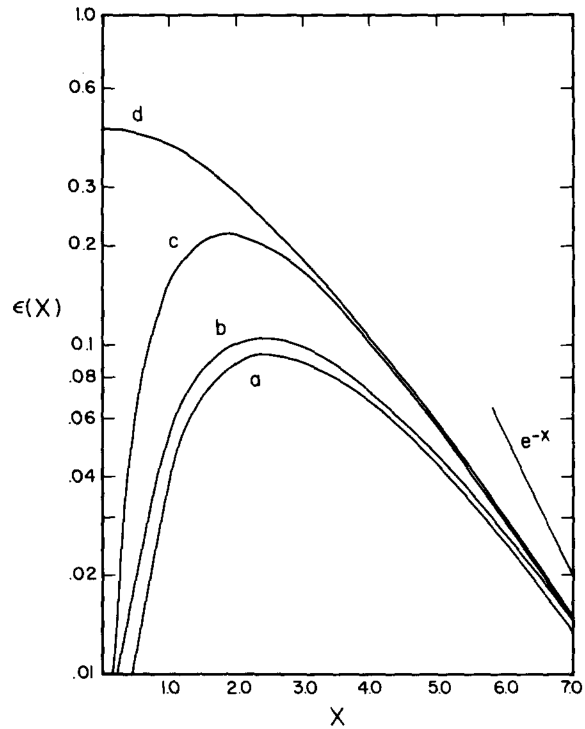

Fig. 1. Contributions to the scaling function $\varepsilon(x)$ for $x = L/\xi$ from a single glueball of mass $m = 1/a\xi$. Curve (a) is the leading term for $x \to \infty$, (b) is the nearest image at distance L, (c) is the first three images at L, $\sqrt{2}L$ and $\sqrt{3}L$ and (d) is the full scaling contribution.

In fig. 1, we plot for $r = 1$ the successive approximations: (a) the asymptotic equation (29) for the nearest image $\varepsilon_0(x)$; (b) the full relativistic contribution of the nearest image $\varepsilon_1(x)$; (c) the first three terms $\varepsilon_3(x)$, and (d) the exact scaling contribution.

We can draw some interesting conclusions. The maximum for $x \simeq 2\frac{1}{2}$ seen in the leading asymptotic term is only present if the first few terms dominate. Clearly as L/ξ gets small, contributions of the high mass spectrum (excited glueballs, threshold, etc.) come in, and our model should not be used. (So the physics of the maximum in the scaling data is an open question, discussed briefly in the conclusion.) On the other hand even for $x \geqslant 3$, the images at $\sqrt{2}L$ and $\sqrt{3}L$ are significant. We shall use the exact formula for definiteness, but fortunately the images at $2L$ and beyond are negligible for $x \geqslant 3$ where we use it. At $2L$ the corrections are of the same order as neglected effects due to glueball scattering and where our free gas should not be trusted.

Our model has two parameters, the degeneracy r and the mass m measured in units of the Λ parameter. For convenience we plot the scaling data of ref. [4] on a log–log plot so that the two parameters represent translating the curve upward for increasing degeneracy, and leftward for increasing mass. The parameter x_0 is defined with an arbitrary normalization

$$x_0 = L/\xi_0 , \qquad \xi_0^{-1} = 100\left(\frac{6\pi^2}{11}\beta\right)^{51/121} e^{-(3\pi^2/11)\beta} . \tag{32}$$

When the first two terms of the renormalization group function dominate the coupling dependence of the correlation length, x_0 is proportional to x. (The value $x = x_0$ corresponds to $m = 1.3\sqrt{\kappa}$ in terms of the string tension [1].)

There is a great deal of ambiguity if both r and m are allowed to vary simultaneously. However, a small degeneracy, such as a single $J = 0$ glueball is inconsistent with the magnitude of ε. If we fix the degeneracy at $r = 5$ (e.g. one $J = 2$ state) the glueball mass is quite tightly constrained to $m \simeq (1.3 \pm 0.2)\sqrt{\kappa}$ as shown in fig. 2, but solutions with larger degeneracies and correspondingly larger masses are allowed. For example, fig. 2 also shows that a degeneracy of $r = 15$ with $m = (1.9 \pm 2)\sqrt{\kappa}$ is certaintly consistent with the data.

These results are not implausible. As you allow for higher degeneracy, the central mass of the glueballs increases. We could state the result as the assertion that the

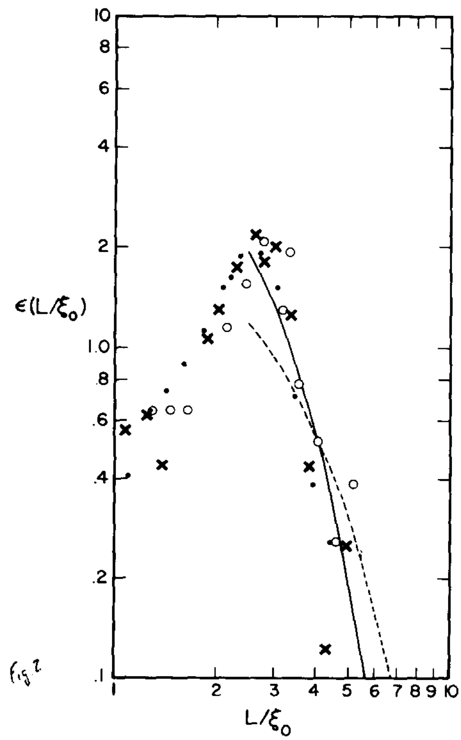

Fig. 2. The finite lattice scaling data for $\varepsilon(x_0)$ for $L = 4$ (·), 5 (×) and 6 (○) with the mass scale $m_0 = 1/a\xi_0$ set equal to $1.3\sqrt{\kappa}$. The dotted line is a fit to $m_{GB} = 1.3\sqrt{\kappa}$ and degeneracy $r = 5$, and the solid line is $m_{GB} = 1.9\sqrt{\kappa}$ and $r = 15$. The scatter in the data points reflects their statistical uncertainty.

spectrum grows to 15 states as you move to a mass on the order of $1\frac{1}{2}$ times the lowest state. Bag models have this property. For example, Donoghue, Johnson and Li find [6] a glueball spectrum with states, m_0, $1.3m_0$, $1.5m_0$, $1.8m_0$ with degeneracies 6, 6, 11, 21, respectively, in agreement with our trend. Indeed, a better fit to our data would be the superposition of a few low mass states plus a large number of higher mass states. This tends to give a steeper envelope. However, we don't feel fits with more parameters are useful. Nonetheless, two features do stand out.

First, rather high degeneracies are favored. Since most glueball models begin with nearly degenerate $J=0$ and $J=2$ states, degeneracies of six or more are sensible. Roughly speaking, this reflects the spin states of adding two or more vector gluons [$(2J+1)\times(2J+1)=9$ for two vectors] to form the color singlet glueball.

Second, we get a much lower mass than earlier Monte Carlo estimates [2, 3]. The earlier estimates of $m \sim (3\text{–}4)\sqrt{\kappa}$ used the two-point function with poorer statistics. In their analysis, power corrections in L/ξ, were omitted. On the other hand, our estimate is consistent with the strong coupling expansion of Münster* $m = (1.8\pm0.8)\sqrt{\kappa}$ for SU(2). Ref. [4] also suggested $m \lesssim 1.2\sqrt{\kappa}$ for the lowest glueball state which is nearly satisfied. Their argument is based on the plausible idea that at the onset of precocious scaling ($\beta > 2.05$) the largest correlation length should exceed one lattice spacing ($\xi > 1$).

If we notice that $m/\sqrt{\kappa}$ increases by as much as a factor of 2 going from SU(2) to SU(3) in the strong coupling estimate [6] our lowest mass becomes $m \sim 1000\pm500$ MeV adjusted to SU(3), where 50% error reflects the uncertainty in $\sqrt{\kappa}$. While the mass looks a little small compared to Regge expectations for a $J=2$ state at about ≈ 1400 MeV, there is no conflict. Perhaps, the lowest glueball is a pseudoscalar and by mixing with the η is usually sensitive to adding light quarks.

Finally, we should emphasize that the peak and turn over in the scaling data on the left (weak coupling) side in fig. 2 is totally beyond the scope of our glueball model. On a lattice with a finite time axis ($L_0 \ll L$) one expects a deconfinement transition into a gluonic phase at high temperature (or weak coupling) [7]. Very likely, our peak in $\varepsilon(x)$ represents an analogous phenomenon. From the peak, we can define a critical "radius" R_c (or inverse effective "temperature" $T_c = 1/R_c$):

$$R_c = aL|_{\text{peak}} = \frac{1}{(0.5\pm0.1)\sqrt{\kappa}}. \tag{33}$$

Curiously, the finite temperature studies of deconfinement have also given the same number, $T_c \approx (0.5\pm0.1)\sqrt{\kappa}$. A more precise understanding of our scaling function near the peak could be of great help in alleviating the degeneracy-mass ambiguity [8].

* Münster [6] estimates by a Padé of strong coupling expansions that $m = (1.8\pm0.8)\sqrt{\kappa}$ for SU(2) and $m = 3\sqrt{\kappa}$ for SU(3).

Another improvement would be to introduce an asymmetric lattice of volume L_0L^3, which is shorter in the time direction ($L_0 < L$). This would suppress the contribution of the spatial images. Since images at or beyond $2L$ give small contributions, asymmetric lattices with $L_0 \leq \frac{1}{2}L$ should be adequate. Moreover, these lattices allow direct contact with the finite temperature studies, so the deconfinement transition can be more easily identified and controlled. The asymmetry also allows some separation of different spin-parity components, which may be very useful in view of the high degeneracy of the glueball spectrum. In this respect, the two-point correlation function also has a distinct advantage, so a combined approach using both methods may be best. Finally, our determination of the glueball mass to string tension ratio $m/\sqrt{\kappa}$ cannot be pushed farther without a parallel improvement in the errors for $\sqrt{\kappa}$.

In conclusion, our finite lattice image technique provides a useful tool for deciphering the glueball spectrum, but better statistics and the accommodation of SU(3) and quark effects are necessary to confront the experiments. Still our results already favor low masses (from $1.2\sqrt{\kappa}$ to $2.0\sqrt{\kappa}$) and high degeneracies (from 5 to 15) as a prediction of SU(2) gauge theory.

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