Lattice Gauge Theory

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Supercomputers have recently become a crucial tool for the quantum field theorist. Applied to the formalism of lattice gauge theory, numerical simulations are providing fundamental quantitative information about the interactions of quarks, the fundamental constituents of those particles which experience nuclear interactions. Perhaps most strikingly, these simulations have provided convincing evidence that the interquark forces can prevent the isolation of these constituents.

Quarks are the primary constituents of particles subject to the strong nuclear force. Their basic interactions are believed to follow from a generalization of the gauge theory of electromagnetism. Instead of a single photon, this theory involves eight spin-1 quanta, referred to as gluons. Furthermore, these eight gluons are themselves charged with respect to one another. This introduces subtle nonlinear effects which appear even in the pure glue theory.

One particularly important consequence of these nonlinearities is that the quark interactions weaken at small separations. This phenomenon, known as "asymptotic freedom", is essential to many of the successes of the simple quark model. As long as the quarks remain near each other, their interactions are small.

In contrast, the behavior of the gauge fields changes dramatically as the quarks are pulled apart. The experimental nonobservance of free quarks has led to the conjecture of the phenomenon of "confinement", wherein interquark forces increase and remain strong as quarks are pulled apart to arbitrary separations. In this picture, it requires an infinite amount of energy to separate a single quark from the other constituents of a physical particle. This explains why free quarks are not produced in nature.

Standard field-theoretical tools are severely hampered in the regime of large distances where these effects come into play. Perturbation theory, the historic mainstay of quantum field theory, begins with free particles and then treats their interaction as a small correction. With confinement, however, the fundamental constituents become increasingly strongly interacting as their separation is increased. In this domain the conventional perturbative approach fails totally.

Lattice gauge theory, originally formulated by K. Wilson, provides a novel framework for calculations in this regime. This approach replaces the relativistic continuum of space and time with a discrete space-time lattice. The quarks move through this scaffolding by a sequence of discrete hops between nearest-neighbor sites. The gluon fields lie on the bonds connecting these sites.

This lattice is a mathematical trick, introduced for calculational purposes only. It should not be taken as requiring a crystalline basis for physical space. At the end of any calculation, one should consider a continuum limit, wherein the lattice spacing is extrapolated to zero. In this limit observable quantities, such as the masses of particles and the forces between them, should approach their physical values.

The lattice artifice, however, has several advantages. First, by replacing an infinite number of space-time points in any given volume by a finite number, the field-theoretical system becomes mathematically considerably simpler and better defined. Continuum quantum field theories are notorious for the appearance of formally infinite quantities. These divergences involve short-distance singularities and must be exorcised by a renormalization procedure. A space-time lattice provides a particularly convenient regulator of such divergences. Indeed, the lattice spacing represents a minimum length and singularities arising from wavelengths shorter than this distance are automatically excluded.

Second, this formulation makes no assumptions on calculational schemes to be applied. Other techniques for controlling the singular behavior of a field theory are usually formulated directly in terms of some calculational method. For example, conventional discussions of renormalization regulate the divergences only after they are encountered in the perturbative expansion. On the lattice the theory is mathematically well defined at the outset.

Finally, the lattice formulation gives a system particularly well suited to numerical simulation. While there are several analytic techniques which have been applied to the strongly interacting lattice gauge problem, numerical simulations by Monte Carlo techniques currently dominate the field. These simulations have given compelling evidence that the confinement phenomenon does indeed occur in the standard gauge theory of the nuclear force. In addition, the approach is now giving quantitative predictions for long-range hadronic properties not accessible to more traditional theoretical methods.

In the Wilson approach, the gauge degrees of freedom are represented by matrices, one of which is associated with each lattice bond. To describe the physical theory of quarks, these are 3×3 unitary matrices with determinant 1; thus, they are elements of the group $SU(3)$. The interactions of these degrees of freedom are most concisely summarized in the "action"

$$S = -\frac{1}{3} \sum_p \mathrm{Re}\,\mathrm{Tr}\, U_p \ .$$

Here the sum is over all elementary squares, or "plaquettes", p, and U_p is the product of the link variables around the plaquette in question. The latter represents the flux of the gauge fields through the corresponding tile. For slowly varying fields, the above sum reduces to the conventional gauge-theory action as the lattice spacing goes to zero.

The numerical techniques used for lattice gauge simulations are borrowed directly from statistical mechanics. Indeed, there is a deep mathematical relationship between quantum field theory and classical statistical mechanics in four dimensions. In this relationship, the strength of the quark coupling to the gauge fields corresponds directly to temperature and the action S corresponds to the classical energy. Thus, a study of confinement and long-distance quark interactions is equivalent to a study of a high-temperature statistical model.

In the Monte Carlo approach to studying such a lattice system, one attempts to create an ensemble of field configurations with a Boltzmann-like distribution where the probability for any given configuration C takes the form

$$P(C) \propto \mathrm{e}^{-\beta S(C)} \ .$$

Here β is proportional to the inverse of the gauge coupling squared. Thus one wants configurations typical of "thermal equilibrium".

The procedure begins with the storage of some initial values for all the lattice fields in the computer memory. These are then updated with pseudorandom changes on the field variables, thus mimicking thermal fluctuations. The structure of such a program is quite simple. On the outside is a set of nested loops overall the system variables. These loops surround calls to the random number generator, so as to simulate a thermal coupling to these degrees of freedom. The field changes are constructed with a bias toward lower values of S so as to obtain the appropriate thermal weighting of configurations.

Having the values of all fields at his disposal, the physicist is free to calculate any quantity of interest. There will, of course, be statistical errors coming from the thermal fluctuations. In addition, there will be errors coming from the requisite extrapolation to the continuum limit and from the practical requirement of working with a finite volume. It is attempts to reduce these uncertainties that have driven the theorists to the most powerful computers available.

Despite the inevitable uncertainties, several important results have been extracted. Perhaps the most dramatic of these is the measurement of the confinement force and how it relates to the weaker interactions of the quarks at short distances. Then there are successful studies of the mass spectra of the bound states of quarks. These calculations are being refined to give information on the distributions of the quarks and on the strong-interaction effects on other processes, such as weak decays.

In addition, there have been quantitative studies of physics at temperatures sufficiently high that strongly interacting particles are created by thermal fluctuations. Here lattice gauge calculations have provided strong evidence that the vacuum undergoes a phase transition at a temperature of $kT \sim 200\,\mathrm{MeV}$. This transition is from a phase of ordinary matter, made up of quarks bound into the familiar nuclear particles, to a new plasma phase where the quarks and their attendant gluon fields form a thermal gas. Indeed, the lattice approach gives the best estimates for the temperature of the transition to this phase, which will be looked for in future accelerator experiments.

Until recently, the bulk of the numerical simulations in lattice gauge theory considered the full dynamics of the gauge fields but only included the quarks as fixed sources. In this way the confinement potential has been mapped out. A more refined approach allows the primary "valence" quarks to carry kinetic energy, but still ignores the creation of matter–antimatter quark pairs by quantum fluctuations. Most of the hadronic spectrum calculations have been done in this approximation.

Thus far, only limited results have been obtained beyond this valence approximation. To proceed in this direction, one must allow for the possibility of unlimited numbers of virtual quarks being created by quantum fluctuations. The most difficult part of the problem is the inclusion of the effects of the

Pauli exclusion principle in the simulations. The development of algorithms to treat the dynamical quarks appropriately forms an area of intense current research. Promising new schemes combine Monte Carlo methods with ideas from molecular-dynamics simulations and stochastic processes.

In conclusion, computer simulations of lattice gauge theory provide a powerful tool for the study of nonperturbative phenomena. The technique provides a first-principles approach to calculating particle properties as well as details of the phase transition to a quark gluon plasma.

See also: Gauge Theories; Hadrons; Quarks; Strong Interactions.

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