The invariant measure for $SU(N)$

Michael Creutz

Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA

Abstract. I present a simple recursive relation between the invariant measure for integration over SU(N) and the product of the measure for a sphere S(2N-1) times the measure for SU(N-1). I also discuss the periodicity factor that appears in non-trivial mappings of S(2N-1) into SU(N).

Parametrizing an $SU(N)$ matrix in the form

$$U = \begin{pmatrix} 1 & \cdots \\ \vdots & g_{N-1} \end{pmatrix} g_s(\vec{z})$$

with $g_{N-1} \in SU(N-1)$ and $g_s \in SU(N)$ a standard form with given top row $\vec{z} \in S_{2N-1}$,

$$g_s = \begin{pmatrix} 1 & 0 & \cdots & \\ 0 & 1 & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & \prod_i p_i^* \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 & \cdots \\ -s_1 & c_1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & c_2 & s_2 & \cdots \\ 0 & -s_2 & c_2 & \cdots \\ \vdots & \vdots & \vdots & 1 \end{pmatrix}$$
$$\cdots \begin{pmatrix} 1 & \vdots & \vdots & \vdots \\ \cdots & 1 & 0 & 0 \\ \cdots & 0 & c_{N-1} & s_{N-1} \\ \cdots & 0 & -s_{N-1} & c_{N-1} \end{pmatrix} \begin{pmatrix} p_1 & 0 & \cdots & \\ 0 & p_2 & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & p_N \end{pmatrix}$$

Then the invariant integration measure takes the form

$$dg_N = dS_{2N-1}\, dg_{N-1}$$

That is, the measure is uniform over an S_{2N-1} and $SU(N-1)$.

To construct $\Pi_{2N-1}(SU(N)$ via mapping an $S(2N-1)$ into the group non-trivially requires covering the above $S(2N-1)$, defined by $\vec{z}$, 4^{N-2} times due to singularities at the poles of the mapping when $s_i = 0$.

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