

Emergent spin

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Abstract

Quantum mechanics and relativity in the continuum imply the well known spin-statistics connection. However for particles hopping on a lattice, there is no such constraint. If a lattice model yields a relativistic field theory in a continuum limit, this constraint must "emerge" for physical excitations. We discuss a few models where a spin-less fermion hopping on a lattice gives excitations which satisfy the continuum Dirac equation. This includes such well known systems such as graphene and staggered fermions.

1 Introduction

As is well known, in quantum field theory the constraints of quantum mechanics with special relativity give rise to the spin-statistics connection. In particular, fermions must have half integer spin. However in a lattice theory, the lattice structure itself breaks relativity. In a lattice model one is free to formulate a model based on spin-less fermions. If one finds that the low energy excitations of such a model have a relativistic spectrum, then these excitations must carry half integer spin. In some sense spin must "emerge" from the dynamics.

Remarkably such models exist. The most famous is based on graphene. The Hamiltonian for spin-less fermions hopping on a hexagonal two dimensional lattice is easily diagonalized [1], and the low energy excitations above the half filled system do indeed mimic a relativistic spectrum. And these excitations satisfy the two dimensional Dirac equation, correspondingly carrying half integer angular momentum [2].

This situation is not unique. Another example, discussed in more detail below, is a two dimensional square lattice subjected to a magnetic field of half a

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flux unit per elementary square. This model generalizes to more dimensions by threading the magnetic field through all elementary plaquettes. This is a route to the well known staggered fermion theory [3, 4, 5].

This paper reviews these models and discusses some of the interesting relations with chiral symmetry and doubling issues. Section 2 goes through the standard solution of the graphene solution in the tight binding limit. Section 3 discusses the close ties between the doubling issues of lattice fermions and topology. Also we see how a chiral symmetry protects masses from an additive renormalization. Section 4 generalizes these ideas to a square lattice in a magnetic field and makes the connection to staggered fermions. Section 5 extends this idea to higher dimensions. Section 6 discusses issues that can arrive in going from the Hamiltonian version of staggered fermions to a Euclidian path integral approach. Section 7 turns to the introduction of gauge fields and an amusing property when the gauge group is $SU(N)$ with N even. In Section 8 we make some general observations on the effects of gauge field topology on the lattice fermion spectrum. In particular we present a variation of the Nielsen-Ninomiya theorem[6] that applies to all lattice actions including mass terms. Finally there are some brief conclusions in Section 9.

2 Graphene

As is well known, the solution to a theory of fermions hopping on a hexagonal lattice displays two Dirac cones. With small excitations around half filling, each of these cones represents a fermion satisfying the Dirac equation. Graphene, basically a two dimensional hexagonal planar structure, represents a realization of this system [1].

In the physical situation the electrons already have spin, so the extra doubling of species with the two cones can be thought of as representing “flavor” or “isospin.” However, from a theoretical point of view one can consider spin-less fermions hopping on the lattice, and then the excitations will formally acquire spin one-half. In this section we review this solution.

To solve this problem, it is useful to use a fortuitous set of coordinates, as sketched in Fig. 1. Orienting the lattice as in the figure, the sites can be considered as being of two types. We consider type a sites on the left side of each horizontal bond, and b sites on the right. These sites are labeled with non-orthogonal coordinates x_1 and x_2 labeling the horizontal bonds. The two axes are not orthogonal, but oriented at 30 degrees from the horizontal. Associated with each site is a pair of creation-annihilation operators, labeled (a^\dagger, a) and (b^\dagger, b) respectively. These satisfy the usual anti-commutation relations

$$\begin{aligned} [a_{x_1, x_2}, a_{y_1, y_2}^\dagger]_+ &= \delta_{x_1, y_1} \delta_{x_2, y_2} \\ [a_{x_1, x_2}, b_{y_1, y_2}^\dagger]_+ &= [b_{x_1, x_2}, a_{y_1, y_2}^\dagger]_+ = 0. \end{aligned} \tag{1}$$

With these coordinates, the nearest neighbor Hamiltonian takes the form

$$H = K \sum_{x_1, x_2} a_{x_1, x_2}^\dagger b_{x_1, x_2} + b_{x_1, x_2}^\dagger a_{x_1, x_2}$$

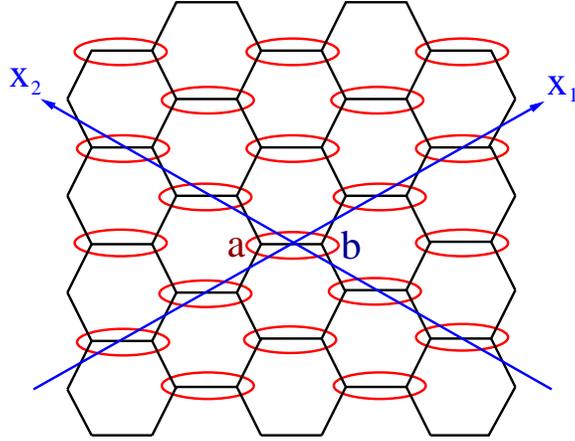


Figure 1: Fermion hopping on a hexagonal lattice is nicely formulated in terms of a and b type sites labeled by non-orthogonal coordinates as indicated here. Figure from Ref. [7].

$$\begin{aligned}
& + a_{x_1+1,x_2}^\dagger b_{x_1,x_2} + b_{x_1,x_2}^\dagger a_{x_1+1,x_2} \\
& + a_{x_1,x_2}^\dagger b_{x_1,x_2+1} + b_{x_1,x_2+1}^\dagger a_{x_1,x_2}.
\end{aligned} \tag{2}$$

The three terms correspond to horizontal, upward to the right, and upward to the left bonds respectively. Here K is usually referred to as the ‘‘hopping’’ parameter, and sets the energy scale.

To solve this system, go to momentum (reciprocal) space and define

$$\tilde{a}_{p_1,p_2} = \sum_{x_1,x_2} e^{-ip_1x_1} e^{-ip_2x_2} a_{x_1,x_2}. \tag{3}$$

These satisfy the commutation relations

$$[\tilde{a}_{p'_1,p'_2}, \tilde{a}_{p_1,p_2}]_+ = (2\pi)^2 \delta(p'_1, p_1) \delta(p'_2, p_2). \tag{4}$$

Similar equations apply for the operators b . Because of periodicity in p , we can restrict $-\pi < p_\mu \leq \pi$. This change of variables breaks the Hamiltonian into two by two blocks

$$H = K \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \begin{pmatrix} \tilde{a}_{p_1,p_2}^\dagger & \tilde{b}_{p_1,p_2}^\dagger \end{pmatrix} \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \begin{pmatrix} \tilde{a}_{p_1,p_2} \\ \tilde{b}_{p_1,p_2} \end{pmatrix} \tag{5}$$

where

$$z = 1 + e^{-ip_1} + e^{+ip_2}. \tag{6}$$

The three terms in this expression correspond to the three types of bonds: horizontal, right leaning, and left leaning.

We see that due to the presence of two types of site, a spinor structure $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ “emerges” naturally. For a given momentum, the Hamiltonian reduces to a two by two matrix

$$\tilde{H}(p_1, p_2) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}. \quad (7)$$

This is diagonalized with the eigenstates being two component spinors

$$\psi = \frac{1}{\sqrt{2|z|}} \begin{pmatrix} \sqrt{z} \\ \pm\sqrt{z^*} \end{pmatrix}. \quad (8)$$

The corresponding eigenvalues are

$$E(p_1, p_2) = \pm K|z|. \quad (9)$$

The fermionic level energy vanishes when $|z|$ does. This occurs at exactly two points

$$p_1 = p_2 = \pm 2\pi/3. \quad (10)$$

Near these zeros, the energy behaves linearly in momentum. This linear behavior is exactly that of the relativistic Dirac equation.

For our problem we want to consider excitations on the half filled system. The “vacuum” has all negative energy states filled. This, of course, has infinite negative energy, which should be subtracted to find the energies of physical states. And for rotations, we are free to define things such that the vacuum state is invariant under such.

In this framework, one particle states are constructed by filling one of the positive energy states. Anti-particles correspond to holes in the Dirac sea. The concept of spin in two space dimensions is somewhat different than in three. In particular, we don’t have helicity states. Rather we have rotations which are naturally thought of as about an axis orthogonal to the spatial plane. A state of definite angular momentum J transforms under a rotation by an angle θ as

$$|\psi\rangle \longrightarrow e^{iJ\theta}|\psi\rangle. \quad (11)$$

The basic idea here is to consider such a rotation on the vacuum with one additional filled positive energy state. We have half integer spin if a rotation by 2π gives the wave function a minus sign.

3 Topology and spin

The way the spin arises in this model is related to a topological behavior within the Brillouin zone. Consider a contour of constant energy near and wrapping around one of the zero points, as sketched in Fig. 2. As we traverse this contour, the phase of z in Eq. (6) wraps nontrivially around the unit circle. This wrapping means one cannot collapse this contour without shrinking it down to a Dirac point at $|z| = 0$, a point where the energy vanishes. Furthermore, as one fully

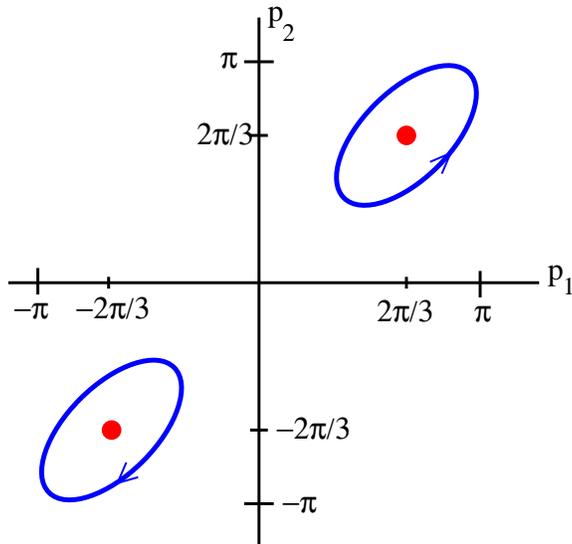


Figure 2: When a curve of constant energy wraps around one of the Dirac points, the phase of the variable z wraps around a unit circle and the spinor wave function acquires a minus sign.

goes around the contour, the spinor expression in Eq. (8) becomes its negative. This is the behavior of a half integer spin system. Indeed, the fermion spin has “emerged.” This is only possible because the momentum eigenstates involved are non-local in terms of the underlying position space operators.

In addition to spin, this model has an emergent chiral symmetry. Since all hoppings couple a and b sites, the sign of the Hamiltonian is reversed if we replace each b operator by its negative. In particular, the Hamiltonian in Eq. (7) anti-commutes with the matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (12)$$

There is a close connection with the no-go theorem of Nielsen and Ninomiya [6]. Because of the periodicity of the Brillouin zone, any contour expanded to the boundaries of this zone cannot wrap z non-trivially. Given any Dirac cone, there must exist another about which the topology unwraps. This requires an even number of Dirac cones. Two such is the minimum possible without breaking the symmetries. And the chiral properties of the two cones must be opposite. Since the complex number z rotates around the origin in opposite directions, in this model the rotation gives conjugate phases to the physical fermion states near the two cones. One might think of the two cones as representing spin up and spin down in the direction orthogonal to the spatial plane.

For the three dimensional case discussed later we will see that we get a separate four component spinor associated with each cone. Each then represents

a Dirac fermion with both spin states present. In that case the two cones can be thought of as representing an isospin symmetry, also emerging from the dynamics.

4 Hopping in a magnetic field

We now turn to a rather different model that exhibits a similar phenomenon to the graphene case. Staying in two dimensions, consider spin-less fermions hopping on a square lattice. Apply to this system a constant magnetic field transverse to the plane, giving phase factors to the fermions as they pass around closed loops. This problem was studied extensively by D. Hofstadter in Ref. [8], showing a rather complex structure depending sensitively on the strength the applied field. In particular, the spectrum with a rational flux p/q passing through each plaquette gives rise to q bands. The strength is measured in the natural units, where the flux gives a phase of $e^{2\pi ip/q}$ in traversing around a fundamental plaquette.

Here we concentrate on the case $p/q = 1/2$; so there are two bands in the spectrum. This situation has some interesting mathematical properties that have been frequently studied in the condensed matter context; see for example Refs. [9, 10, 11]. We will see that the two bands touch at two Dirac points, in direct analogy with graphene. We implement the magnetic field by placing a static phase $Z_x(x, y)$ ($Z_y(x, y)$) representing a $U(1)$ gauge field on each link in the positive x (y) direction. We label the sites by the integer coordinates (x, y) . The flux condition means that the product of the link variables around any plaquette to be -1 . We will see that this system is equivalent to what are known as staggered fermions [3, 4, 5].

The model leaves free a gauge freedom in selecting the link variables. Since the spectrum does not depend on the gauge choice, we are free to pick a convenient one. To proceed, we take

$$Z_x(x, y) = 1, \quad Z_y(x, y) = (-1)^x. \quad (13)$$

This arrangement of phases is sketched in Fig. 3. Since this is just a gauge choice, the underlying physics retains the underlying square symmetry.

With this choice we see that, as with graphene, there are two types of site. Let (a^\dagger, a) label the fermion creation and annihilation operators on the top of the negative y bonds, and correspondingly (b^\dagger, b) for sites on top of positive y bonds, as labeled in the figure.

Again we find the spectrum of states via Fourier transform. For this gauge choice the translation symmetry in the y direction remains that of the underlying lattice. On the other hand, the periodicity in the x direction involves a translation by twice the lattice spacing. Thus at fixed momenta we find translationally invariant eigenstates satisfy

$$\begin{aligned} \psi(2x, y) &= e^{2ip_x x + ip_y y} \psi(0, 0) \\ \psi(2x + 1, y) &= e^{2ip_x x + ip_y y} \psi(1, 0). \end{aligned} \quad (14)$$

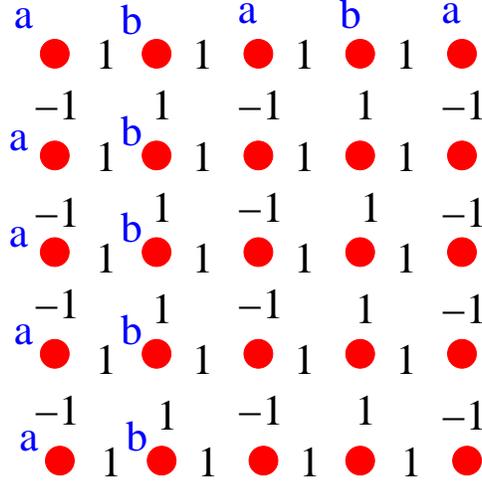


Figure 3: Placing phases on the links of a two dimensional lattice so as to produce a static magnetic field of $1/2$ flux unit per plaquette.

In this way we wind up with the emergence of a two component “base spinor”

$$\Psi = \begin{pmatrix} \psi(0,0) \\ \psi(1,0) \end{pmatrix}. \quad (15)$$

Because of the modified translation symmetry, the Brillouin zone is effectively “half sized” and we can take

$$\begin{aligned} 0 &\leq p_y < 2\pi \\ 0 &\leq p_x < \pi. \end{aligned} \quad (16)$$

At fixed momentum we have a two by two Hamiltonian matrix

$$H = K \begin{pmatrix} 2 \cos(p_y) & 1 + e^{2ip_x} \\ 1 + e^{-2ip_x} & 2 \cos(p_y) \end{pmatrix} \quad (17)$$

This is easily diagonalized to give

$$E = \pm 2K \sqrt{\cos^2(p_x) + \cos^2(p_y)} \quad (18)$$

for the individual fermionic levels. With this spectrum, the positive and negative energy bands touch at just two points $(p_x, p_y) = (\pi/2, \pi/2)$ and $(p_x, p_y) = (\pi/2, 3\pi/2)$. The system displays two Dirac cones, just as in the graphene case.

As mentioned above, this is a rewriting of staggered fermions. The sign factors on the links arise from our gauge choice. As this is just a choice, the square symmetry of the underlying lattice is preserved.

5 Higher dimensions

This procedure immediately generalizes to three spatial dimensions. Thread a half unit of magnetic flux through every plaquette of a cubic lattice. As before, it is useful to select a convenient gauge for the phases Z on the lattice bonds. For this we take

$$\begin{aligned} Z_x(x, y, z) &= 1 \\ Z_y(x, y, z) &= (-1)^x \\ Z_z(x, y, z) &= (-1)^{x+y} \end{aligned} \quad (19)$$

where we label the sites by the integer coordinates (x, y, z) . Translation invariance is by one site in the z direction but by two in the x and y directions. Use translation invariance to look for a solution of form

$$\begin{aligned} \psi(2x, 2y, z) &= \exp(2ip_x x + 2ip_y y + ip_z z) \psi(0, 0, 0) \\ \psi(2x + 1, 2y, z) &= \exp(ip_x(2x + 1) + 2ip_y y + ip_z z) \psi(1, 0, 0) \\ \psi(2x, 2y + 1, z) &= \exp(2ip_x x + ip_y(2y + 1) + ip_z z) \psi(0, 1, 0) \\ \psi(2x + 1, 2y + 1, z) &= \exp(ip_x(2x + 1) + ip_y(2y + 1) + ip_z z) \psi(1, 1, 0). \end{aligned} \quad (20)$$

This translates any fixed momentum wave function down to four base components

$$\begin{pmatrix} \psi(0, 0, 0) \\ \psi(1, 0, 0) \\ \psi(0, 1, 0) \\ \psi(1, 1, 0) \end{pmatrix}. \quad (21)$$

A four component spinor structure emerges.

The complete Brillouin zone is covered by $0 \leq p_x < 2\pi, 0 \leq p_{y,z} < \pi$, i.e. one quarter of the naive zone. This gives a four by four reduced Hamiltonian

$$H = 16 \begin{pmatrix} \cos(p_z) & \cos(p_x) & \cos(p_y) & 0 \\ \cos(p_x) & -\cos(p_z) & 0 & -\cos(p_y) \\ \cos(p_y) & 0 & -\cos(p_z) & \cos(p_x) \\ 0 & -\cos(p_y) & \cos(p_x) & \cos(p_z) \end{pmatrix}. \quad (22)$$

This can be written more compactly as

$$H = 16(\cos(p_x)I \otimes \sigma_1 + \cos(p_y)\sigma_1 \otimes \sigma_3 + \cos(p_z)\sigma_3 \otimes \sigma_3). \quad (23)$$

The three terms anti-commute; so, energy eigenvalues are

$$E = \pm 16 \sqrt{\cos^2(p_x) + \cos^2(p_y) + \cos^2(p_z)}. \quad (24)$$

In the restricted Brillouin zone, we again have two Dirac cones, one at $\vec{p} = (\pi/2, \pi/2, \pi/2)$ and the other at $\vec{p} = (3\pi/2, \pi/2, \pi/2)$. We have a Hamiltonian version of staggered fermions. In three dimensions we have only two doublers, the minimum required if there is to be a chiral symmetry.

For connection with the more usual continuum Hamiltonian, identify $i\gamma_0\vec{\gamma} = (I \otimes \sigma_1, \sigma_1 \otimes \sigma_3, \sigma_3 \otimes \sigma_3)$. Given that γ_0 should anti-commute with these and

γ_5 commute, we are led to the gamma matrix convention

$$\begin{aligned}
\gamma_1 &= \sigma_2 \otimes \sigma_2 \\
\gamma_2 &= \sigma_3 \otimes I \\
\gamma_3 &= \sigma_1 \otimes I \ . \\
\gamma_0 &= \sigma_2 \otimes \sigma_3 \\
\gamma_5 &= \sigma_2 \otimes \sigma_1.
\end{aligned}
\tag{25}$$

Since the Hamiltonian only contains nearest neighbor hoppings, its sign would be changed by changing the signs on the creation and annihilation operators on all sites of a given parity. Thus it anti-commutes with $(-1)^{x+y+z}$ which in the continuum should be represented by $\gamma_0\gamma_5$.

Near the zeros in the Brillouin zones, the Dirac cones display a linear dispersion. These come with opposite effective chirality. Thus the theory which has emerged has two “flavors” with opposite chirality. The physical chiral symmetry is actually a “flavored” symmetry. This is consistent with the anomaly, which forbids a flavor singlet chiral symmetry.

6 Path integrals

So far the discussion has been in terms of a hopping Hamiltonian. Usually lattice gauge theory is discussed in terms of the Euclidean space path integral including a Dirac operator appearing in a quadratic form $\bar{\psi}D\psi$. Here $\bar{\psi}$ and ψ are independent Grassmann fields. In a continuum discussion D is formally an anti-Hermitian operator, although this is not generally true on the lattice.

The procedure in the previous section of inserting phase factors on the lattice links generalizes immediately to a four dimensional lattice, and in this way we can find a corresponding Hermitian Hamiltonian H . One can try to use this form directly in a path integral, using $D = iH$. With a corresponding gauge choice, we have periodicity by two lattice sites in three of the dimensions and periodicity of one in the remaining. In this way the basic spinor that emerges has 8, rather than the 4 components of the usual Dirac theory. In terms of the relativistic fermions that appear, there is an extra doubling. This gives rise to a total of four “tastes,” a famous aspect of staggered fermions. It is possible to break the degeneracy of these states by adding Hermitian terms to D that break the chiral symmetry [12, 13]. That procedure is similar in spirit to the Wilson fermion construction.

A minimally doubled formulation of staggered fermions with only two “tastes” appears in [14, 15]. This form preserves the hyper-cubic symmetry, but suffers from a non-positive determinant. There are a variety of other known 4d minimally doubled chiral formulations for fermions [16] that do not suffer from a sign problem. All, however, appear to break hyper-cubic symmetry in some way. This in general introduces new renormalization counterterms; in particular, the “speed of light” for both the fermions and the gauge fields can be renormalized [17]. Whether this is always necessary for a four dimensional minimally doubled and chiral discretization is not known.

Since we have a three dimensional Hamiltonian formulation with only two doublers, it is natural to ask if this can lead to a minimally doubled staggered action for the path integral. This would be particularly convenient because the physical world has two light quark species, rather the four that are natural in the usual staggered approach.

This procedure is possible, but one must use care to properly treat the connection between the Hamiltonian and path integral approaches. One method is to use an old exact relationship [18] for the trace of a product of n normal ordered operators in a Hilbert space generated by fermionic creation and annihilation operators to a path integral over a set of Grassmann variables.

The resulting action is not what is usually used for staggered fermions and has some interesting features. First, it contains both Hermitean and anti-Hermitean parts. In this it shows a certain similarity to Wilson fermions [19]. Second, the time direction is treated differently than the spatial coordinates. The presence of a non-symmetric hopping in this direction explicitly breaks $4d$ hyper-cubic symmetry. The breaking of hyper-cubic symmetry indicates that on adding the gauge fields one should expect to require additional counter-terms to maintain the proper Lorentz symmetry. Third, when gauge fields are present, the corresponding fermion determinant need not be positive. This introduces a potential “sign problem” into any Monte Carlo treatment. It is unclear how severe this problem would be in practice. And fourth, the explicit chiral symmetry of the Hamiltonian becomes hidden and it is unclear whether lattice artifacts from the discretization in the time direction would introduce the need for an additional mass counterterm.

7 Including gauge fields

The above discussion has been in terms of free fermions. Nonetheless, it is straightforward to insert gauge degrees of freedom. If we consider an $SU(N)$ gauge group, traditionally this would be done by inserting group matrices on the lattice links. These are independent of the fixed phases Z on the links. Since the latter are all plus or minus one, we might as well consider them as an auxiliary Z_2 gauge field giving a factor of Z_{ij} on the link from site i to site j . When a fermion hops it picks up the product $U_{ij}Z_{ij}$ with U_{ij} being the link variable from the gauge group.

One can think of the theory as having two gauge couplings, a $\beta \sim \frac{1}{g^2}$ for the non-Abelian $SU(N)$ and a second β_z for the Z_2 factors. The above model arises in taking the limit $\beta_z \rightarrow -\infty$, which drives all Z_2 plaquettes to -1 . As mentioned above, this limit is exactly staggered Hamiltonian lattice gauge theory

A rather interesting special case occurs when the color group is even. Although this is not relevant to the physical $SU(3)$ of color, it does apply to the simple $SU(2)$ gauge group. With an even number of colors, then the phase factor -1 is in the gauge group. In the link product $U_{ij}Z_{ij}$, one can then absorb the Z factor into the U matrices. The $SU(N)$ measure is invariant under

this. After such an absorption, the only remnant of the Z_2 factors is that the coefficient β for the $SU(N)$ plaquette term changes sign. We conclude that for any even N , a conventional gauge theory of spin-less fermions at negative β is equivalent to staggered fermions. This does not work for $SU(3)$ because -1 is not an element of the gauge group.

8 Gauge fields and topology

In this section we change subject slightly and discuss the connections between lattice fermion fields and the topology of gauge fields. In a continuum field theory there is a well known index theorem [20], where

$$\nu = n_+ - n_- . \tag{26}$$

Here n_{\pm} counts the number of zero modes of the continuum Dirac operator of \pm chirality. The other side of the equation, ν represents the topological index of the gauge field.

This relation represents the heart of the anomaly. If the numbers of zero modes of each chirality are unequal, formally $\text{Tr } \gamma_5 = \nu$. This comes about because all non-zero eigenvalues of the Dirac operator come in complex conjugate pairs that cancel in this trace. The main consequence is that the naive chiral rotation

$$\psi \rightarrow e^{i\gamma_5\theta}\psi \tag{27}$$

changes the integration measure in the path integral [21]

$$d\psi \rightarrow e^{i\theta\text{Tr}\gamma_5} d\psi = e^{i\nu\theta} d\psi . \tag{28}$$

Because of this, a chiral rotation of the mass term

$$m \bar{\psi}\psi \rightarrow m \bar{\psi}e^{i\gamma_5\theta}\psi \tag{29}$$

results in an inequivalent theory. This is the famous “theta vacuum,” which violates CP symmetry when the rotation is non-trivial.

While all this is standard in a continuum formulation, on the lattice it becomes a bit trickier. With the cutoff in place γ_5 is always a finite traceless matrix, in apparent conflict with the above. To emphasize this point, consider any lattice Dirac operator D . For simplicity assume this satisfies gamma five hermiticity

$$\gamma_5 D \gamma_5 = D^\dagger . \tag{30}$$

All the fermion operators used in practice in lattice gauge theory satisfy this (except for twisted mass, which brings in an additional isospin rotation).

Now consider dividing D into Hermitean and anti-Hermitean parts $D = K + M$

$$\begin{aligned} K &= (D - D^\dagger)/2, \\ M &= (D + D^\dagger)/2. \end{aligned} \tag{31}$$

These automatically satisfy

$$\begin{aligned} [K, \gamma_5]_+ &= 0, \\ [M, \gamma_5]_- &= 0. \end{aligned} \tag{32}$$

With these definitions,

$$M \rightarrow e^{i\theta\gamma_5} M \tag{33}$$

is automatically an exact symmetry of the determinant

$$|K + M| = |e^{i\theta\gamma_5/2}(K + M)e^{i\theta\gamma_5/2}| = |K + e^{i\theta\gamma_5} M|. \tag{34}$$

This seems to contradict the continuum discussion. Indeed, where is the anomaly? In the above models, the answer lies with the doublers. Half of them use γ_5 and half $-\gamma_5$ for chiral rotations. In this case the naive chiral symmetry is actually flavored, and there is no contradiction.

Note that the above discussion was completely general and applies even to Wilson fermions, which are normally thought of as breaking chiral symmetry. What happens in this case depends crucially on the doublers, which have been given masses of order the cutoff. The above rotation $M \rightarrow e^{i\theta\gamma_5} M$ also rotates their phases. The important point is that the physical Θ is a relative angle arising when one independently rotates the fermion mass and the Wilson term; this was pointed out long ago by Seiler and Stamatescu [22]. In a sense, we still have a flavored chiral symmetry.

This interpretation also applies to the overlap operator [23]. In this case the eigenvalues of the Dirac operator lie on a complex circle. For every zero eigenmode there exists a heavy counterpart on the opposite side of the circle. The above rotation of Hermitian part rotates both the low mode and the heavy mode as well. Formally the anomaly brings in another chiral matrix $\hat{\gamma}_5$ defined so that $D\gamma_5 = -\hat{\gamma}_5 D$. In this approach the index theorem becomes

$$\nu = \text{Tr}(\gamma_5 + \hat{\gamma}_5)/2 \tag{35}$$

and need not vanish when $\hat{\gamma}_5$ is not traceless.

This discussion reveals an interesting message for continuum QCD. The physical parameter Θ can be moved around and placed on the mass term for any one flavor at will. In particular, Θ can be entirely moved into the phase of the top quark mass. This has the non-intuitive consequence that the top quark properties are relevant to the low energy physics of QCD. The traditional decoupling theorems [24] don't apply non-perturbatively when the masses have phases.

9 Conclusions

We have discussed a variety of models for spinless fermions hopping on simple lattices for which the excitations on the Dirac sea can carry spin. This is required by the relativistic form of spectrum. The phenomenon has close

connections with chiral symmetry and the topological protection from additive mass renormalization. For the phenomenon to occur, doublers are required to appear. And the entire picture is intimately entwined with the possibility of a CP violating parameter Θ in QCD.

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