Eigenvalues popular for discussion
- chiral condensate and density of small eigenvalues
- Banks-Casher formula
- approximations to the Ginsparg Wilson relation
- eigenvalues near “circles”
- projection issues for the overlap/domain wall operators
  - undefined when $D_W^+ D_W$ not invertible
  - need a gap in the Wilson operator spectrum

Dangers
- eigenvalues depend on gauge fields
- gauge fields depend on eigenvalues
- Highly non-linear system!
Generic path integral

\[ Z = \int (dA)(d\psi)(d\bar{\psi}) \, e^{-S_G(A) + \bar{\psi}D(A)\psi} \]

Integrate out fermions

\[ Z = \int (dA) \, |D(A)| \, e^{-S_G(A)} \]

Determinant is product of eigenvalues

\[ D(A)\psi_i = \lambda_i \psi_i \]

\[ Z = \int (dA) \, e^{-S_G(A)} \, \prod_i \lambda_i \]
Eigenvalue “density”

\[ \rho(x+iy) = \frac{1}{VZ} \int (dA) |D(A)| e^{-S_G(A)} \sum_i \delta(x - \text{Re}\lambda_i(A))\delta(y - \text{Im}\lambda_i(A)) \]

- \( V \) dimension of \( D \); proportional to system volume
- May not be positive if \( |D| \) is not
- Hermiticity condition (no chemical potential)
  - \( \gamma_5 D \gamma_5 = D^\dagger \)
  - \( \rho(z) = \rho(z^*) \)

Repeat warning:
- \( \lambda \) depends on \( A \) which is weighted by \( \lambda \) which depends on \( A \) . . .
Continuum

\[ D = \gamma_\mu (\partial_\mu + igA_\mu) + m \]

\[ \rho(x + iy) = \delta(x - m)\tilde{\rho}(y) \]

Banks and Casher, multiple flavors, vanishing mass
- \( \langle \overline{\psi}\psi \rangle \neq 0 \) correlates with \( \tilde{\rho}(0) \neq 0 \)

Index theorem: consider eigenmodes with real eigenvalues
- \( \gamma_5 \) commutes with \( D \) when restricted to this set
- chirality \( \pm 1 \)
- winding number \( \nu = n_+ - n_- \)
- matches winding from smooth gauge field topology
Lattice
- free Wilson fermions
- doublers given large real part

Turn on gauge fields
- $D$ no longer normal, i.e. $[D, D^\dagger] \neq 0$
- eigenvalues spread out, remain in complex conjugate pairs
- some eigenvalue pairs collide and become real
  - continuous spectrum of eigenvalues along real axis
  - index theorem for smooth fields
Overlap: project Wilson eigenvalues onto circle

- $D = 1 + V$
- $V = (D_W D_W^\dagger)^{-1/2} D_W$
- $V^\dagger V = 1$  \textit{Ginsparg-Wilson condition}
- normality restored
- $m < 0$: Wilson hopping parameter “supercritical”

Exact chiral symmetry

$$
\psi \rightarrow e^{i\theta \gamma_5} \psi \\
\bar{\psi} \rightarrow \bar{\psi} e^{i\theta \hat{\gamma}_5} \\
\hat{\gamma}_5 = -V \gamma_5 \\
\nu = \frac{1}{2} \text{Tr} \hat{\gamma}_5
$$
Consider the overlap

- Eigenvalues in complex conjugate pairs on a circle
  - $D = 1 + V$
  - $V^\dagger V = 1$
- Calculate the condensate

$$\langle \bar{\psi} \psi \rangle = \langle \text{Tr} D^{-1} \rangle = \left\langle \sum \frac{1}{\lambda_i} \right\rangle = \left\langle \sum \text{Re} \frac{1}{\lambda_i} \right\rangle$$
Inverting a complex circle gives another circle.

Circle for $D$ touches the origin
- inverses collapse onto line $\text{Re} \frac{1}{\lambda} = \frac{1}{2}$
- For all eigenvalues!

For the condensate

$$\langle \bar{\psi} \psi \rangle = \sum \text{Re} \frac{1}{\lambda_i} = \sum \frac{1}{2} = \frac{N}{2} \neq 0$$

- $N$ is the dimension of $D$
- Independent of any dynamics!?
Do we have the wrong operator?

- $\bar{\psi}\psi$ nontrivial under generalized chiral symmetry
- is $\langle \bar{\psi}(1 - D/2)\psi \rangle$ better?
  - goes to its negative on chiral rotation

The second term is also easy to calculate

$$\langle \bar{\psi} D \psi \rangle = \text{Tr} D^{-1} D = \text{Tr}1 = N$$

Combining:

$$\langle \bar{\psi}(1 - D/2)\psi \rangle = N/2 - N/2 = 0$$

Oops, the condensate is gone?
Resolution: $V \to \infty$ and $m \to 0$ limits don’t commute

- add a small mass
- $\langle \psi \psi \rangle = \sum \frac{1}{\lambda + m}$
- look for a jump as $m$ passes through zero

Contour integral around the GW circle $\lambda = 1 + e^{i\theta}$

$$i \int_{0}^{2\pi} d\theta \frac{\rho(\theta)}{1 + e^{i\theta} + m}$$

- pole at $-m$ moves from inside to outside the circle
- residue $\rho(0) \equiv \lim_{\theta \to 0} \rho(\theta)$
- integral jumps by $2\pi \rho(0)$

The Banks-Casher relation for the overlap
Another Puzzle

Two flavors
- expect spontaneous chiral symmetry breaking, light pions
- should have $\rho(0) \neq 0$

One flavor
- anomaly breaks all chiral symmetry
- $\langle \bar{\psi}\psi \rangle$ behaves smoothly at $m \sim 0$
- should have $\rho(0) = 0$
  - note: zero modes give smooth contribution to $\langle \bar{\psi}\psi \rangle$ (see later)

But
- one flavor has one power of $|D|$
- two flavors have two powers
  - Two flavors should naively suppress small eigenvalues more!

How can two flavors have the bigger $\rho(0)$???
\( \rho \) depends on distribution of \( A \) depends on \( \rho \)

Not just low eigenvalues are relevant

- fermions tend to smooth out gauge fields
- more fermions smooth things more
- involves all scales
- smoother fields give more low eigenvalues
- overcomes suppression from more powers of the determinant

\[
\int dA \ |D|^N f \ e^{-S_g(A)}
\]

Increasing \( N_f \) can increase density of small eigenvalues!
Zero modes?

Again insert a small mass

\[ Z = \int dA \, e^{-S_g} \prod (\lambda_i + m) \]

As \( m \) goes to zero any configurations involving a \( \lambda = 0 \) drop out

- are “instantons” irrelevant in the chiral limit?
No: add sources $\eta, \bar{\eta}$

$$Z(\eta, \bar{\eta}) = \int dA \, d\psi \, d\bar{\psi} \, e^{-S_g + \bar{\psi}(D+m)\psi + \bar{\psi}\eta + \bar{\eta}\psi}$$

- integrate out fermions

$$Z = \int dA \, e^{-S_g + \bar{\eta}(D+m)^{-1}\eta} \prod (\lambda_i + m)$$

If source overlaps with the zero mode eigenvector $(\psi_0, \eta) \neq 0$
- $1/m$ in source term cancels $m$ from determinant
- With multiple flavors
  - need source factor from each flavor: “t’Hooft vertex”

Instantons drop out of $Z$
- but survive in correlation functions
  - small mass extrapolations are numerically difficult
Masses and topology

One massless flavor
- 't Hooft vertex quadratic in fermion fields
- generates smooth contribution to $\langle \bar{\psi}\psi \rangle$
  - $1/m$ term in $\sum 1/\lambda_i$ cancels $m$ from $|D|$ 
  - an additive mass shift "renormalon"
  - non-perturbative
  - depends on scale and regulator

Overlap operator is not unique
- depends on
  - particular input $D$ chosen
  - Wilson mass (domain wall height)

Scheme dependent additive mass shift
- $m = 0$ is not a physical concept for a single flavor
- $m_u = 0$ cannot solve strong $CP$ problem
Conventional variables

- $\Lambda$
- $|M|$
- $\theta : \tan(\theta) = \frac{\text{Im} M}{\text{Re} M}$

With one flavor these are singular coordinates
- scheme dependent additive shift in $\text{Re} M$ changes $\theta$

$\text{Re} M$ and $\text{Im} M$ are better independent variables
- CP symmetry protects the real axis
- imaginary axis can shift
  - Di Vecchia and Veneziano (1980)
\[ m = 0 \iff \text{vanishing topological susceptibility?} \]

Winding number ambiguous when \( D_W D_W^\dagger \) not invertible

- occurs with eigenvalues near domain wall height

Admissibility condition

- strong constraint on allowed plaquettes
- disallows rough configurations, making winding unique
- Violates reflection positivity!

Is the topological susceptibility a well defined observable?

- do we care?
Final remarks

Eigenvalues can give some insight
- Banks-Casher

But can be misleading
- adding flavors enhances low eigenvalues

Unresolved issues
- do we understand non-perturbative ambiguities?
  - is topological susceptibility an observable?
  - are rough gauge fields essential?
- how do these issues interplay with quark masses?
  - is $m_u = 0$ a definable concept?