Playing with sandpiles

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Some dissipative systems naturally flow to a critical state

- Bak, Tang, Weisenfeld 1987
- physics on all scales (fractal)
- no parameter tuning

Self-organized criticality complements chaos

Chaos:

- complex behavior
- few degrees of freedom

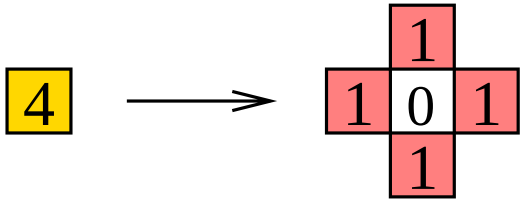
SOC:

- many degrees of freedom
- simple general features

The BTW model

Sandpile cellular automaton

- $d = 2$ square lattice
- integers Z_i on sites

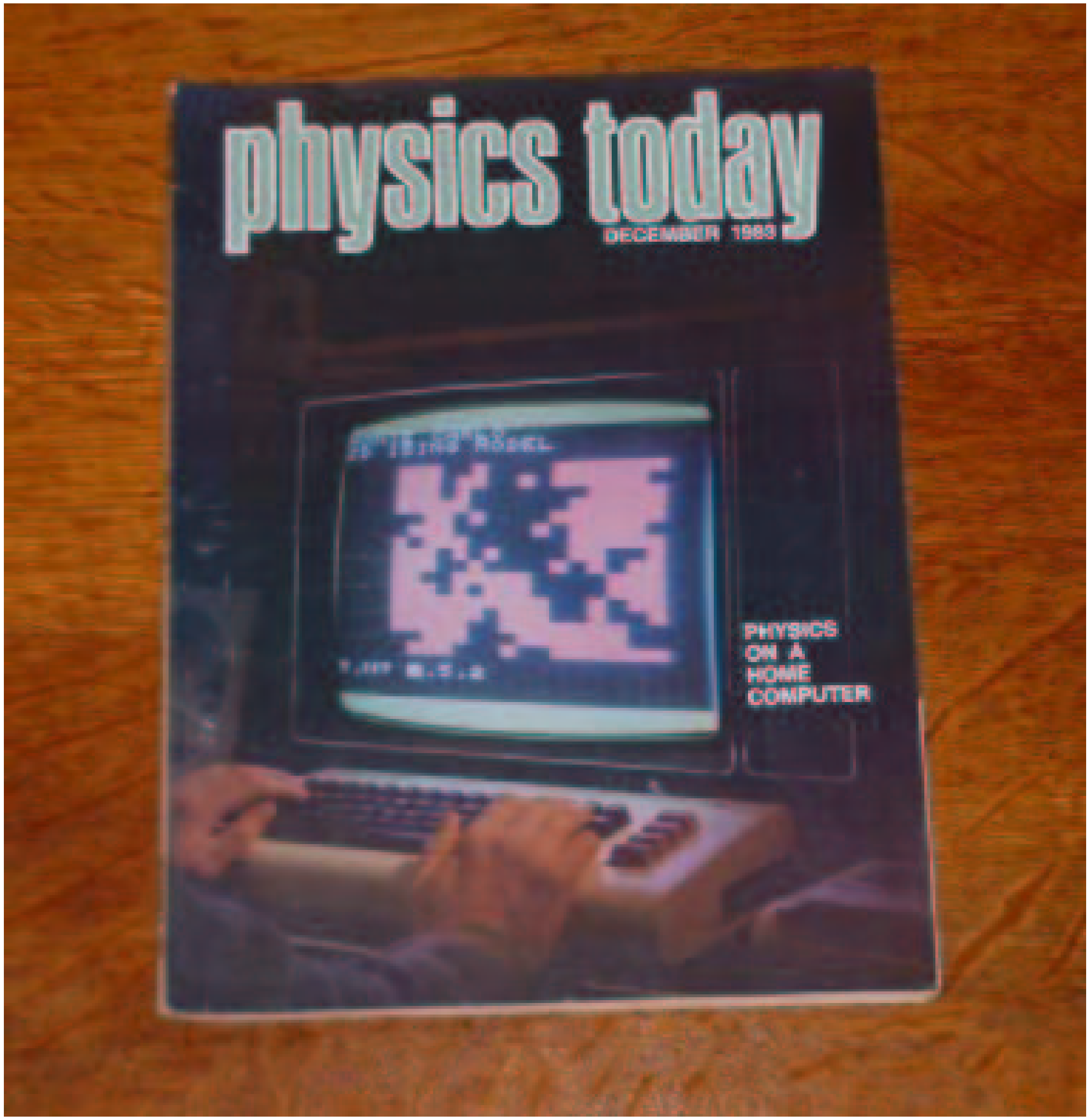


$Z_i > 3$ is unstable and tumbles

- $Z_i \to Z_i - 4$
- $Z_j \to Z_j + 1 \qquad j = \text{nearest neighbor}$

Update all sites simultaneously

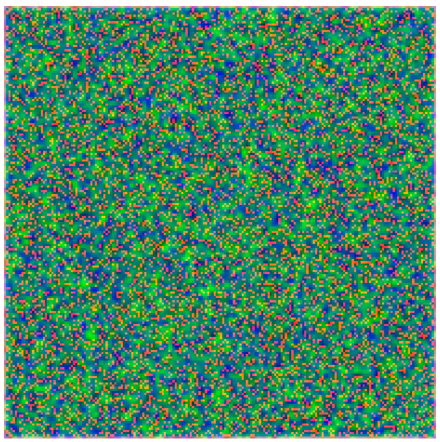
- repeat until all $Z_i < 4$
- sand lost at boundaries

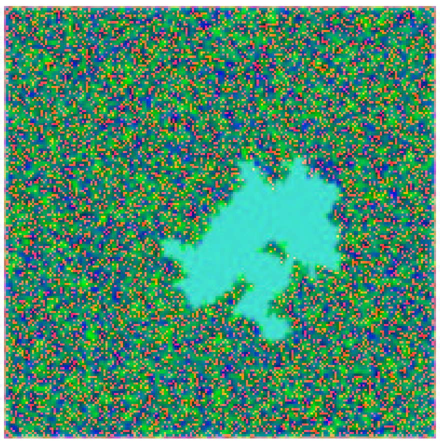


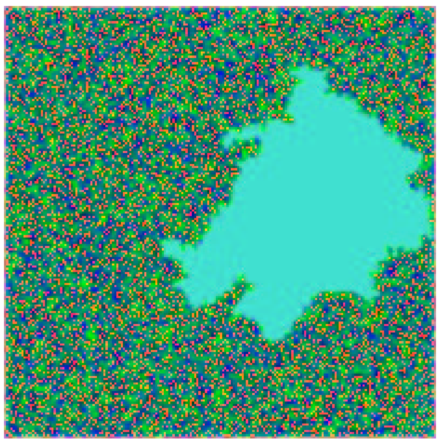
Per Bak’s hands

Simulator for X:

- http://thy.phy.bnl.gov/www/xtoys/xtoys.html
- (also windoze and amiga versions)







0 1 2 3 4 5 6 7

198 by 198 lattice

Sand model special -- Many exact results (Dhar)

Definitions:

Configuration C

- set of integer heights $Z_i \geq 0$
- on $d = 2$ finite square lattice

Tumbling $t_i C$

- $Z_j \rightarrow Z_j - \Delta_{ij}$, $\Delta_{ij} = \begin{cases} 4 & i = j \\ -1 & i,j \text{ neighbors} \\ 0 & \text{otherwise} \end{cases}$

Relaxation

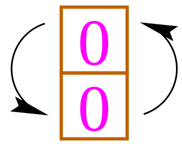
- tumble all sites with $Z_i > 3$
- repeat to stability
- sand spreads $\Rightarrow$ avalanche will stop

Addition operator $a_i C$

- $Z_i \to Z_i + 1$ and then relax

After dumping lots of sand, cannot reach some states

- two adjacent $Z_i = 0$



Recursive set R

- configurations reachable from any state
- set not empty; contains at least:

Minimally stable state C^*

- all $Z_i = 3$
- C^* is in R

Dhar proved two remarkable theorems

The a_i commute

- $a_i a_j C = a_j a_i C$
- proof uses linearity of t_i
- order of tumblings rearranged
- like addition: tumbling $\sim$ carrying

On the recursive set, a_i is invertable

- given C in R and specified site i
- there exists <u>unique</u> C' in R
- such that $a_i C' = C$
- $C' \equiv a_i^{-1} C$

Consequences

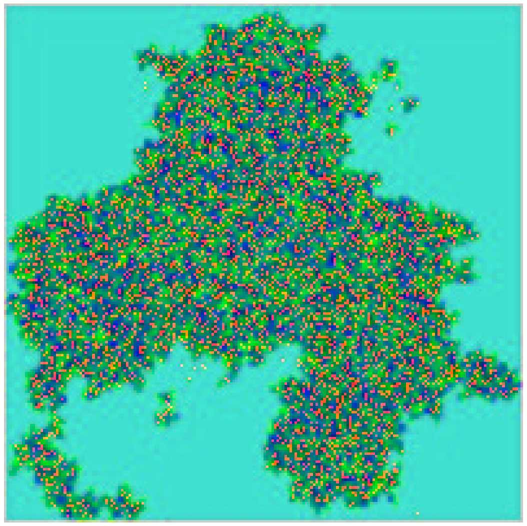
In "critical ensemble"

- all recursive states equally likely
- number of recursive states $|\Delta|$
- large system of N sites $|\Delta| \sim (3.2102\ldots)^N$
- versus 4^N total stable states

How to determine if a configuration is recursive?

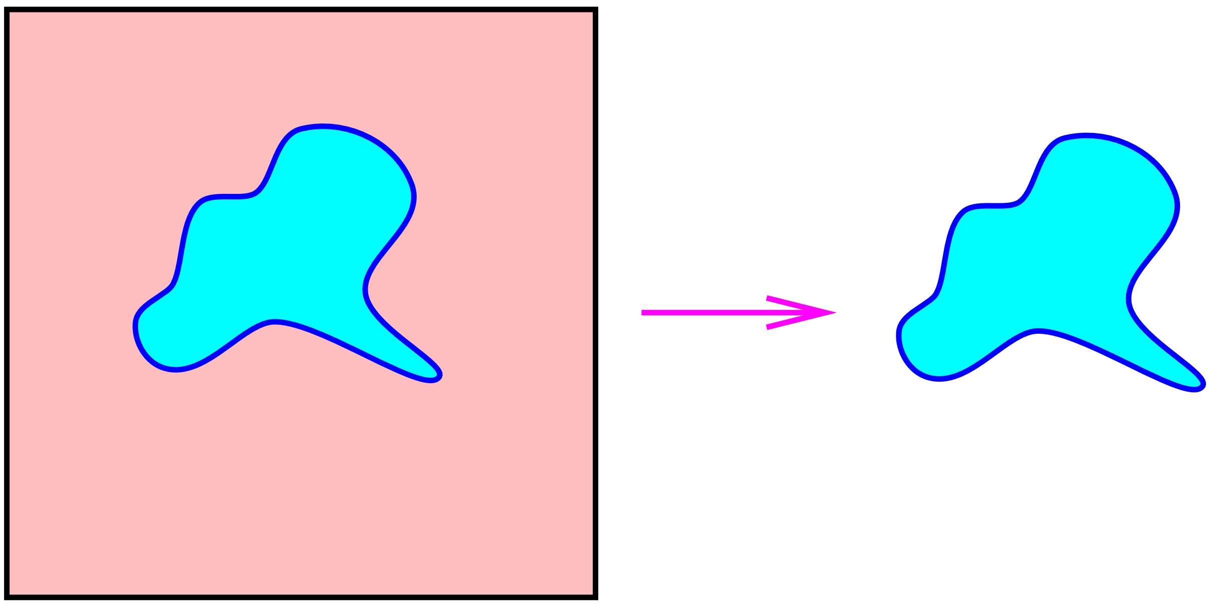
- "Burning algorithm" (Dhar)
- add one sand grain to C from every open edge
- "sandy boundaries" for one time step
- dump one grain on edges, two on corners

C recursive iff each site then tumbles exactly once



Recursive sublattices

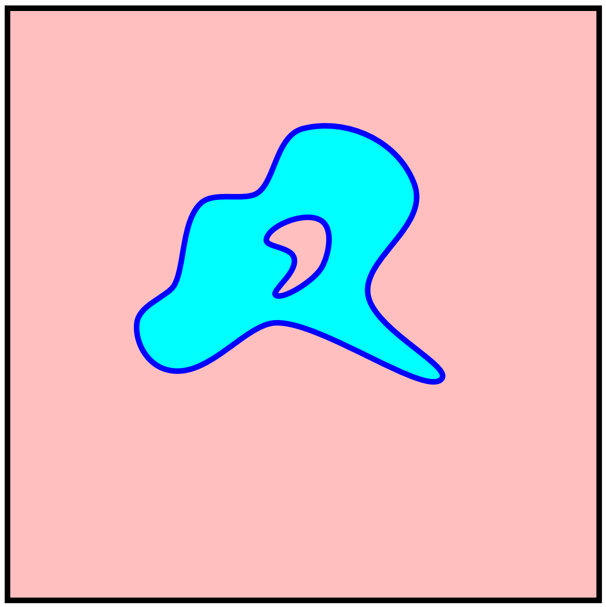
Extract a sublattice L from a recursive state C



The resulting subsystem is recursive on sublattice L

- follows from the burning algorithm
- avalanche surrounding L will burn it

Avalanches on a recursive state are simply connected



forbidden

- not true for non-recursive states

Adding states

Given stable configurations C, C'

with slopes Z_i, Z_i'

Define $C \oplus C'$ by relaxing $Z_i + Z_i'$

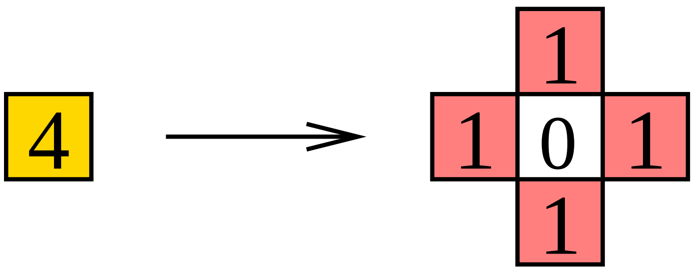
Under $\oplus$ recursive states $\rightarrow$ Abelian group

What is the "identity" state I?

- $I \oplus C = C$ iff C is recursive
- unique nontrivial configuration with $I \oplus I = I$

To construct I:

$$a_i^4 = \prod_{\text{neighbors}} a_j$$

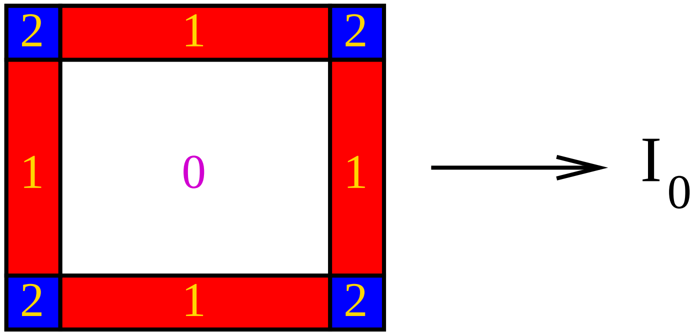


a_i invertable on recursive set

$$1 = a_i^4 \prod_{\text{neighbors}} a_j^{-1}$$

Multiply over all sites

- a_i factors cancel on interior sites
- one power of a_i left on each edge
- two grains on each corner
- basis of burning algorithm



Iff C recursive

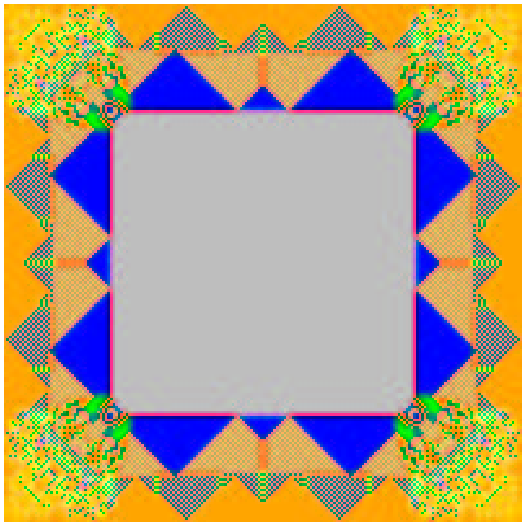
$$I_0 \oplus C = C$$

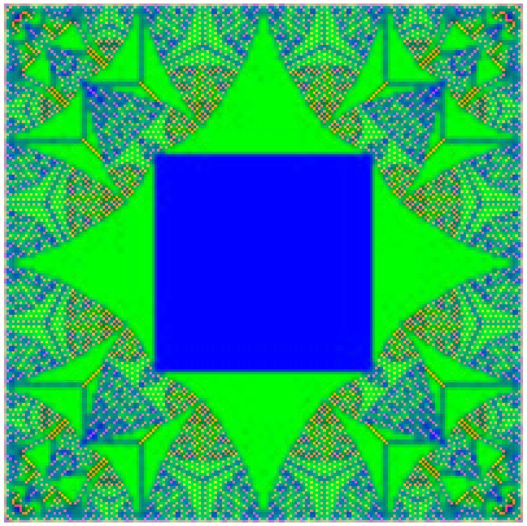
But I_0 is not itself recursive

- $I_0 \neq I$

Iterate:

- sandy boundaries
- fill empty table
- return to open boundaries
- relaxes to the identity





Other exact results

1. If C recursive, in constructing $C \oplus I$ the number of topplings at any given site is independent of C.

2. A single added grain n sites from the edge can tumble no site more than n times.

3. A single grain added anywhere can tumble a site n steps from the edge no more than n times.

4. Some averages known

- tumblings from a given addition
- sites of given height   Priezzhev

These slides:

- http://thy.phy.bnl.gov/∼creutz/slides/copenhagen.pdf