Chiral symmetries and lattice fermions

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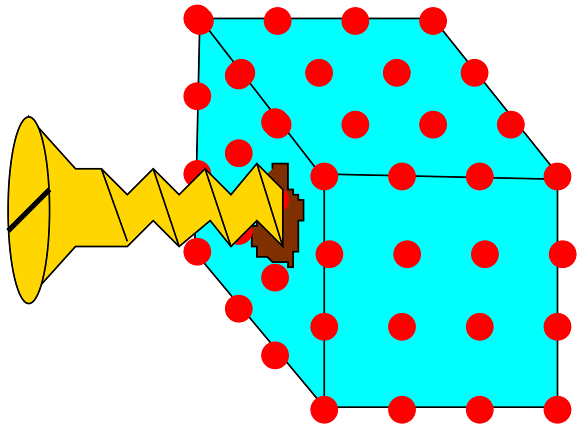
BNL



Two powerful tools for non-perturbative QCD

- chiral symmetry
- the lattice

Combining these has a tortuous history

- lattice formulation must properly include chiral anomalies

Source of persistent and bitter controversies

Two flavor QCD with light but non-degenerate masses

Pseudoscalars

$\overline{u}\gamma_5 d \sim \pi_+$ $\qquad M^2_{\pi_+} \sim (m_u + m_d)/2$

$\overline{d}\gamma_5 u \sim \pi_-$ $\qquad M^2_{\pi_-} \sim (m_u + m_d)/2$

$\overline{u}\gamma_5 u$

$\overline{d}\gamma_5 d$

Helicity conservation naively suggests mixing of

$$\overline{u}\gamma_5 u = \overline{u}_L\gamma_5 u_R + \overline{u}_R\gamma_5 u_L \quad \text{with} \quad \overline{d}\gamma_5 d = \overline{d}_L\gamma_5 d_R + \overline{d}_R\gamma_5 d_L$$

- suppressed by $m_u m_d$

$M^2_{\overline{u}\gamma_5 u} \sim m_u$ $\qquad M^2_{\overline{d}\gamma_5 d} \sim m_d$

Wrong: the anomaly strongly mixes $\overline{u}\gamma_5 u$ and $\overline{d}\gamma_5 d$

topology induces the effective “t’Hooft vertex” $\sim \overline{u}\gamma_5 u\overline{d}\gamma_5 d$

- physical $\eta' \sim \overline{u}\gamma_5 u + \overline{d}\gamma_5 d$ not a pseudo-Goldstone boson

$$M_{\eta'} \sim \Lambda_{qcd} + O(m_u + m_d)$$

Leaves the orthogonal combination $\pi_0 \sim \overline{u}\gamma_5 u - \overline{d}\gamma_5 d$

$$M_{\pi_0}^2 \sim \frac{m_u + m_d}{2}$$

- isospin breaking suppressed to higher order

$$M_{\pi_0}^2 = M_{\pi_\pm}^2 - O((m_u - m_d)^2) - \text{e.m. effects}$$

Fix m_d , vary m_u

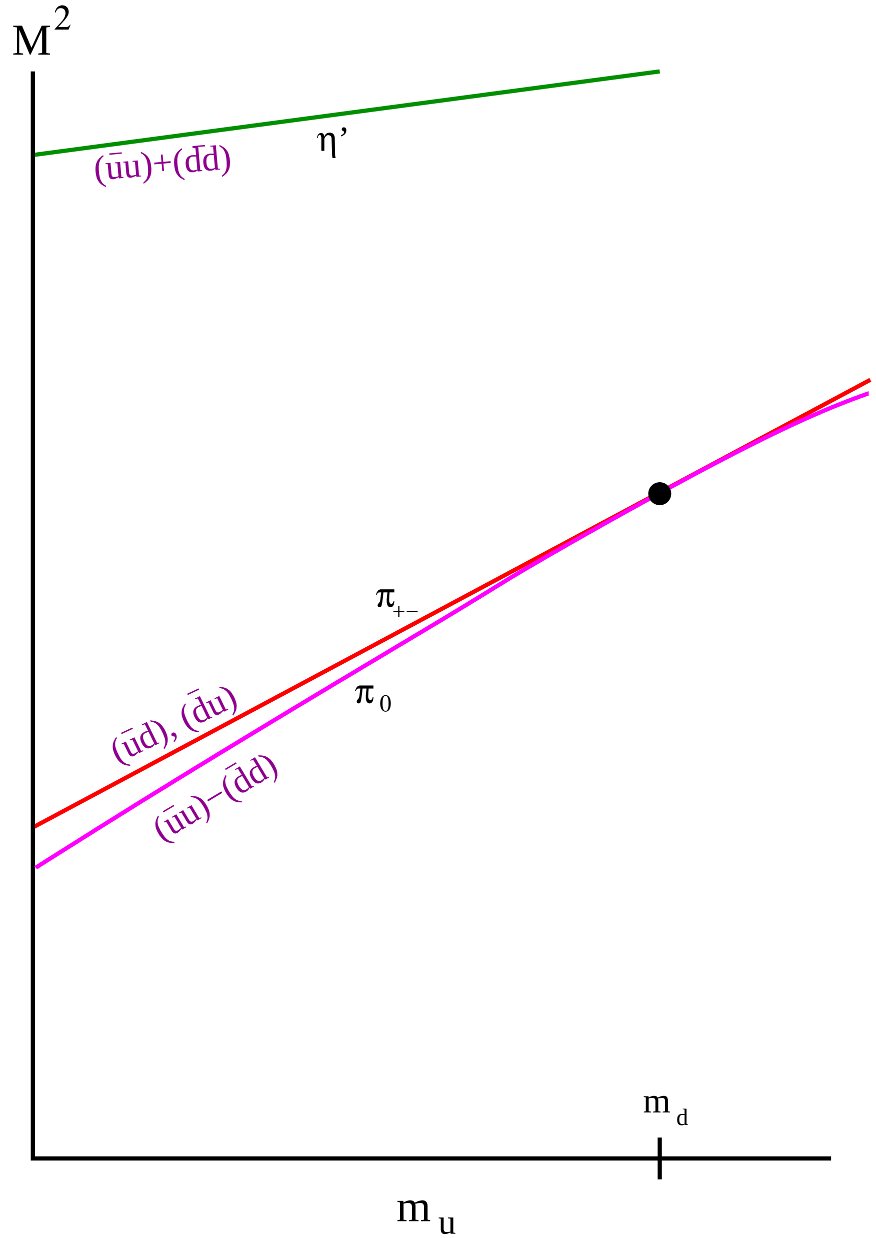
$$M_\pi^2 \propto \frac{m_u+m_d}{2} + O(m_q^2)$$

$$M_{\eta'} \sim \Lambda_{qcd}$$

With isospin broken

$$M_{\pi_\pm}^2 - M_{\pi_0}^2 \propto (m_d - m_u)^2$$

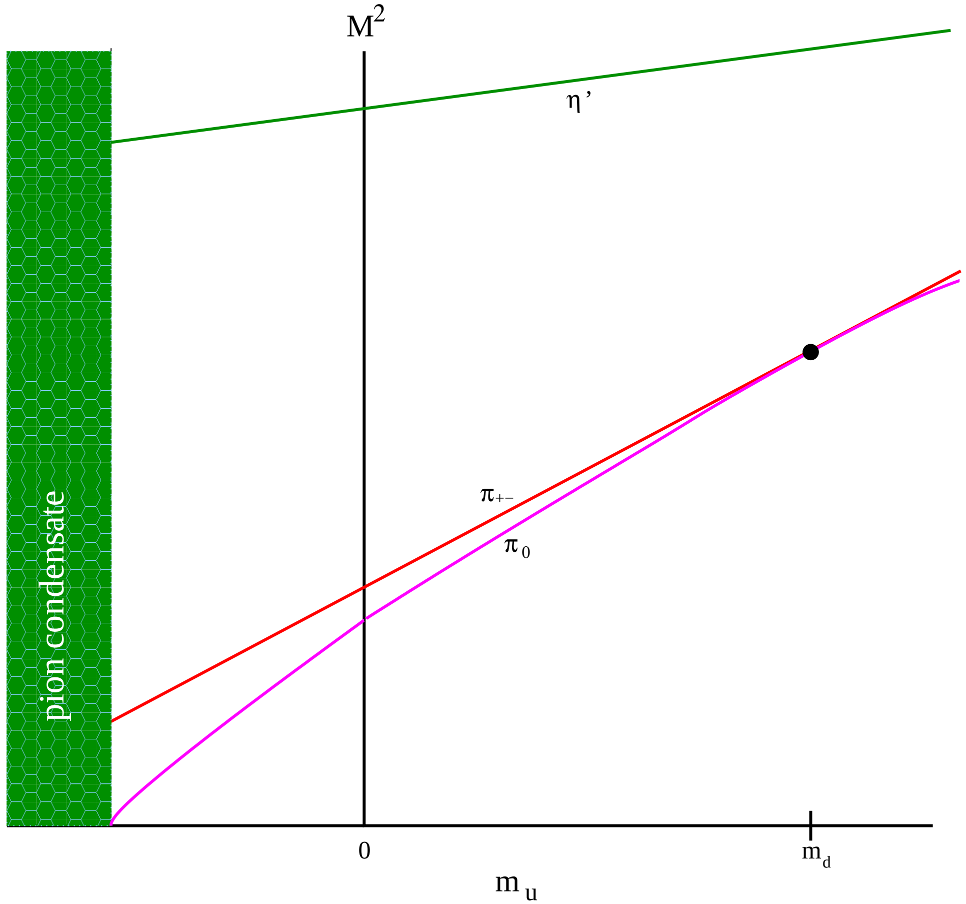
Mass gap survives at $m_u = 0$



The Dashen phenomenon

Mass gap at $m_u = 0$

- No singularity as the up quark mass passes through zero



Continue to negative m_u

- $M^2_{\pi_0}$ can go negative
- pion condensate forms

  $\langle \pi_0 \rangle \neq 0$

  CP broken

- formally at $\Theta = \pi$

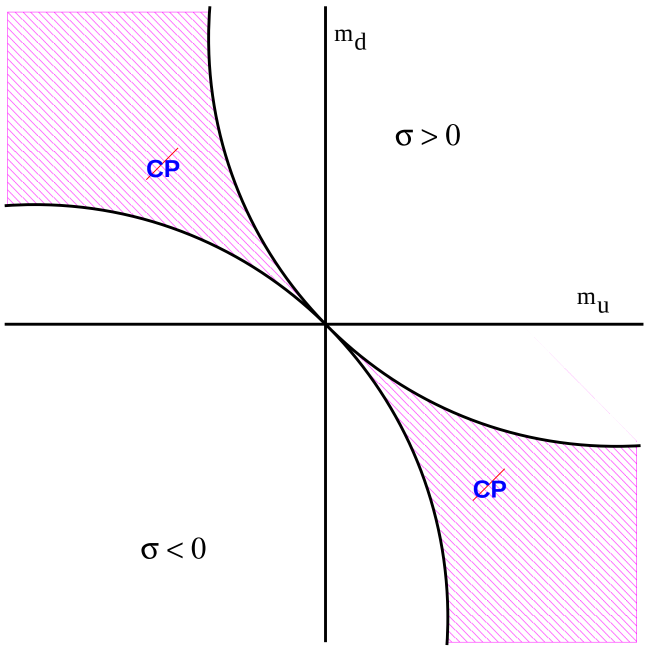
  $\prod_q m_q < 0$

Dashen 1971

Structure explicit in both “linear” and “nonlinear” sigma models

Ising-like transition at $m_u < 0$

- order parameter $\langle \pi_0 \rangle \neq 0$
- breaks CP spontaneously



No symmetry between $m_d + m_u$ and $m_d - m_u$

Message 1: A mass gap persists when only one quark is massless

Long distance physics falls exponentially

- $\langle\phi(x)\phi(0)\rangle_c <\sim e^{-m_{\pi_0}|x|}$
- for any gauge invariant operator ϕ
- no long distance physics despite possible small Dirac eigenvalues

2 or more massless quarks: pions become massless

$m_u = 0$ has no clear experimental consequences

Is $m_u = 0$ universal between non-perturbative schemes? $\Leftarrow$ Not Proven!

No Banks-Casher singularity at $m_u = 0$ with $m_d > 0$

- small Dirac eigenvalue density vanishes for one massless quark

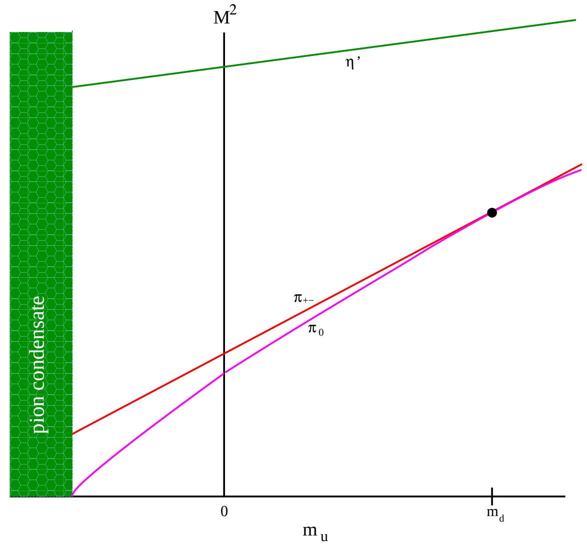
Raises issues for partial quenching

- valence quarks can't condense if the sea doesn't

  as $m_{valence} \to m_u$ valence and up quark propagators become equal

- valence pions will not become massless as $m_{valence} \to 0$

Usual partial quenching assumptions fail if $m_{valence} < \langle m_{sea} \rangle$



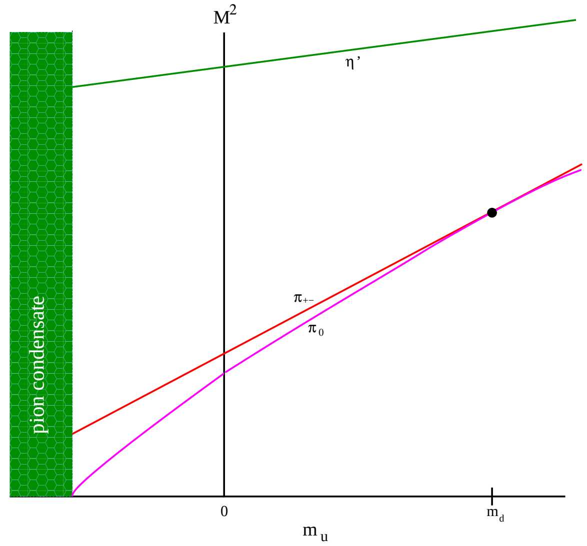
Message 2: The sign of a quark mass is physically relevant

Not a property of perturbation theory

Topology: configurations at $m_u < 0$ have a $(-1)^\nu$ in their weight

- More generally, $m_u \to e^{i\Theta\gamma_5} m_u$ gives inequivalent physics: “Theta vacuum”

Matching lattice and $\overline{MS}$ masses dangerous for non-degenerate quarks



Message 3: Divergent correlation length possible when no quarks are massless

Second order transition at non-vanishing m_u and m_d of opposite sign

Mass gap can vanish without small Dirac eigenvalues

- notes:

  first order transition at $\Theta = \pi$ only within Dashen phase

  topological susceptibility $\to -\infty$ at boundary of Dashen phase

Lattice Fermions

Consider some arbitrary lattice Dirac operator D

- assume gamma five hermiticity $\gamma_5 D \gamma_5 = D^\dagger$
- all operators in practice satisfy this (except twisted mass)

Divide D into Hermitean and antihermitean parts

$$D = K + M$$

$$K = (D - D^\dagger)/2$$

$$M = (D + D^\dagger)/2$$

Then by construction

$$[K, \gamma_5]_+ = 0$$

$$[M, \gamma_5]_- = 0$$

On a lattice everything is finite; so $\mathrm{Tr}\gamma_5 = 0$

$M \to e^{i\theta\gamma_5} M$ is an exact symmetry of the determinant

$$|K + M| = |e^{i\gamma_5\theta/2}(K + M)e^{i\gamma_5\theta/2}| = |K + e^{i\theta\gamma_5} M|$$

Message 4: Any lattice action is symmetric under the chiral rotation $M \to e^{i\theta\gamma_5} M$

Where did the anomaly hide?

This must be a flavored chiral symmetry

All lattice actions must bring in extra structure

Naive, staggered, and minimally doubled fermions have doublers

- half use γ_5 and half $-\gamma_5$
- the naive chiral symmetry is actually flavored

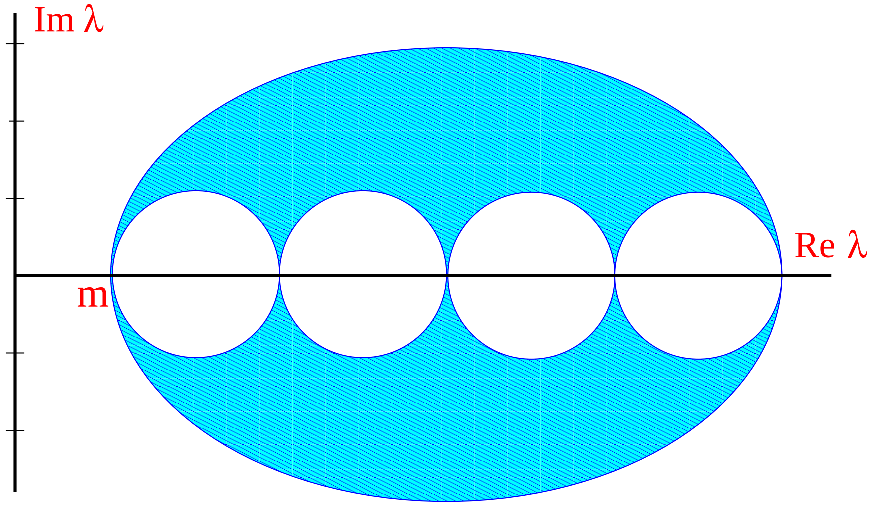
Wilson and overlap fermions

- The Hermitean part of M is not a constant
- heavy states appear to cancel the anomaly

Wilson: Add a momentum dependent mass

$$\overline{\psi} D_W \psi = \overline{\psi} \left(\frac{1}{a} \sum_\mu (i\gamma_\mu \sin(p_\mu a) + 1 - \cos(p_\mu a)) + m \right) \psi.$$

The eigenvalues form a set of "nested circles"

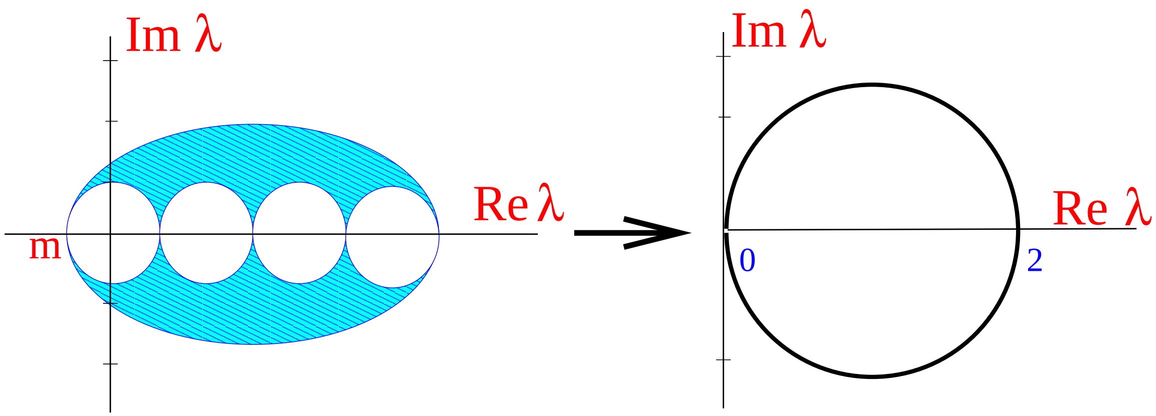


Notes

- $m \leftrightarrow -m$ not a symmetry
- naive chiral symmetry broken: $[D_W, \gamma_5]_+ \neq 0$

The overlap operator: Project D_W onto a circle

$$D_W \to D_O = 1 - V, \qquad V^\dagger V = 1$$



- a modified exact chiral symmetry

$$\psi \to e^{i\theta\gamma_5}\psi$$

$$\overline{\psi} \to \overline{\psi} e^{i\theta\hat{\gamma}_5}$$

- where $\hat{\gamma}_5 = V\gamma_5$.

$$\nu = \frac{1}{2}\mathrm{Tr}\hat{\gamma}_5 = \mathrm{Tr}\frac{\gamma_5+\hat{\gamma}_5}{2}$$

Where the anomaly went

With Wilson fermions

- doublers given masses of order the cutoff
- the rotation $M \to e^{i\theta\gamma_5} M$ also rotates their phases

Physical Θ is a relative angle

- independently rotate the fermion mass and the Wilson term

  Seiler and Stamatescu

The overlap operator

- eigenvalues on a circle
- zero eigenmodes have heavy counterpart
- rotation of Hermitean part rotates heavy mode as well

This hiding of the anomaly also happens in the continuum

- physical Θ can be moved around
- placed on any one flavor at will

Θ can be entirely moved into the top quark phase

- top quark properties relevant to low energy physics!

Decoupling theorems do not apply non-perturbatively

Messages

1: Mass gap persists when only one quark is massless

2: The sign of a quark mass is physically relevant

3: Divergent correlation length possible when no quarks are massless

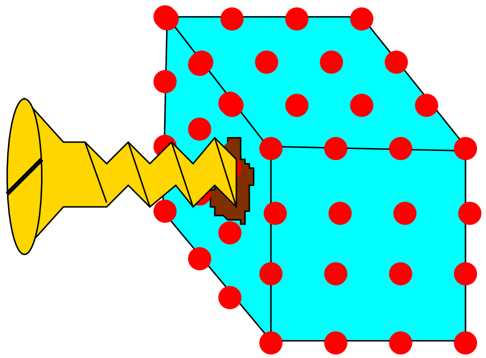
4: Any lattice action is symmetric under the chiral rotation $M \to e^{i\theta\gamma_5} M$

Consequences

- $m_u = 0$ need not be universal when other quarks are massive
- Partial quenching can fail if $m_{valence} < m_{sea}$
- Matching lattice results to $\overline{MS}$ can miss non-perturbative effects
- Decoupling theorems do not apply non-perturbatively

Related issues not discussed here:

- Topological susceptibility is not a universal observable
- Chiral symmetry of staggered fermions invalidates rooting



Addendum: Theta dependence and $m_u = 0$?

No symmetry between $m_d + m_u$ and $m_d - m_u$

- non-perturbative renormalization factors can differ
- $m_u = 0$ is not a renormalization group invariant

(m, Θ) are singular coordinates

$$m\overline{\psi}\psi \rightarrow m_r \cos(\Theta)\overline{\psi}\psi + im_i \sin(\Theta)\overline{\psi}\gamma_5\psi$$

m_r and m_i are independent parameters

- CP symmetry only requires $m_i = 0$
- only protected physical point for m_r is the Dashen phase boundary