Thermodynamics of (2+1)-flavor QCD from the lattice

Christian Schmidt

BROOKHAVEN NATIONAL LABORATORY

for the RBC-Bielefeld Collaboration

--- results from QCDOC ---

RIKEN BNL
Saumen Datta
Frithjof Karsch
Chulwoo Jung
Peter Petreczky
Wolfgang Söldner
Takashi Umeda

Columbia
Michael Cheng
Norman Christ
Robert Mawhinney

NBI
Kostya Petrov

Bielefeld
Olaf Kaczmarek
Edwin Laermann
Chuan Miao
Jan van der Heide

BU
Michael Clark

The Machines

US/RBRC QCDOC

20.000.000.000.000 ops/sec

BI – apeNEXT

5.000.000.000.000 ops/sec

- critical temperature
- equation of state
- hadron properites in matter

today: 1.6 TFlops

http://quark.phy.bnl.gov/~hotqcd

Outline

- **Motivation**
- **The Lattice Setup**
  - The p4-action
- **The critical temperature of QCD**
  - The flavor dependence
- **The Equation of State (preliminary results)**
- **Lattice QCD at non-zero density (the Taylor expansion method)**
  - The EoS non-zero chemical potential
  - The isentropic EoS
  - The speed of sound
  - The critical point ?
- **Summary**

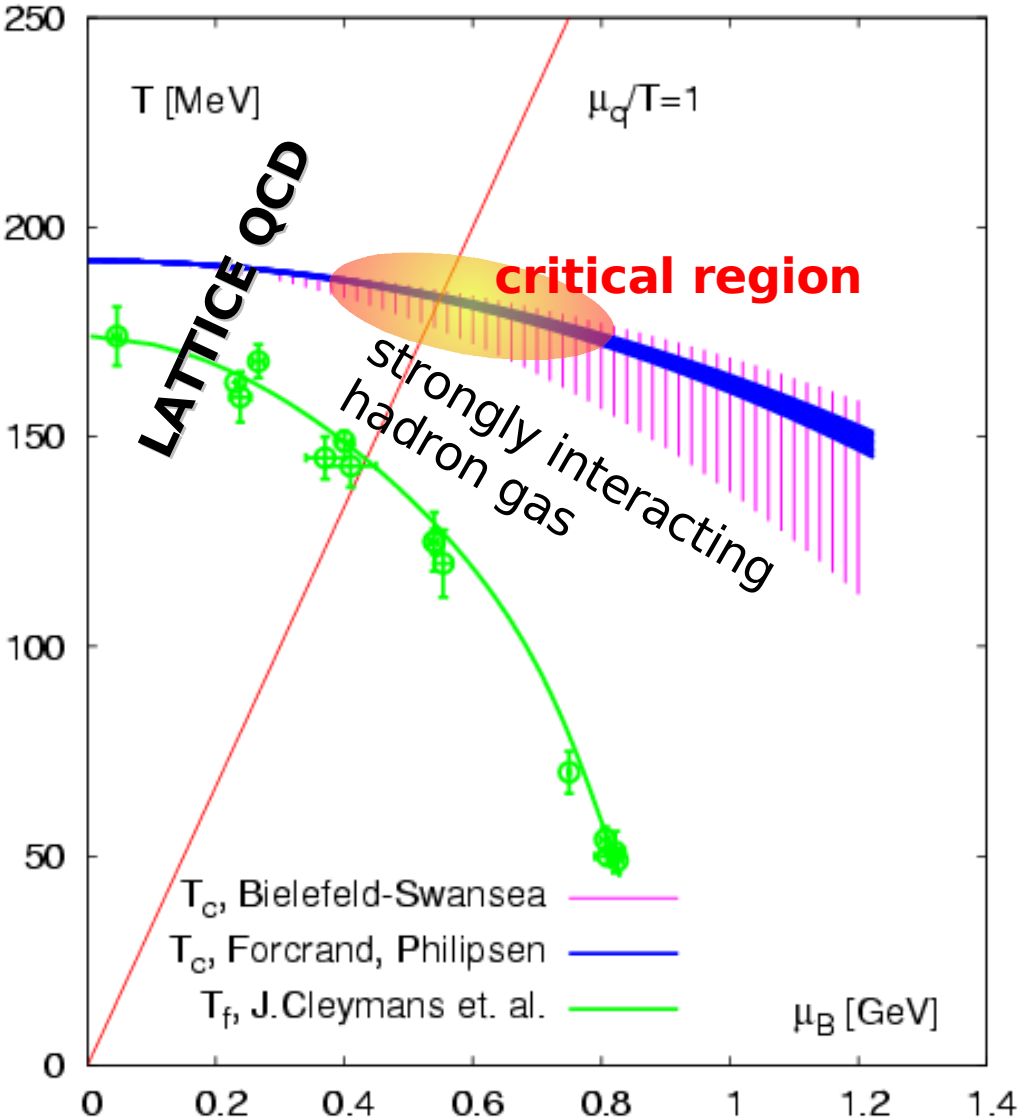
The Motivation

- QCD phase diagram calculations from first principals are important for Heavy Ion Phenomenology (and Cosmology and Astrophysics)

- Improving the accuracy on Tc (as a function of quark mass and chemical potential) is mandatory to make contact to HIC phenomenology

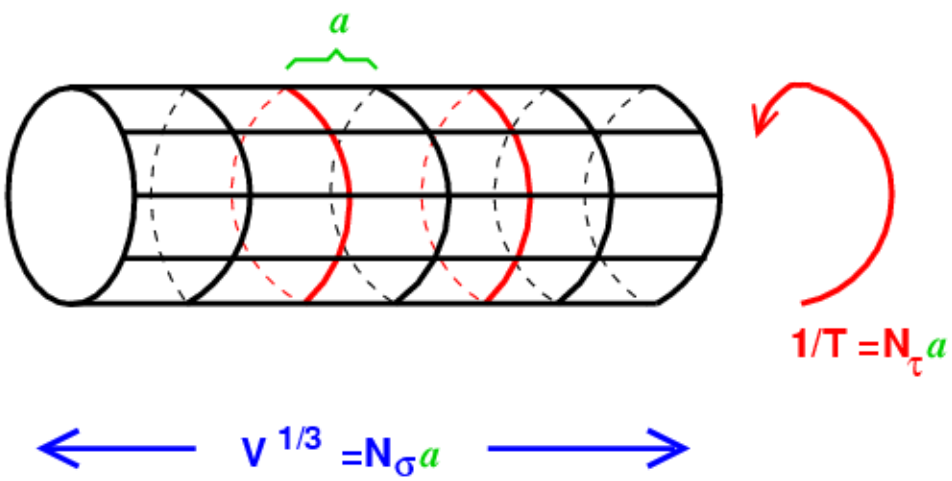
- Do we have a window of a strongly interacting hadron gas in the phase diagram ?

- How large is the size of the critical region?



The Lattice Setup

Analyzing hot and dens matter on the lattice: $N_\sigma^3 \times N_\tau$



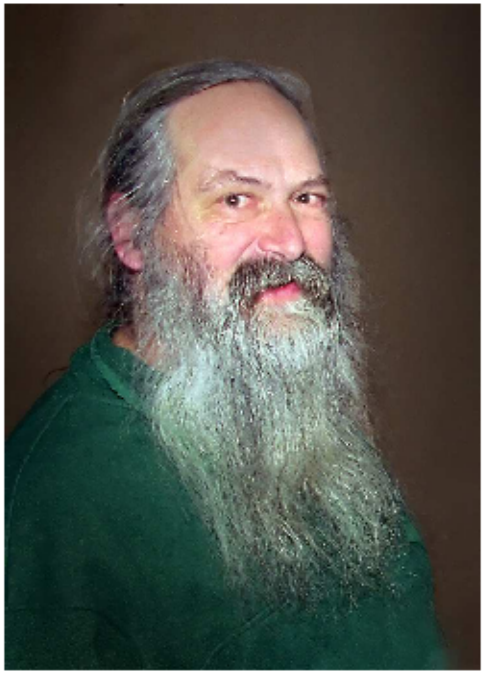
Quantum Chromo Dynamics

partition function: $Z(V,T,\mu) = \int \mathcal{D}\mathcal{A}\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S_E}$

$$S_E = \int_0^{1/T} dx_0 \int_V d^3x\, \mathcal{L}_E(\mathcal{A},\psi,\bar{\psi},\mu)$$

temperature volume chemical potential

Michael Creutz



Phys. Rev. D21 (1980) 2308

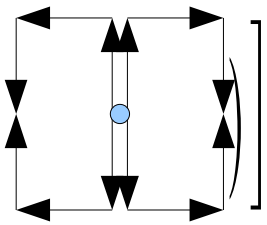
10^6 grid points, 10^8 d.o.f., ➡ integrate eq. of motion

The Lattice Setup

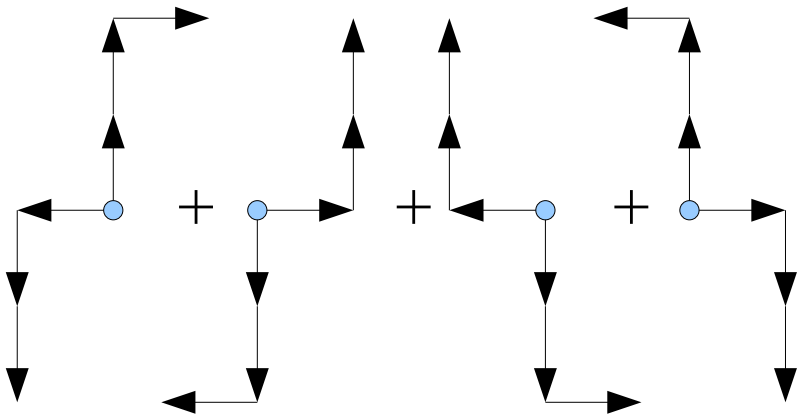
The p4-Action (fermionic part): an improved staggered fermion action

- Remove cutoff-effects and improve rotation symmetry by adding irrelevant operators
- Improve flavor symmetry by smearing the one link term

$$S_F(N_\tau, N_\sigma) = \sum_{n,\acute{n}} \sum_{\mu} \eta(n_\mu)\, \bar{\chi}_n \Bigg(\frac{3}{8} \Bigg[\frac{1}{1+6\omega} \Bigg(\longleftrightarrow + \omega \sum_{\nu \neq \mu}$$



$$+ \frac{1}{48} \sum_{\nu \neq \mu} \Big[$$



$$\Big] \Bigg) \chi_{n'} + m_q \sum_n \bar{\chi}_n \chi_n$$

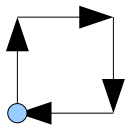
[Karsch, Heller, Sturm (1999)]

The Lattice Setup

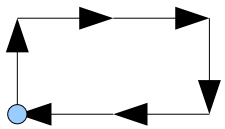
The p4-Action (gluonic part): Symanzik improvement scheme

- Remove cut-off effects of order $O(a^2)$ (tree-level improvement $O(g^0)$)

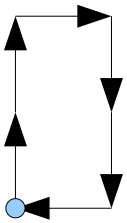
$$S_G(N_\tau, N_\sigma) = \sum_n \sum_{\mu,\nu>\mu} \left(\frac{5}{3}\left[1 - \frac{1}{3}\Re\, Tr \right.\right.$$



$$\left.\left. - \frac{1}{6}\left[1 - \frac{1}{6}\Re\, Tr\left[\right.\right.\right.$$



$$+$$



$$\left.\left.\left.\right]\right]\right)$$

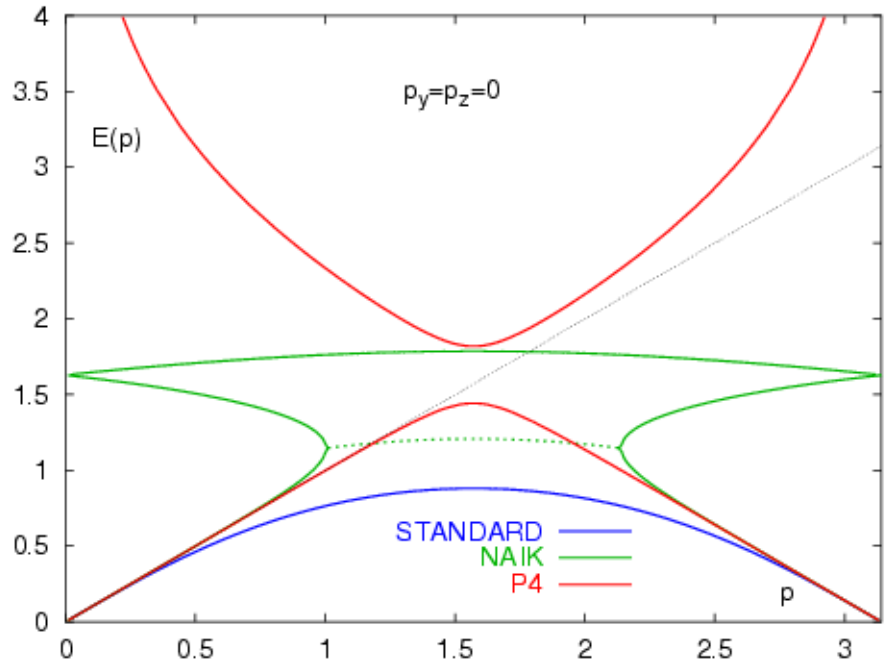
[Weisz, Wohlert (1984)]

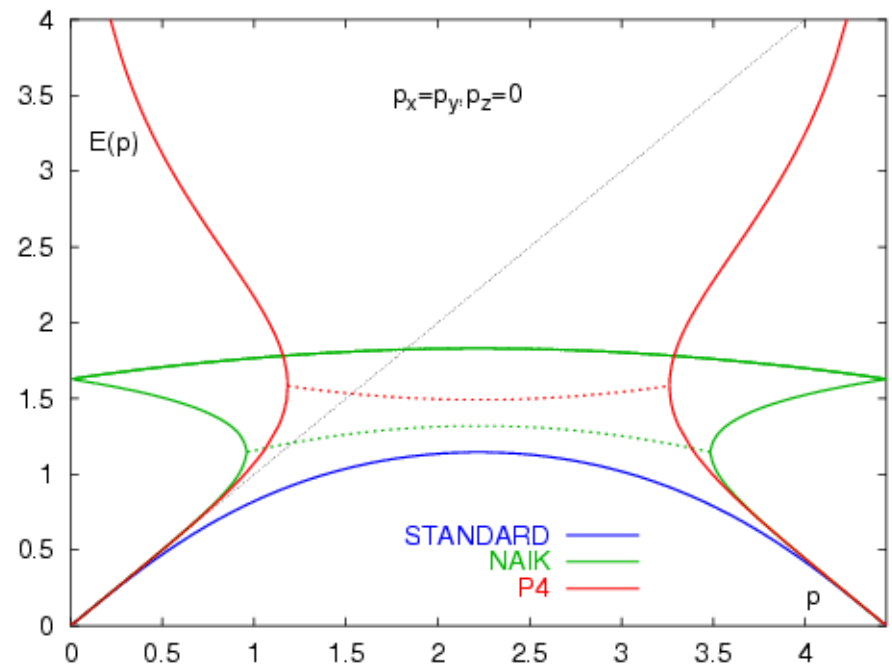
The Lattice Setup

Properties of the p4-Action: the rotational symmetry

- The free quark propagator is rotational invariant up to order $O(p^4)$
- Rotational symmetry of the heavy quark potential improved

Dispersions relation:





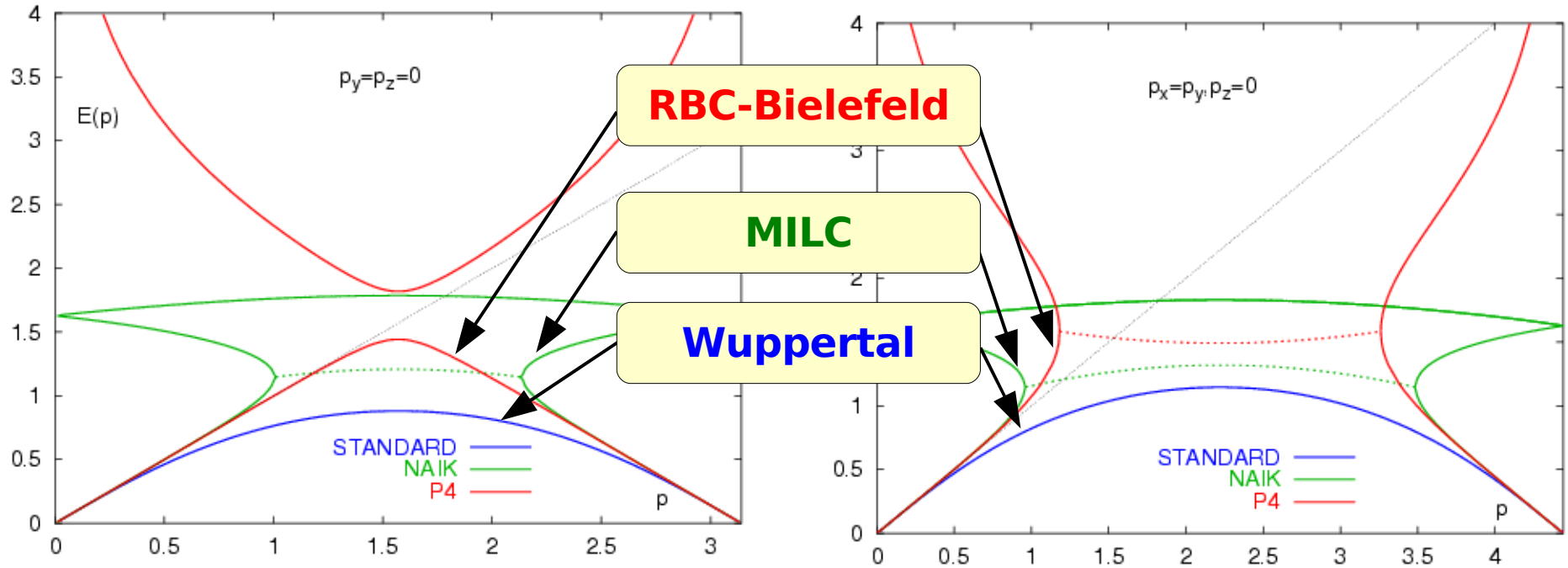
[Karsch, Heller, Sturm (1999)]

The Lattice Setup

Properties of the p4-Action: the rotational symmetry

- The free quark propagator is rotational invariant up to order $O(p^4)$
- Rotational symmetry of the heavy quark potential improved

Dispersions relation:



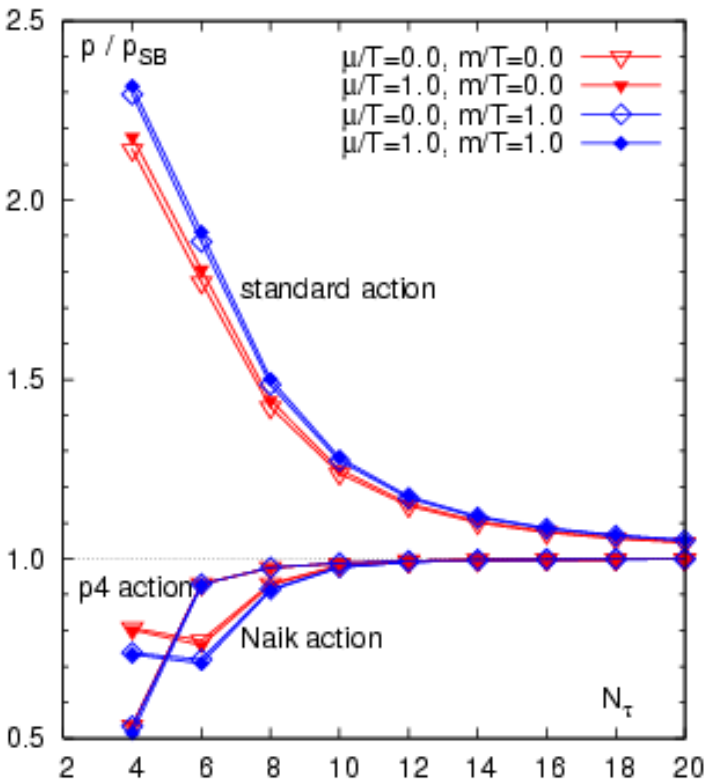
[Karsch, Heller, Sturm (1999)]

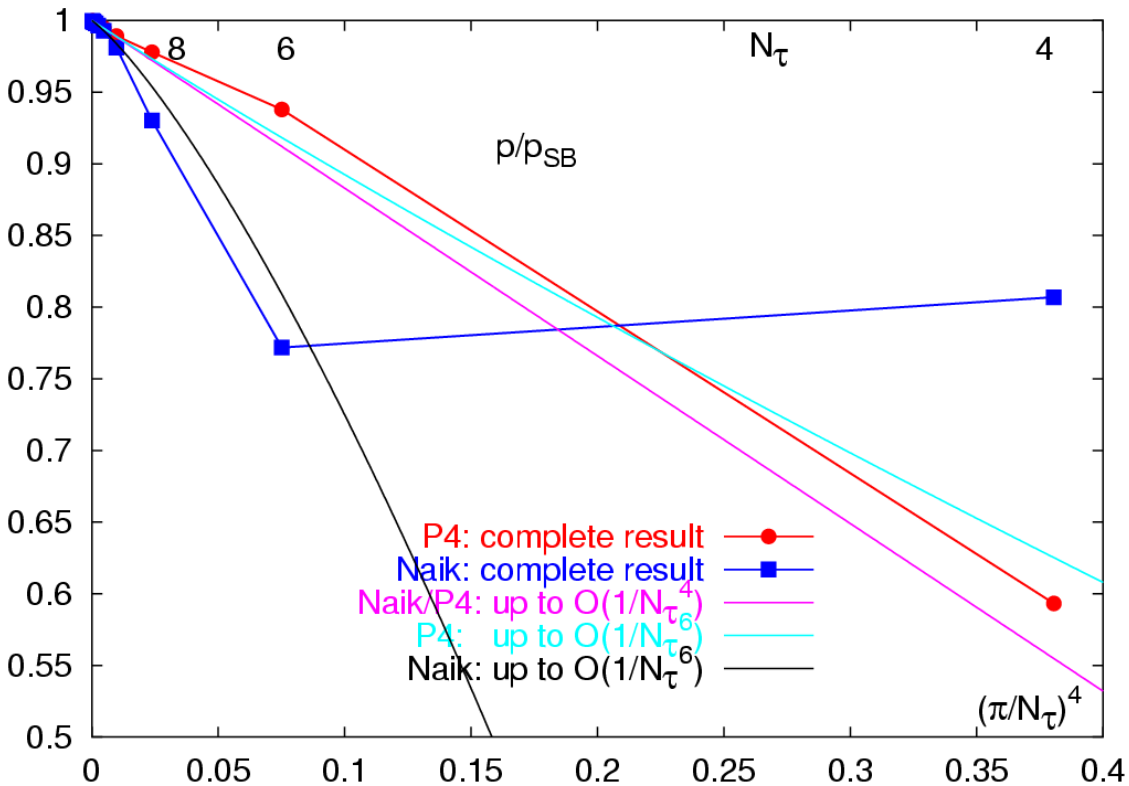
The Lattice Setup

Properties of the p4-Action: the cut-off effects

- Bulk thermodynamic quantities (pressure, energy density, ...) show drastically reduced cut-off effects

Continuum limit of the pressure for the free lattice gas:





The Lattice Setup

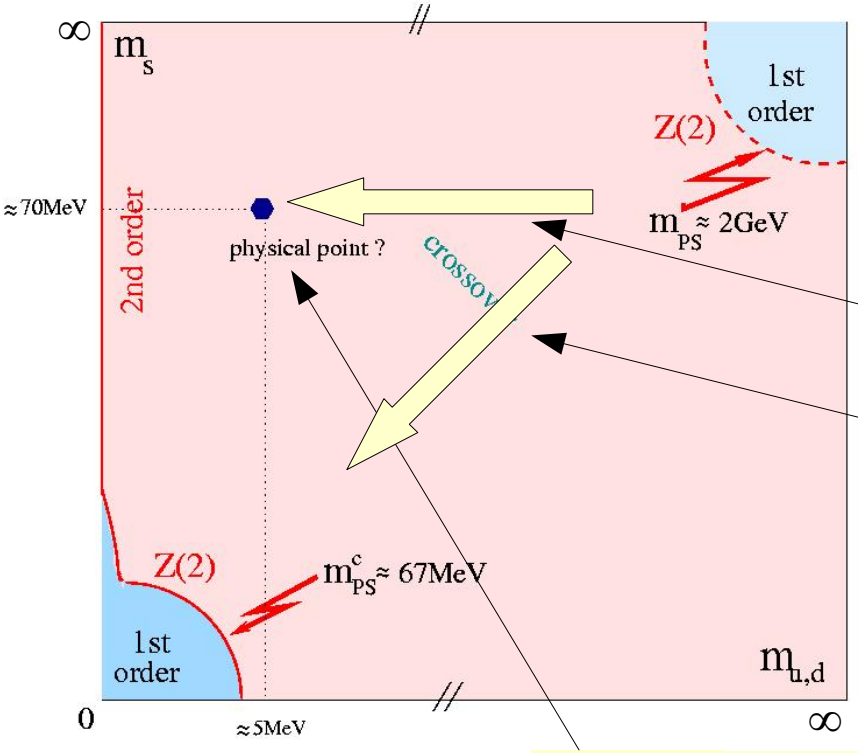
The final goal:

- Almost realistic quark mass spectrum
  (2+1 flavor, Pion mass ~ 200 MeV, Kaon mass ~500 MeV)
- Exploring the continuum limit
  (a ~ 0.1 fm – 0.2 fm $\rightarrow$ $N_\tau = 4, 6, 8$)
- Analyzing the thermodynamic limit
  (V ~ 500 fm³, $\rightarrow$ $N_\sigma = 8, 16, 32$)

The current status:

- 3-flavor QCD
  (masses: $m_\pi / m_\rho \geq 0.2$)
  (lattices: $N_\sigma = 8, 16, 32\,;\ N_\tau = 4, 6$)
- (2+1)-flavor QCD
  (masses: $am_q \geq 0.05\, am_s$)
  (lattices: $N_\sigma = 8, 16\,;\ N_\tau = 4, 6$)

The nature of the QCD transition ($\mu = 0$)



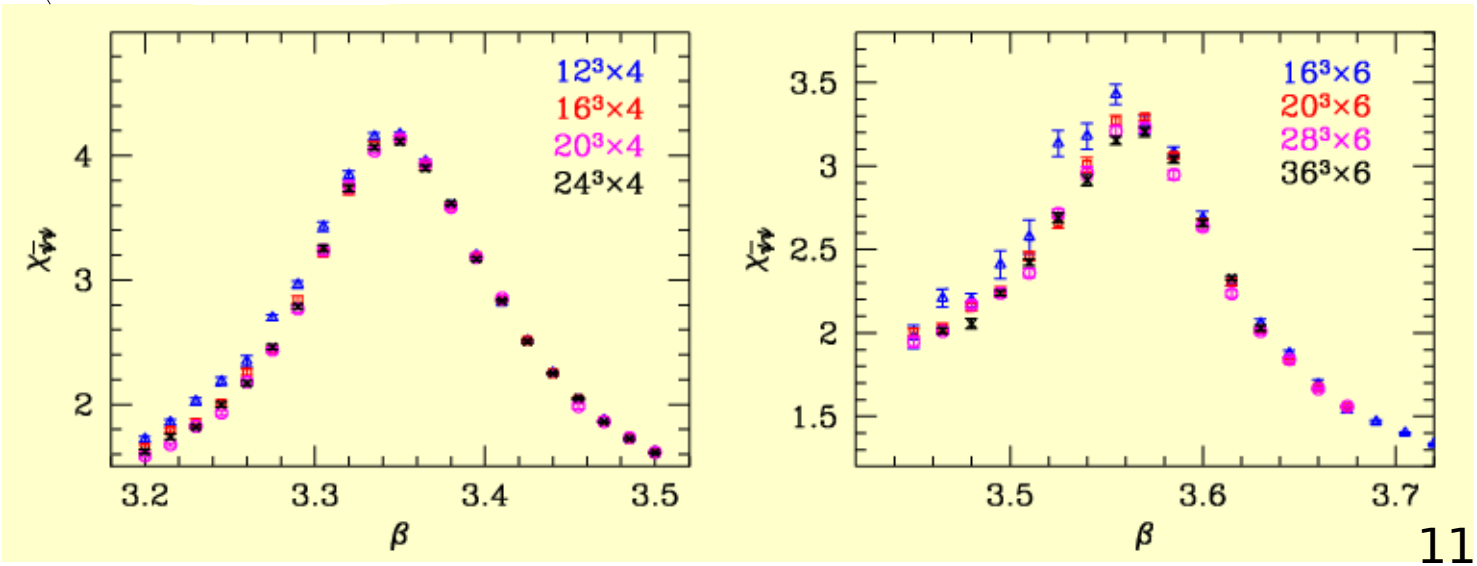
Mass range of the RBC-Bielefeld Collaboration for (2+1)-flavor

Mass range of the RBC-Bielefeld Collaboration for 3-flavor

Aoki et al., Nature 443 (2006) 675.

Std. staggered + stout smearing

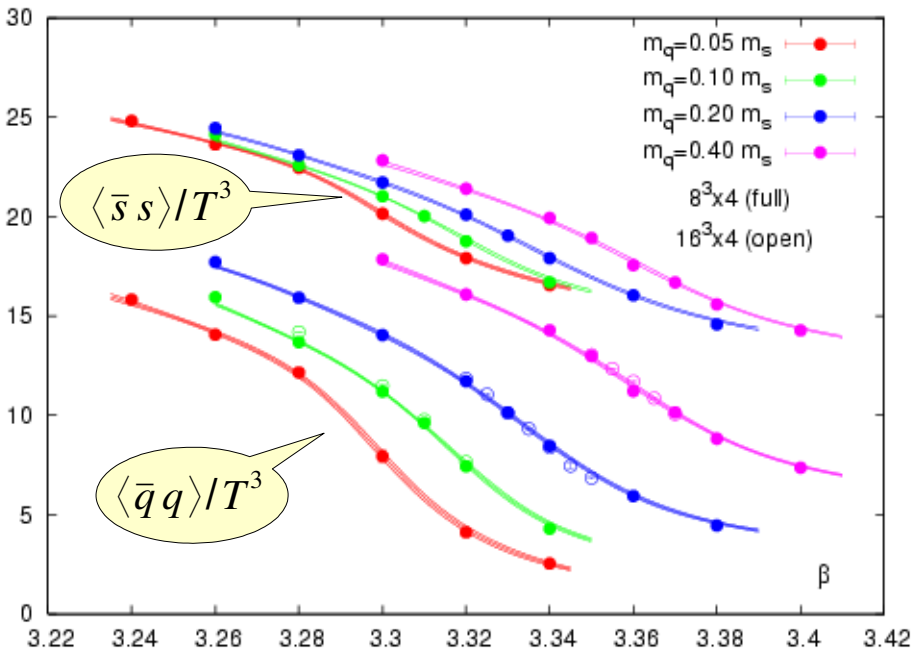
crossover!



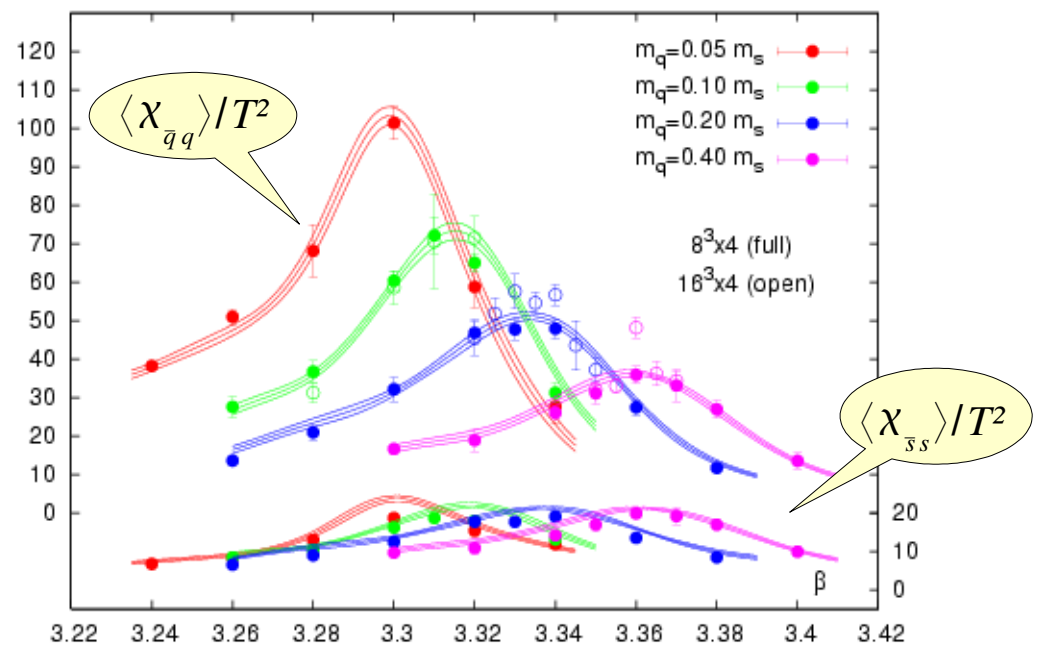
The critical temperature of QCD --- order Parameters and susceptibilities

(2+1)-flavor, 8^3x4 and 16^3x4 lattices

Strange- and light-quark chiral condensate:



Strange- and light-quark chiral susceptibility:



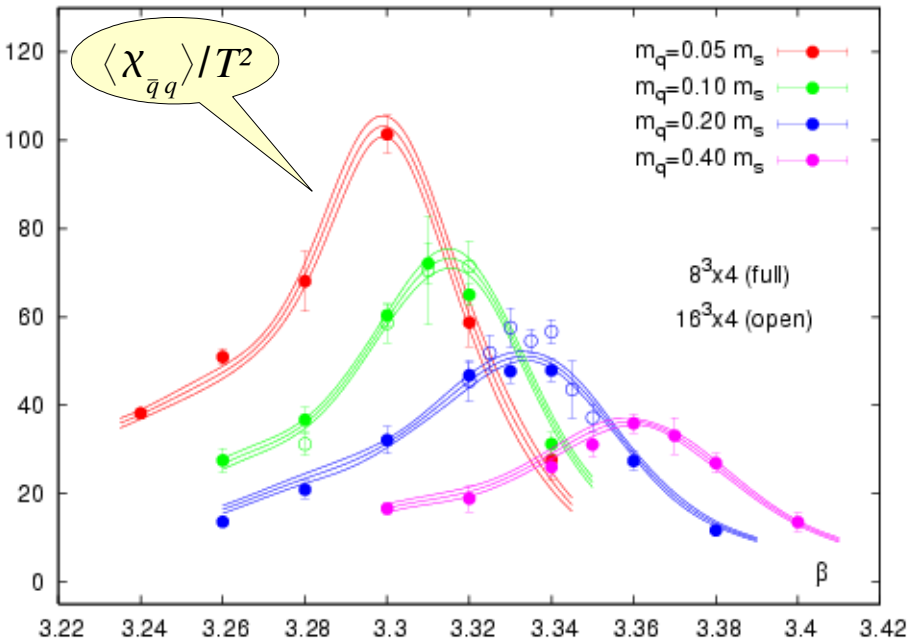
- Transition becomes stronger for *smaller* light quark masses.
- Critical couplings are determined by peak positions of the susceptibilities.
- All susceptibilities peak at the same critical coupling.
- We see no significant volume dependence ➡ *crossover*

- We use a multi-histogram analysis method (Ferrenberg-Swendson)

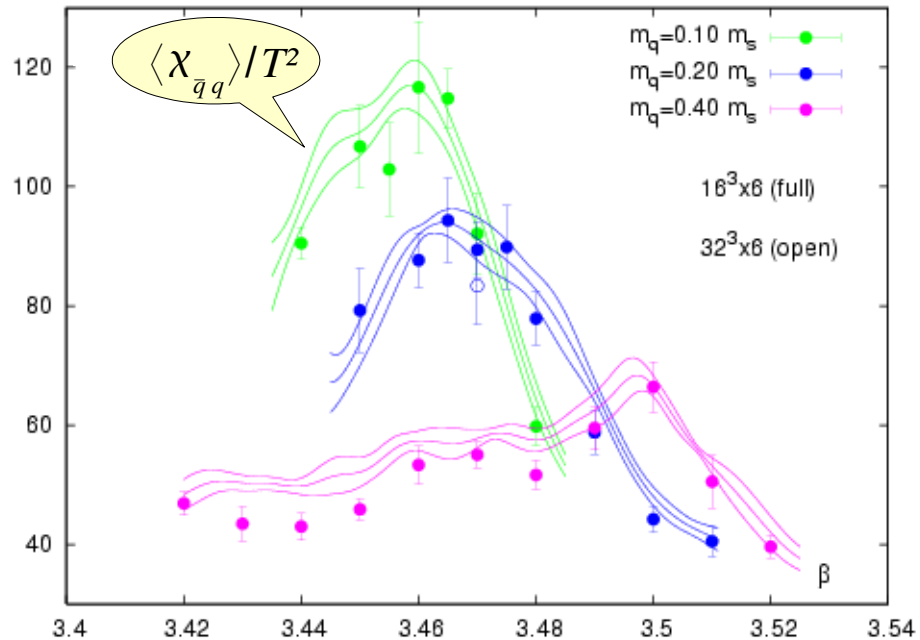
The critical temperature of QCD --- order Parameters and susceptibilities

(2+1)-flavor

8^3x4 and 16^3x4



16^3x6 and 32^3x6



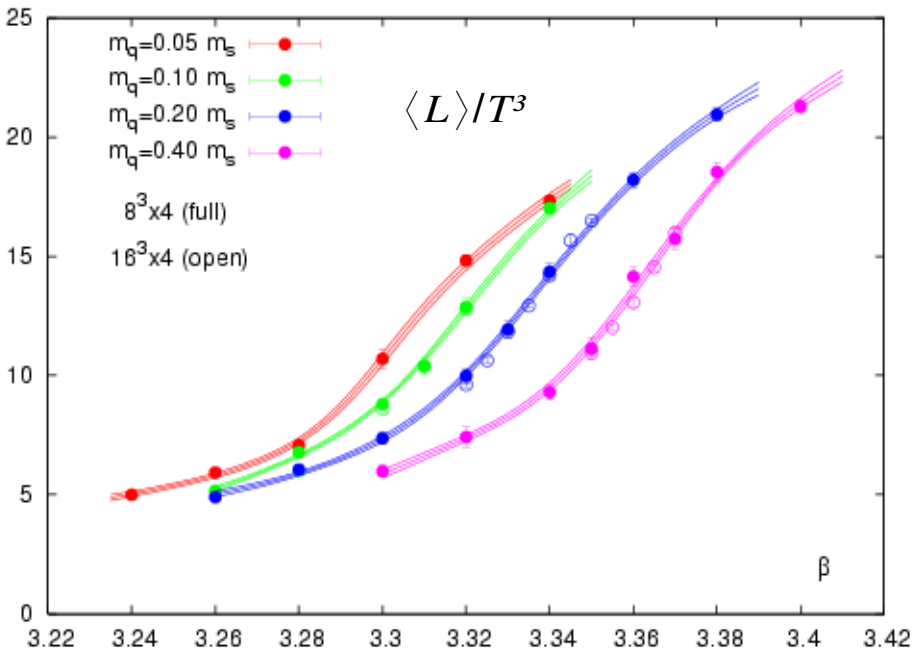
- Transition becomes stronger for *smaller* light quark masses.
- Critical couplings are determined by peak positions of the susceptibilities.
- All susceptibilities peak at the same coupling.
- We see no significant volume dependence ➡ *crossover*

- We use a multi-histogram analysis method (Ferrenberg-Swendson)

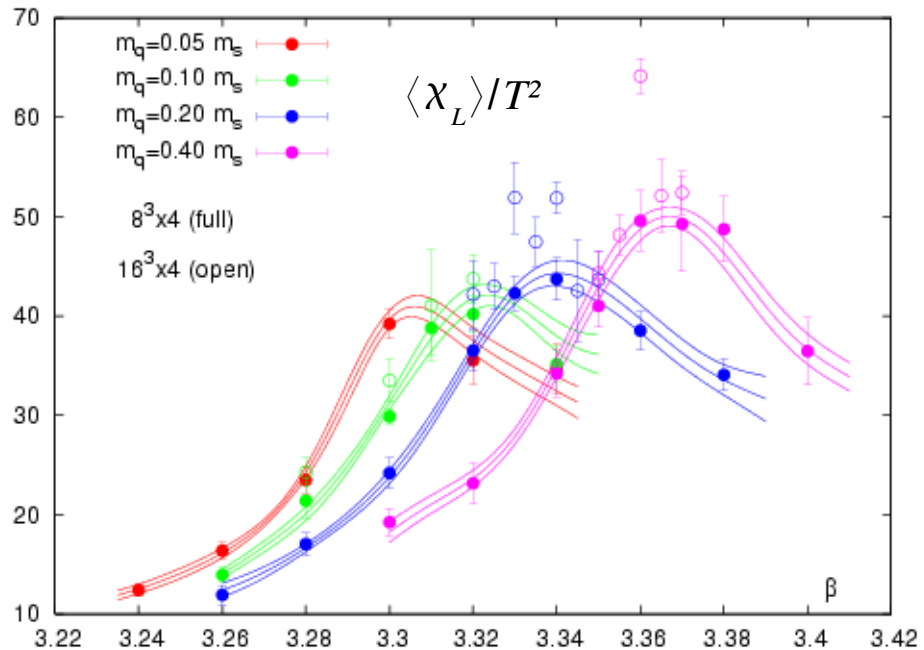
The critical temperature of QCD --- order Parameters and susceptibilities

(2+1)-flavor, 8^3x4 and 16^3x4 lattices

Polyakov Loop:



Polyakov Loop Susceptibility:



- Transition becomes stronger for *larger* light quark masses.
- Critical couplings are determined by peak positions of the susceptibilities.
- Critical couplings from chiral and Polyakov Loop susceptibilities coincide.

- We use a multi-histogram analysis method (Ferrenberg-Swendson)

The critical temperature of QCD --- determination of the crossover point

First Step:

- Determination of the critical couplings from peak positions of the chiral susceptibilities

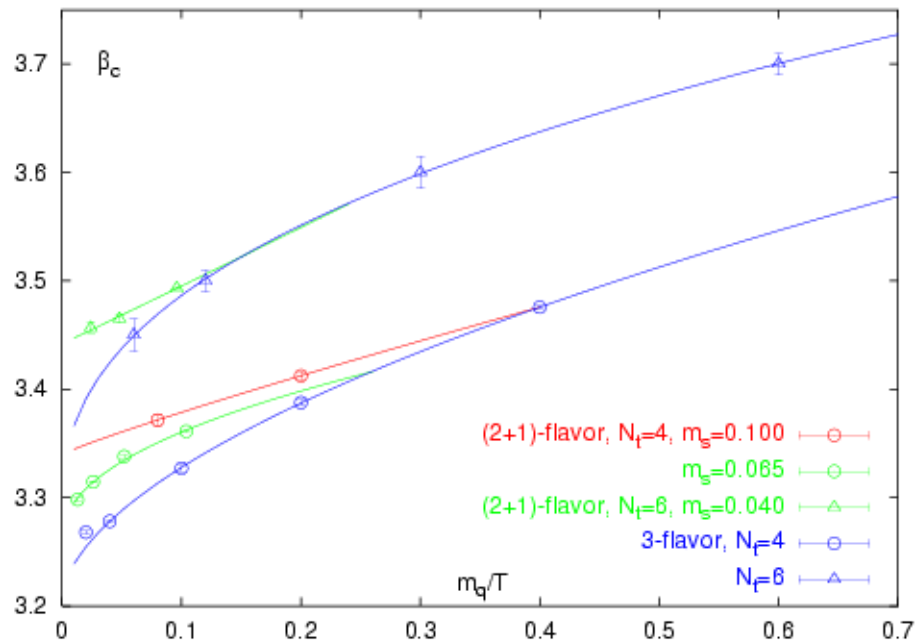
- Fits have been performed with power laws
- We find mass a moderate mass dependence of the beta-separation:

$$\Delta\beta \approx 0.14 - 0.18$$

3-favor and (2+1)-flavor results

$N_\tau = 4\,;\, N_\tau = 6$

Critical couplings:



The critical temperature of QCD --- the static quark potential

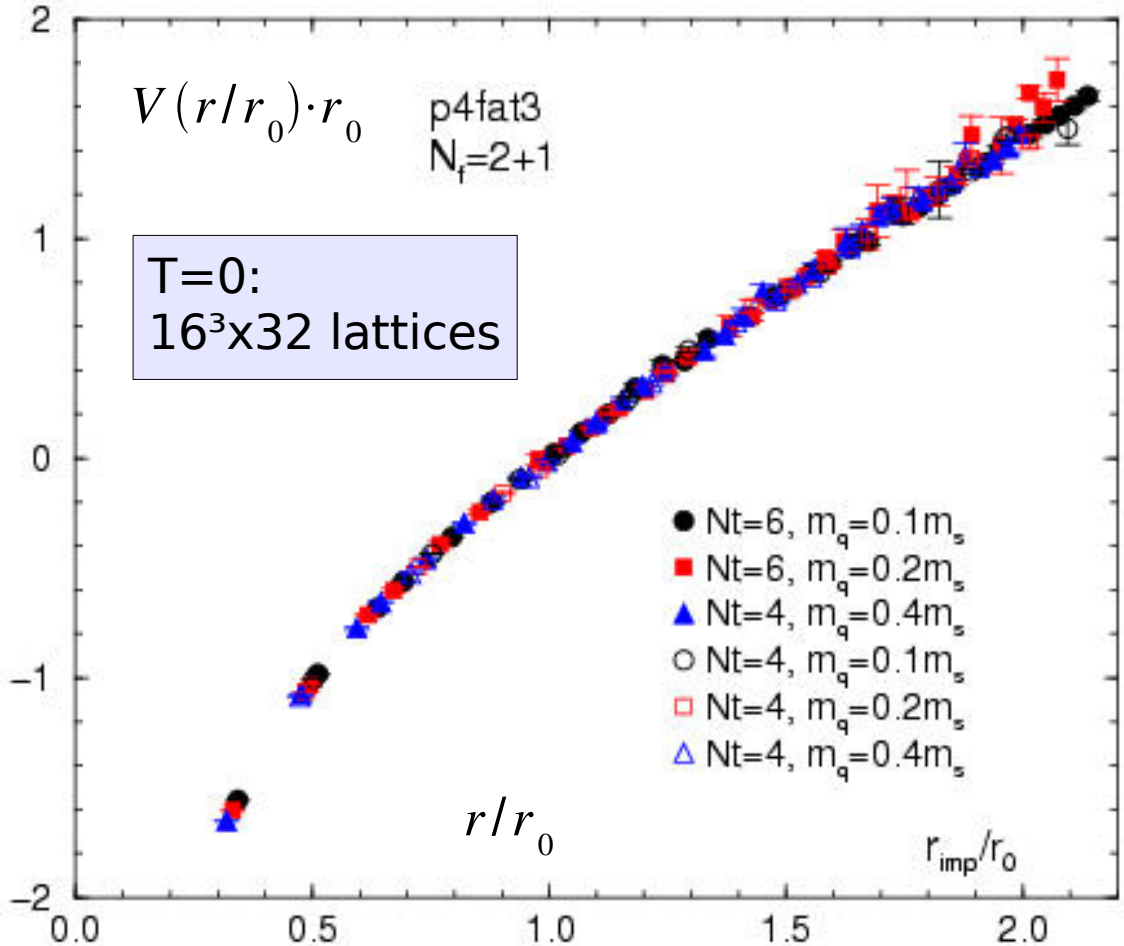
Static Quark Potential: (2+1)-flavor results

Second Step:

- Scale determination at each critical coupling
- We use the Sommer scale: r_0 and the string tension: σ

$$r^2 \frac{\partial V}{\partial r}\bigg|_{r=r_0} = 1.65$$

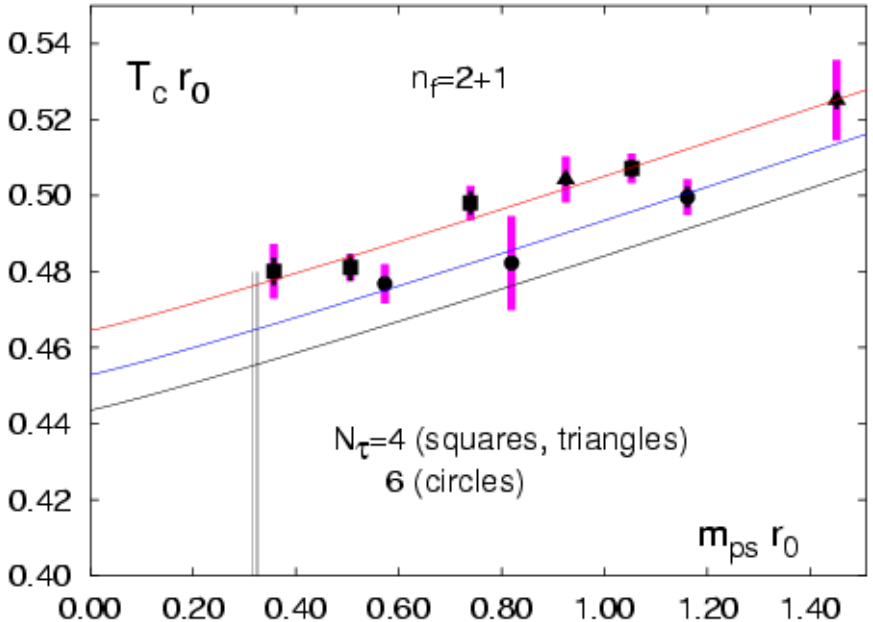
$$r_0 \approx 0.5 \ fm$$
$$\sigma \approx 425 \, MeV$$

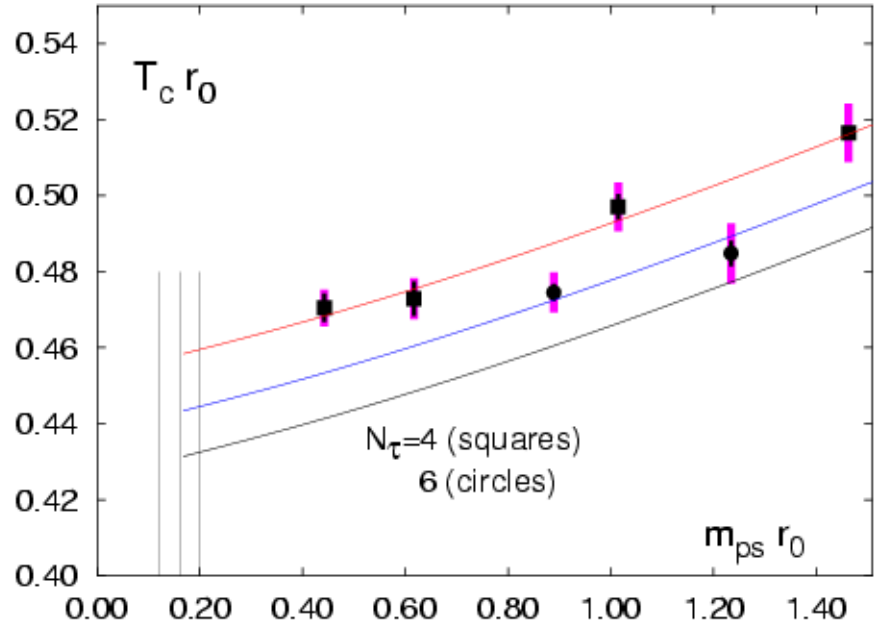


- Almost no mass or cut-off dependence in the scaled static quark potential

The critical temperature of QCD --- combined chiral and continuum extrapolation

The critical Temperature: (2+1)-flavor versus 3-flavor results





$$T_C r_0(chiral) = 0.444(6)$$
$$T_C r_0(\ phys.) = 0.457(7)$$

$$T_C r_0(r_0 m_\pi^e) = 0.429(8)$$

extrapolation Ansatz:

$$T_C r_0 = (r_0 T_C(m_\pi^e))_{cont} + a((r_0 m_\pi)^2 - (r_0 m_\pi^e)^2)^d + c/N_\tau^2$$

3-flavor: $m_\pi^e = 0.00,\ d = 0.64\ \ (Z(2))$

(2+1)-flavor: $m_\pi^e = 0.16,\ d = 0.54\ \ (O(4))$

- Similar cut-off effects for (2+1)-flavor and 3-flavor
- Flavor dependence is about 5% or less.

The equation of state for (2+1)-flavor QCD

the line of constant physics

Goal: calculate the pressure and interaction measure on a line of constant physics (LCP)

LCP determined by:

$$m_\pi / m_K = const. \qquad r_0 m_\pi = const.$$

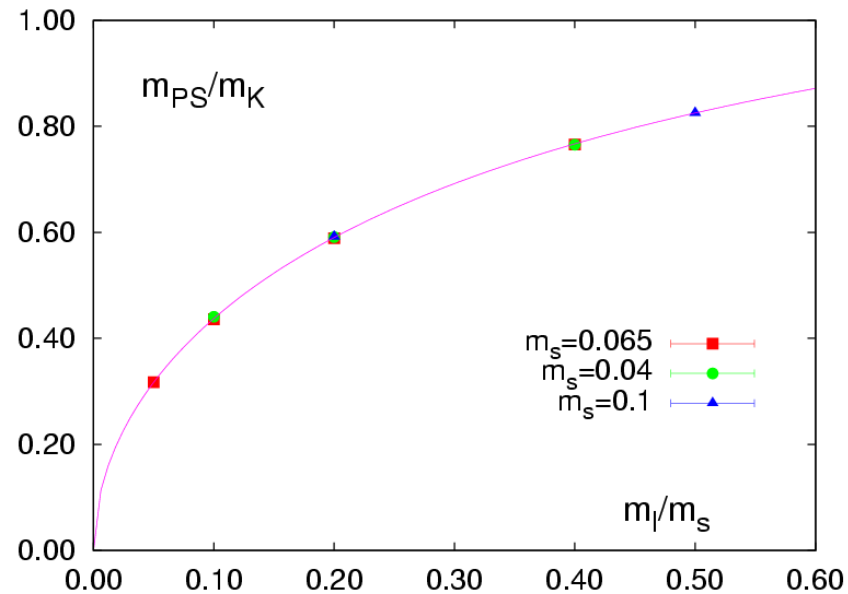
In good approximation we find:

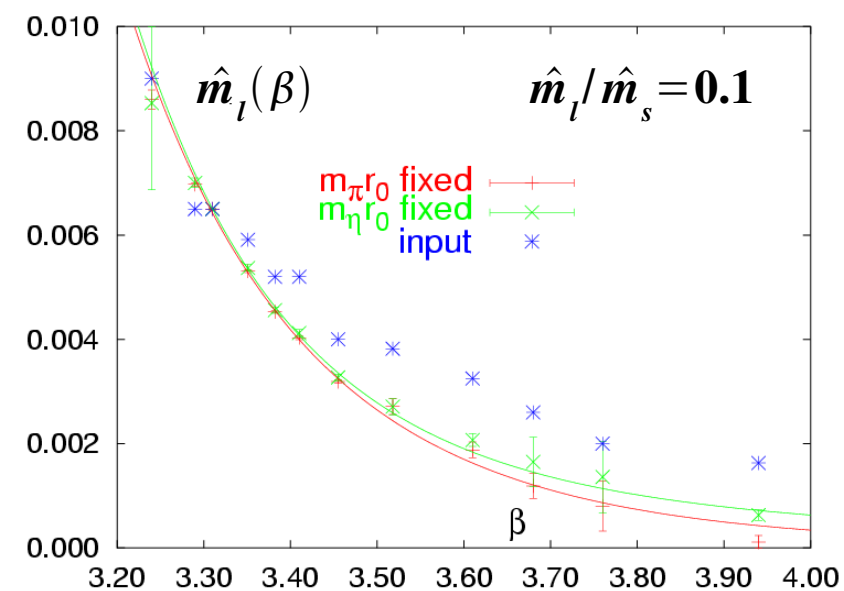
$$m_\pi / m_K = const. \;\Leftrightarrow\; \hat{m}_l / \hat{m}_s = const.$$

Remains to be determined:

$$\hat{m}_l(\beta) = d_0 \left(\frac{6b_0}{\beta} \right)^{-4/9} \times R(\beta) \left(1 + d_2 \hat{a}^2(\beta) + d_4 \hat{a}^4(\beta) \right)$$

(RG inspired Ansatz)





- Integrate the derivative $\frac{\partial}{\partial \beta} \ln Z$ over β along the line of physics to get the pressure

$$\frac{p}{T^4}\Big|_{\beta_0}^{\beta} = N_\tau^4 \int_{\beta_0}^{\beta} d\beta' \Bigg[\frac{1}{N_\sigma^3 N_t} (\langle S_g \rangle_0 - \langle S_g \rangle_T) - \left(2(\langle \bar\psi\psi \rangle_{l0} - \langle \bar\psi\psi \rangle_{lT}) + \frac{\hat m_s}{\hat m_l} (\langle \bar\psi\psi \rangle_{s0} - \langle \bar\psi\psi \rangle_{sT}) \right) \left(\frac{\partial \hat m_l}{\partial \beta'} \right)_{\hat m_s/\hat m_l} - \hat m_l \left(\langle \bar\psi\psi \rangle_{s0} - \langle \bar\psi\psi \rangle_{sT} \right) \left(\frac{\partial \hat m_s/\hat m_l}{\partial \beta'} \right)_{\hat m_l} \Bigg]$$

- The β-function is required to calculate the interaction measure

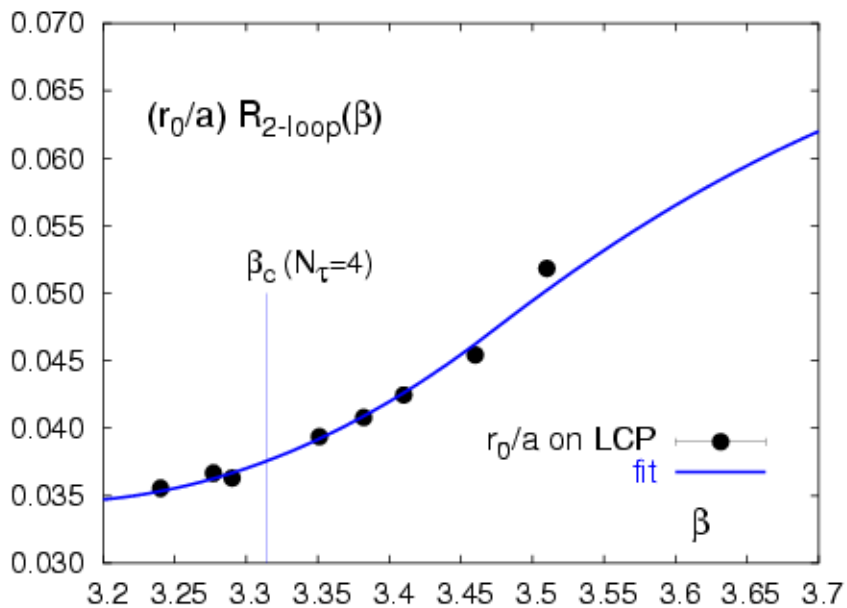
$$\frac{\epsilon - 3p}{T^4} = T \frac{\mathrm{d}}{\mathrm{d}T} \left(\frac{p}{T^4} \right) = \left(a \frac{\mathrm{d}\beta}{\mathrm{d}a} \right)_{LCP} \frac{\partial p/T^4}{\partial \beta}$$

$$= \left(\frac{\epsilon - 3p}{T^4} \right)_{gluon} + \left(\frac{\epsilon - 3p}{T^4} \right)_{fermion} + \left(\frac{\epsilon - 3p}{T^4} \right)_{\hat m_s/\hat m_l}$$

The equation of state for (2+1)-flavor QCD

the β-function

- a precise knowledge of $r0/a$ is mandatory to fix the temperature scale and the β-function, $(a\mathrm{d}\beta/\mathrm{d}a)_{LCP}$
- RG-inspired ansatz for $\beta > 3.47$; rational polynomial ansatz for $\beta < 3.47$
- particularly important in the transition region on coarse lattices such as $N_\tau = 4,\ 6$



RBC-Bielefeld, preliminary

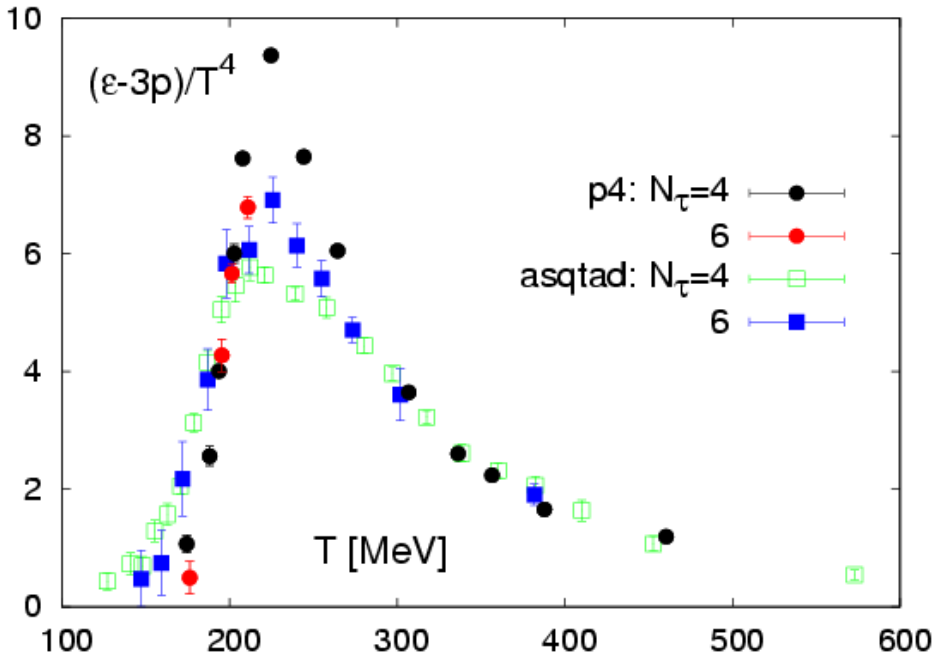
NOTE:

$r_0 T = (r_0/a)/N_\tau$ defines the temperature scale irrespective of any ambiguities in locating T_c.

The non-perturbative β-function:

$$R_\beta = a\frac{\mathrm{d}\beta}{\mathrm{d}a}\bigg|_{\hat{m}_l,\hat{m}_s} = \frac{a}{r_0}\left(\frac{\partial(a/r_0)}{\partial\beta}\bigg|_{LoP}\right)^{-1}$$

- Using an RG-inspired 2-loop β-function underestimates $(\epsilon - 3p)/T^4$ in the transition region and stretches the temperature interval in the low temperature regime artifically, i.e. makes the transition region look broader than it is.



RBC-Bielefeld, preliminary

differences in the transition region partly arise from differences in the β-functions used in the crossover region

- overall good agreement

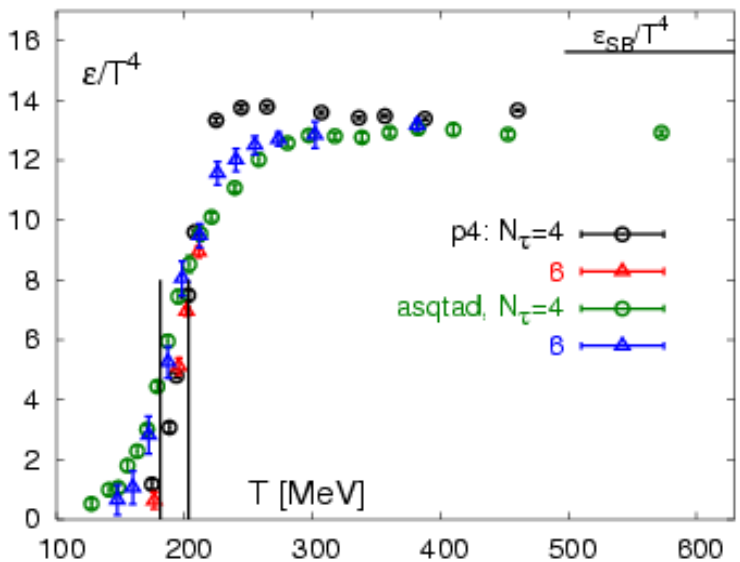
Note:
T-scale is not dependent on T_c determination

asqtad data:
C. Bernard et al., hep-lat/0610017 (Lattice'06)

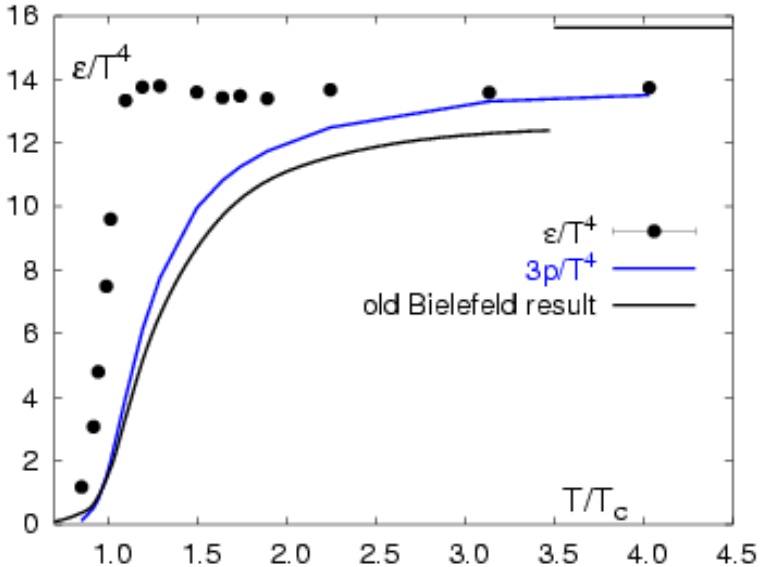
The equation of state for (2+1)-flavor QCD

the interaction measure

- RBC-Bielefeld vs. MILC: the RBC-Bi energy density on $N_\tau = 4$ lattices rises more steeply;
  direct consequence of the use of a non-perturbative β-function directly deduced from calculated r_0/a values
- overall good agreement for $N_\tau = 4,\ 6$,
  Note: T-scale does not depend on T_c determination!!



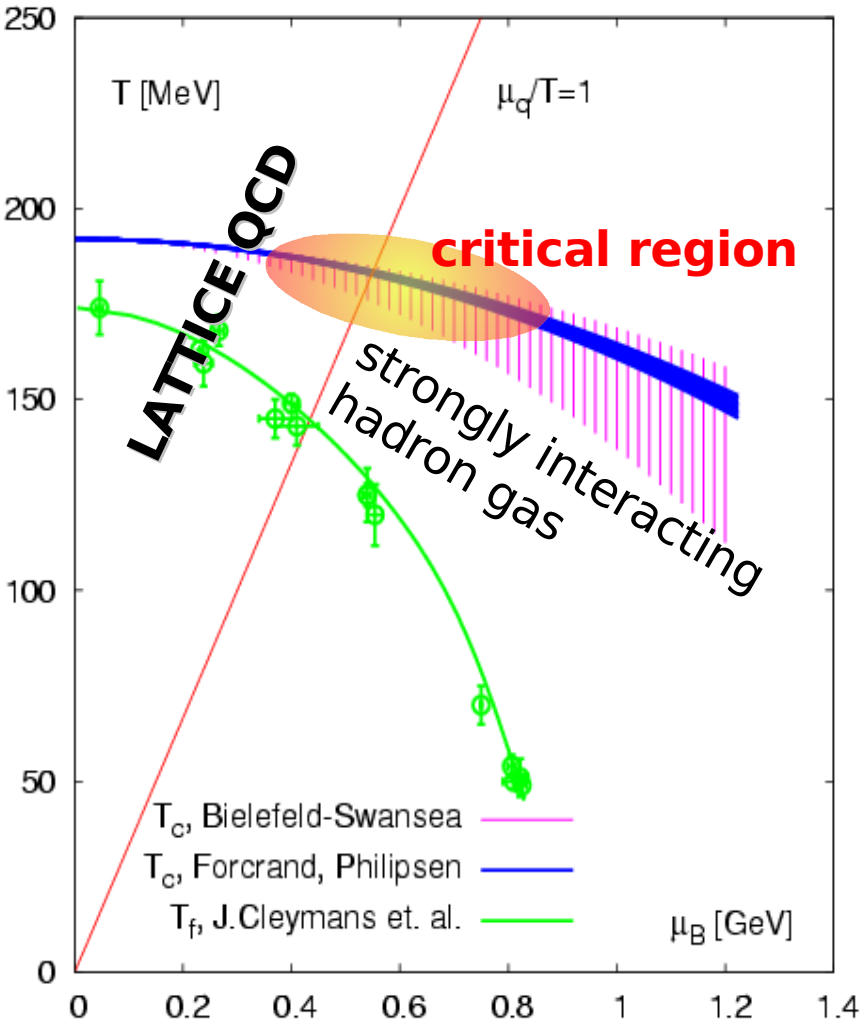
band marks $T = (192 \pm 11)$ MeV
RBC-Bielefeld, preliminary



pressure increased slightly with smaller quark mass

Taylor coefficients with respect to the chemical potential

Why Taylor coefficients ?



QCD partition function on the lattice:

$$Z(N_t, N_s, \beta, am_q, a\mu) = \int DU \, det[U](am_q, a\mu) \exp\{-S_G[U]\}$$

complex for $\mu > 0$

Sign Problem!

known tricks:

- re-weighting the $\mu = 0$ ensemble
  Fodor, Katz (2002)
- Taylor expansion around $\mu = 0$
  Bielefeld-Swansea (2002)
- imaginary chemical potential
  + analytic continuation de Forcrand, Philipsen (2002)
  D'Elia, Lombardo (2002)
- canonical approach
  de Forcrand, Kratochvila (2003)
  Alexandru et al. (2005)

Taylor coefficients with respect to the chemical potential

- expanding the pressure

$$\frac{p}{T^4} \equiv \Omega(T, \mu_q, \mu_q) = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n$$

down quark chemical potential
up quark chemical potential

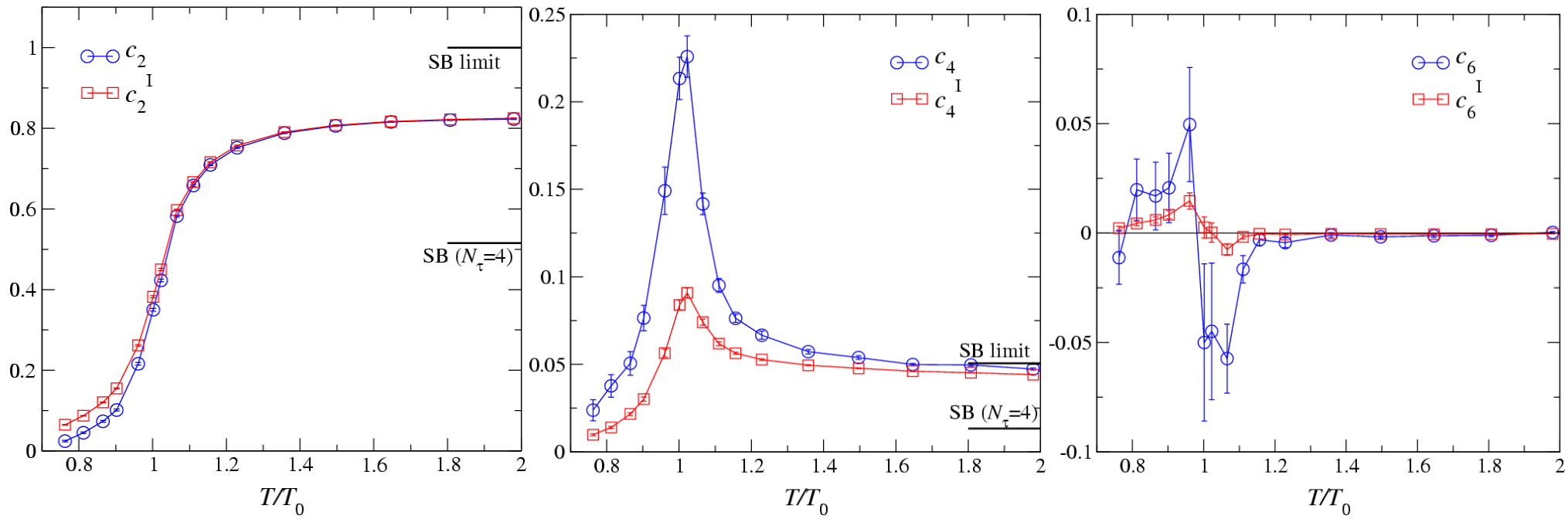
Allton et al. PRD 71, 054508 (2005)

$$c_n(T) = \frac{1}{n!} \frac{\partial^n \Omega}{\partial(\mu_q/T)^n}\bigg|_{\mu_q=0} = \frac{1}{n!} \frac{N_\tau^3}{N_\sigma^3} \frac{\partial^n \ln \mathcal{Z}}{\partial(\mu N_\tau)^n}\bigg|_{\mu=0}$$

$N_f = 2$

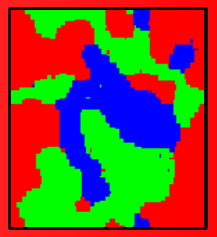
$m/T = 0.4$

$N_t = 4$



Taylor coefficients with respect to the chemical potential

- bulk thermodynamics at small μ_q/T



Pressure, Quark number density,

Quark number susceptibility,...

Allton et al. PRD 71, 054508 (2005)

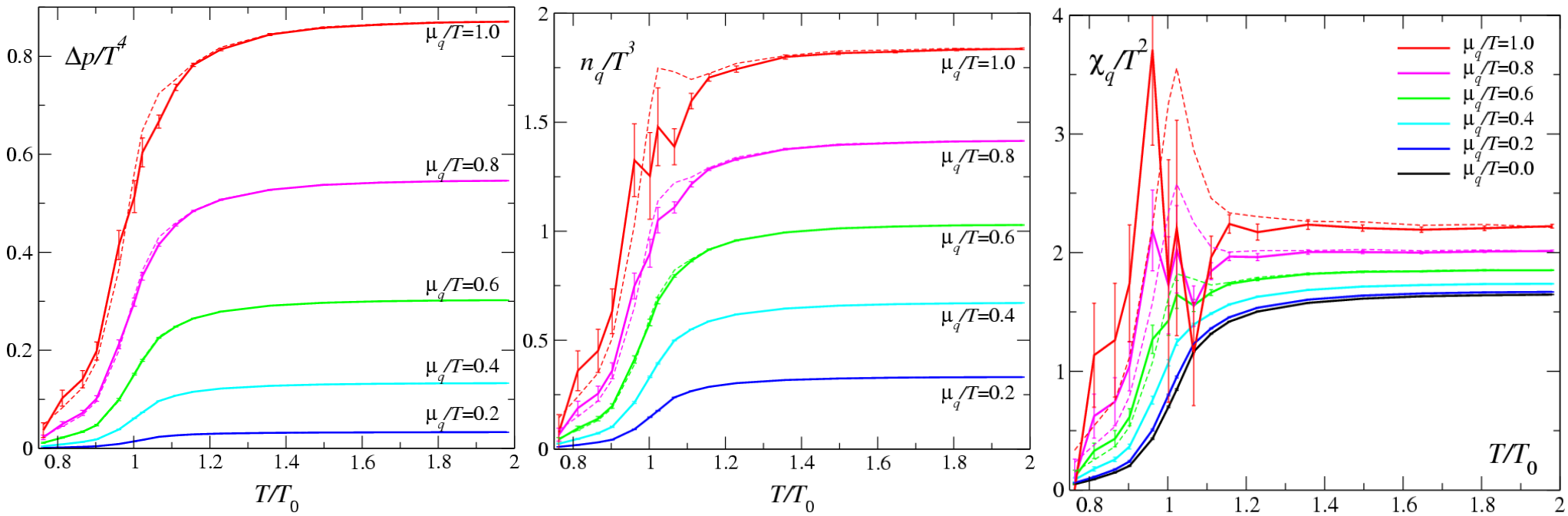
$N_f = 2$

$m/T = 0.4$

$N_t = 4$

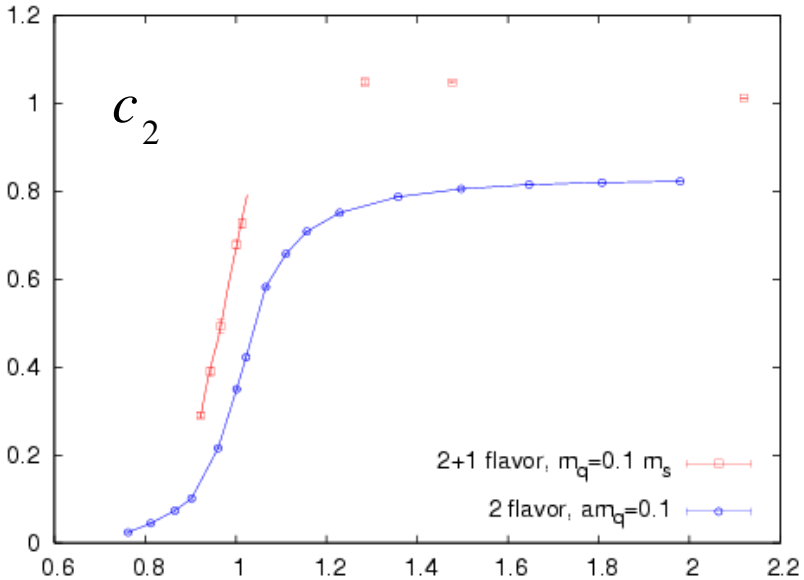
$$\frac{n_q(T,\mu_q)}{T^3} = \frac{\partial\Omega(T,\mu_q,\mu_q)}{\partial\mu_q/T} = 2c_2\left(\frac{\mu_q}{T}\right) + 4c_4\left(\frac{\mu_q}{T}\right)^3 + 6c_6\left(\frac{\mu_q}{T}\right)^5 + \cdots$$

$$\frac{\chi_q(T,\mu_q)}{T^2} = 2c_2 + 12c_4\left(\frac{\mu_q}{T}\right)^2 + 30c_6\left(\frac{\mu_q}{T}\right)^4 + \cdots$$



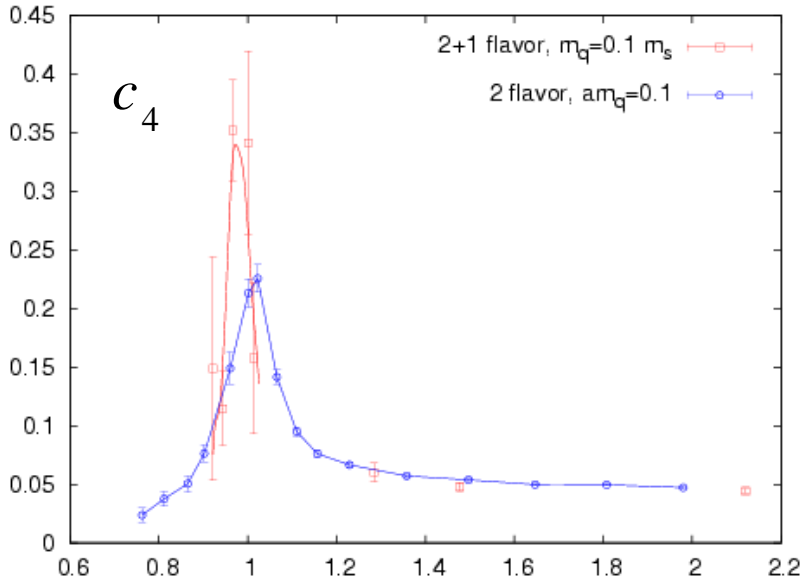
Taylor coefficients with respect to the chemical potential

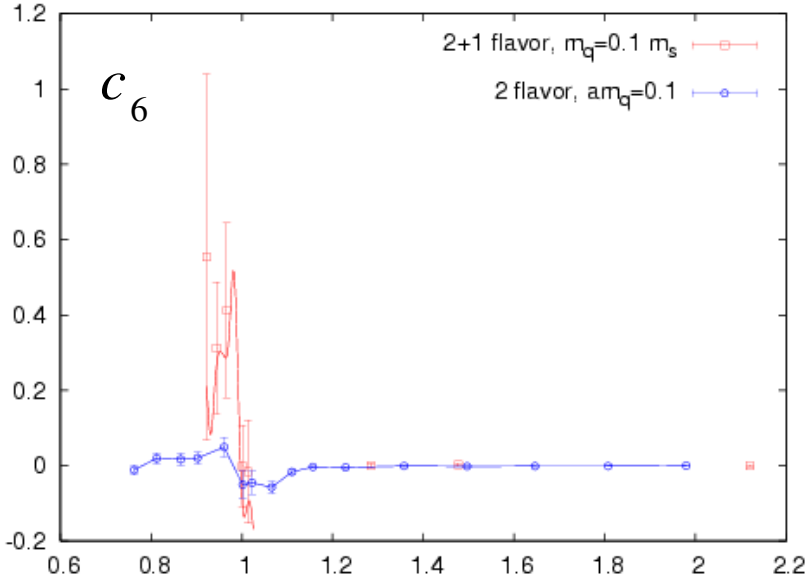
- the mass dependence



- Fluctuations increase over the resonance gas level for smaller quark masses

RBC-Bielefeld preliminary

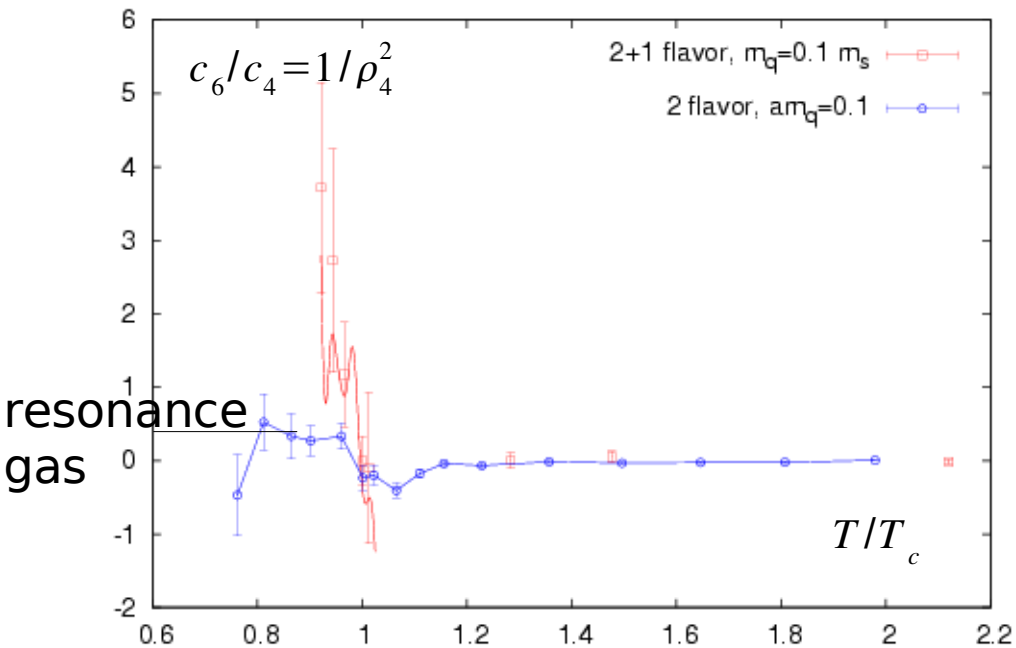




Taylor coefficients with respect to the chemical potential

- the radius of convergence

- The radius of convergence can be estimated by ratios of expansion coefficients



RBC-Bielefeld preliminary

$$\rho_n = \sqrt{\frac{c_n}{c_{n+2}}}$$

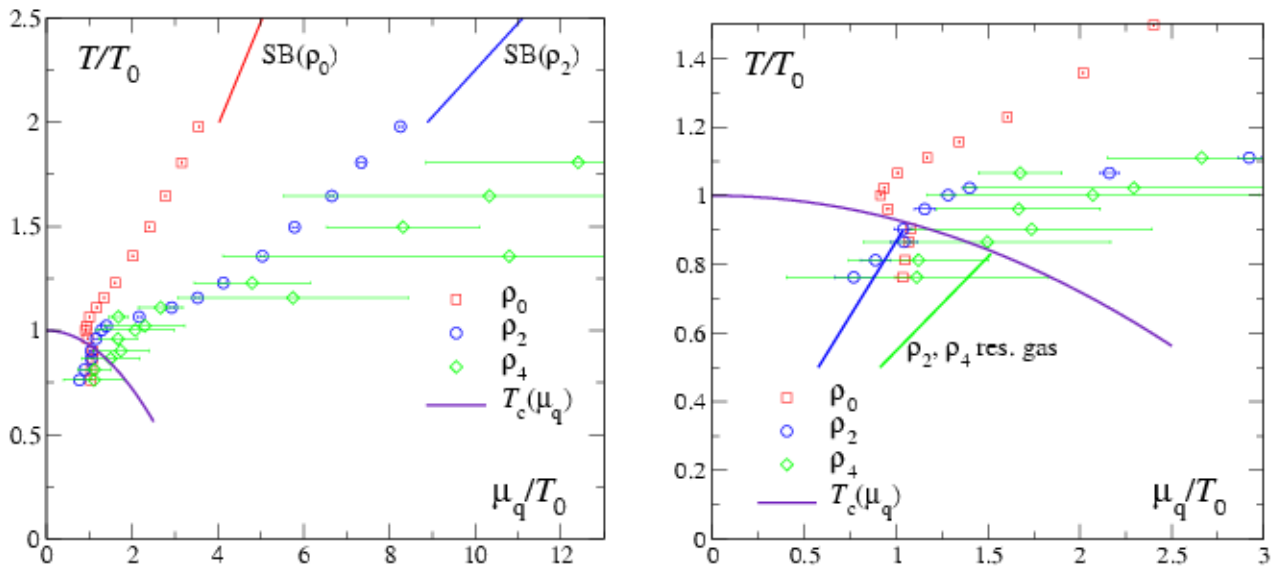
- The radius of convergence gives the Distance to the closest singularity in the complex chemical potential plane
- If all coefficients are positive the singularity lies on the real axis
- The critical point will show up as dip in ρ (as peak in $1/\rho$)

- $c_6/c_4 \approx 3$ corresponds to $\mu_B^{crit} \approx 330\,MeV$

Taylor coefficients with respect to the chemical potential

- the radius of convergence

- Taylor expansion $\Rightarrow$ estimates for radius of convergence $\rho_{2n} = \sqrt{\left|\frac{c_{2n}}{c_{2n+2}}\right|}$



$T < T_0$: $\rho_n \simeq 1.0$ for all n $\Rightarrow$ $\mu_B^{crit} \simeq 500$ MeV

HOWEVER still consistent with resonance gas!!!

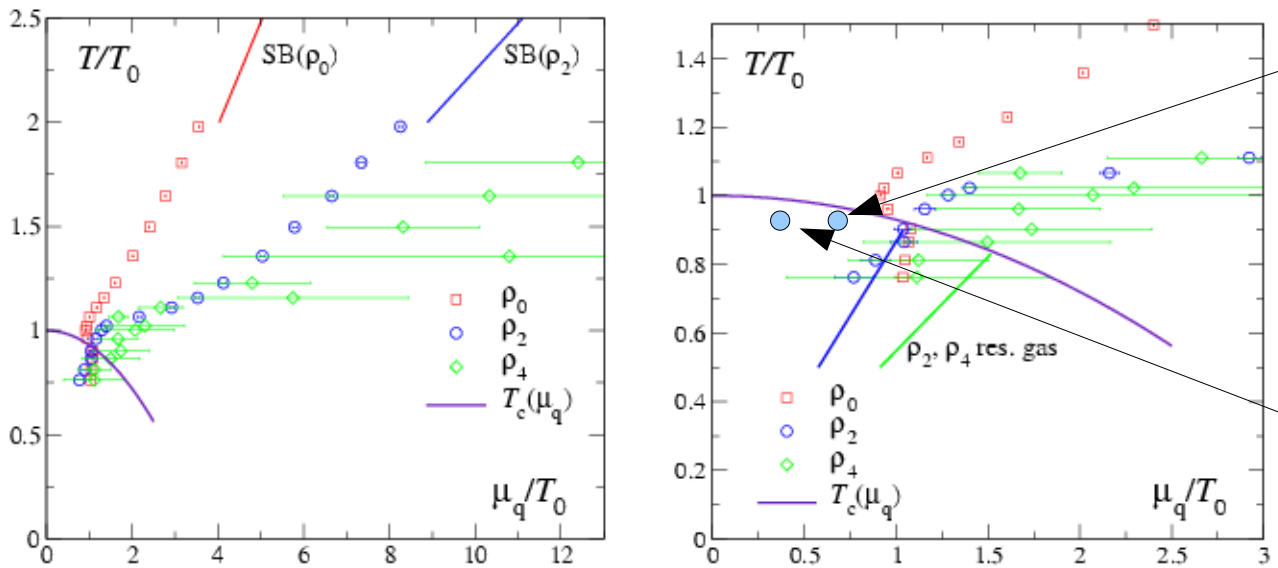
HRG analytic, LGT consistent with HRG $\Rightarrow$ infinite radius of convergence not yet ruled out

Allton et al. PRD 71 (2005) 054508.

Taylor coefficients with respect to the chemical potential

- the radius of convergence

- Taylor expansion $\Rightarrow$ estimates for radius of convergence $\rho_{2n} = \sqrt{\left|\frac{c_{2n}}{c_{2n+2}}\right|}$



Fodor, Katz, JHEP 0404 (2004) 050.
(re-weighting)

Gavai, Gupta, PRD 71 (2005) 114014.
(Taylor-expansion)

$T < T_0$: $\rho_n \simeq 1.0$ for all n $\Rightarrow$ $\mu_B^{crit} \simeq 500$ MeV

HOWEVER still consistent with resonance gas!!!

HRG analytic, LGT consistent with HRG $\Rightarrow$ infinite radius of convergence not yet ruled out

Allton et al. PRD 71 (2005) 054508.

Taylor coefficients with respect to the chemical potential

- bulk thermodynamics at small μ_q/T

Pressure → Energy and Entropy density

- pressure $$\frac{p}{T^4} \equiv \frac{1}{VT^3} \ln Z(T,\mu_q) = \sum_{n=0}^{\infty} c_n(T,\hat{m}) \left(\frac{\mu_q}{T}\right)^n$$

- energy density from "interaction measure"

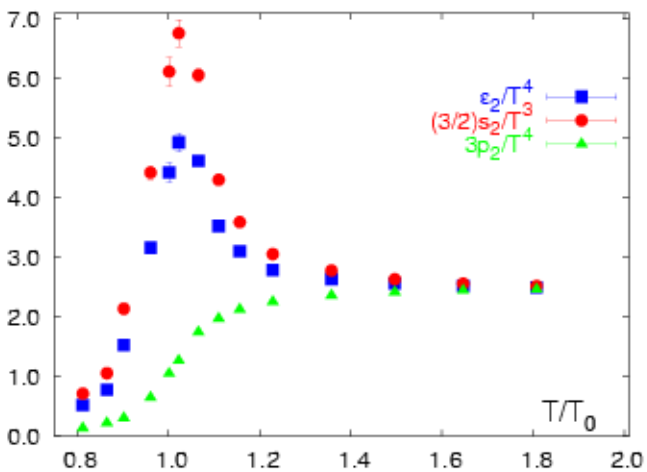
$$\frac{\epsilon - 3p}{T^4} = \sum_{n=0}^{\infty} c'_n(T,\hat{m}) \left(\frac{\mu_q}{T}\right)^n \quad , \quad c'_n(T,\hat{m}) \equiv T\frac{\mathrm{d}c_n(T,\hat{m})}{\mathrm{d}T}$$

- entropy density

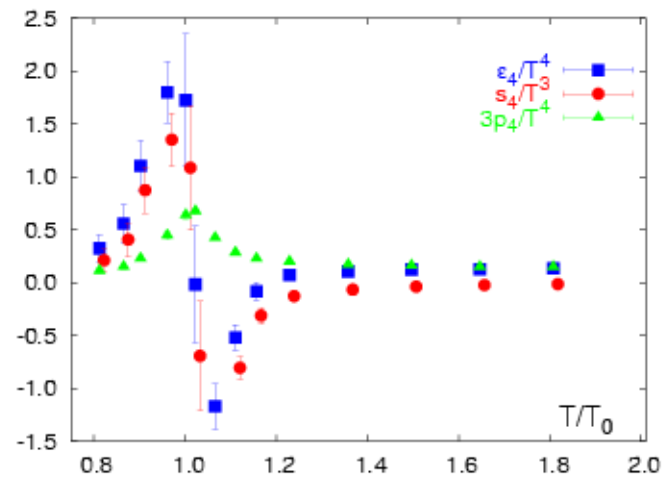
$$\frac{s}{T^3} \equiv \frac{\epsilon + p - \mu_q n_q}{T^4} = \sum_{n=0}^{\infty} \left((4-n)c_n(T,\hat{m}) + c'_n(T,\hat{m})\right) \left(\frac{\mu_q}{T}\right)^n$$

Taylor coefficients with respect to the chemical potential

- bulk thermodynamics at small μ_q/T



$\mathcal{O}((\mu_q/T)^2)$



$\mathcal{O}((\mu_q/T)^4)$

$N_f = 2$

$m/T = 0.4$

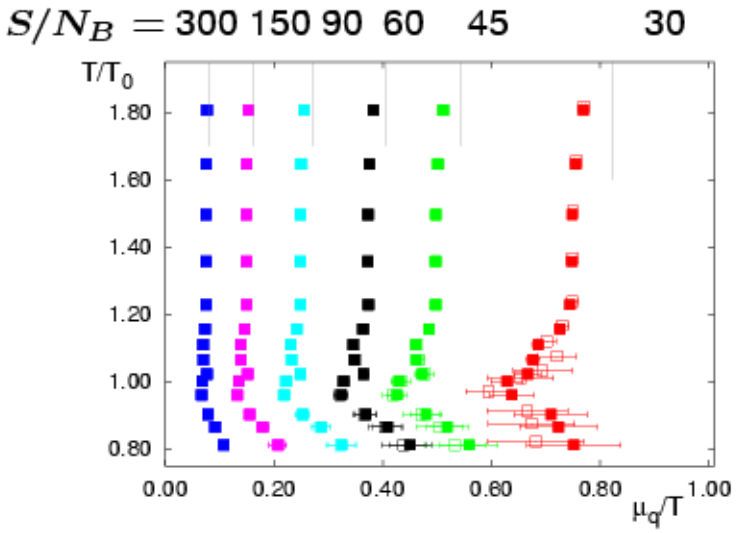
$N_t = 4$

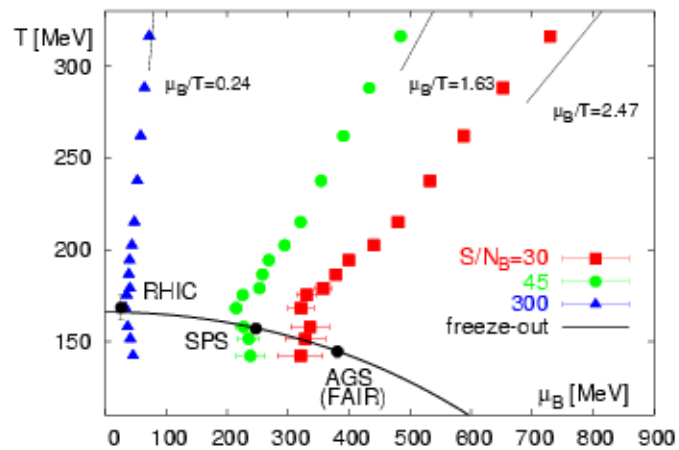
- magnitudes similar for all three quantities;
- expansion of ϵ/T^4 and s/T^3 also dominated by $\mathcal{O}((\mu_q/T)^2)$ for $\mu_q/T \lesssim 1$ (except close to T_c)

S. Ejiri, F. Karsch, E. Laermann, CS, PRD 73 (2006) 054506.

Taylor coefficients with respect to the chemical potential

- lines of constant S/N_B





$N_f = 2$

$m/T = 0.4$

$N_t = 4$

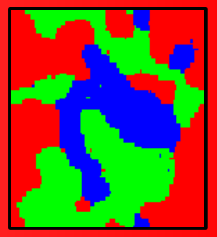
- isentropic trajectories close to ideal gas behavior for $T > T_c$
- trajectories bend towards larger μ_q for $T < T_c$
- $\mathcal{O}(\mu_q^6)$ correction (open sym.) is small for $\mu_q/T \lesssim 0.8$ (despite large errors)

- RHIC corresponds to $S/N_B \simeq 300 \simeq \infty$
- SPS corresponds to $S/N_B \simeq 45$
- FAIR will operate at $S/N_B \simeq 30$ or $\mu_q/T \lesssim 0.9$

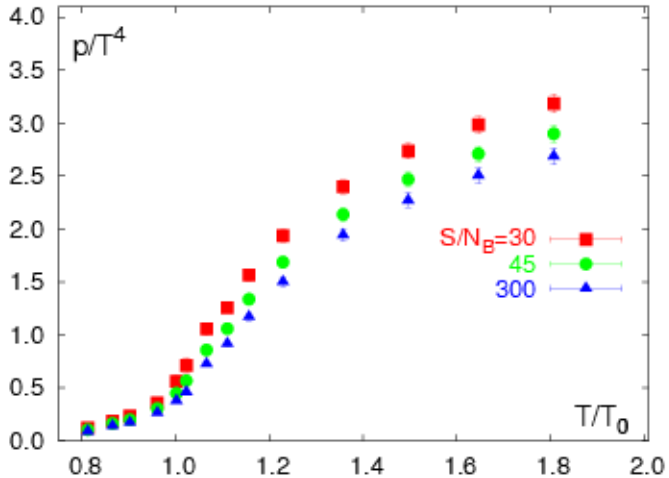
S. Ejiri, F. Karsch, E. Laermann, CS, PRD 73 (2006) 054506.

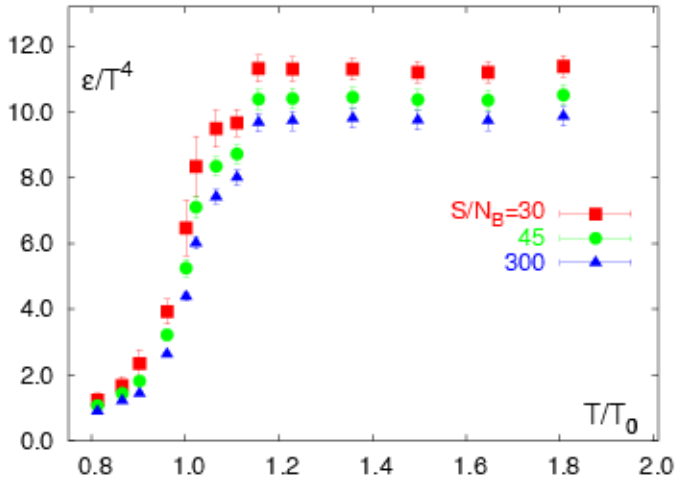
Taylor coefficients with respect to the chemical potential

- the isentropic equation of state



2-flavor QCD





$N_f = 2$

$m/T = 0.4$

$N_t = 4$

pressure
on lines of constant S/N_B

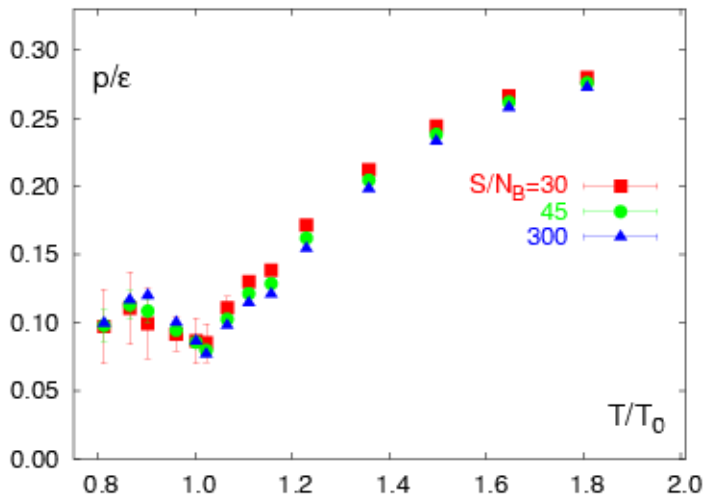
energy density
on lines of constant S/N_B

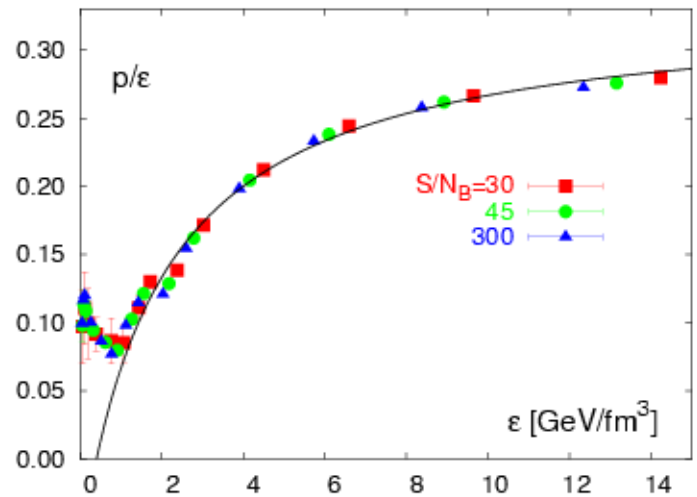
- μ_q-dependent contribution added on top of the $\mu_q = 0$ result
- RHIC EoS essentially coincides with the $\mu_q = 0$ EoS
- EoS at SPS and RHIC differ at high T by less than 10%
- FAIR: changes at high $T \sim 30\%$; but need better resolution at low T

S. Ejiri, F. Karsch, E. Laermann, CS, PRD 73 (2006) 054506.

Taylor coefficients with respect to the chemical potential

- the isentropic equation of state





$N_f = 2$

$m/T = 0.4$

$N_t = 4$

- p/ϵ vs. ϵ shows almost no dependence on S/N_B
- softest point: $p/\epsilon \simeq 0.075$
- phenomenological EoS for $T_0 \lesssim T \lesssim 2T_0$

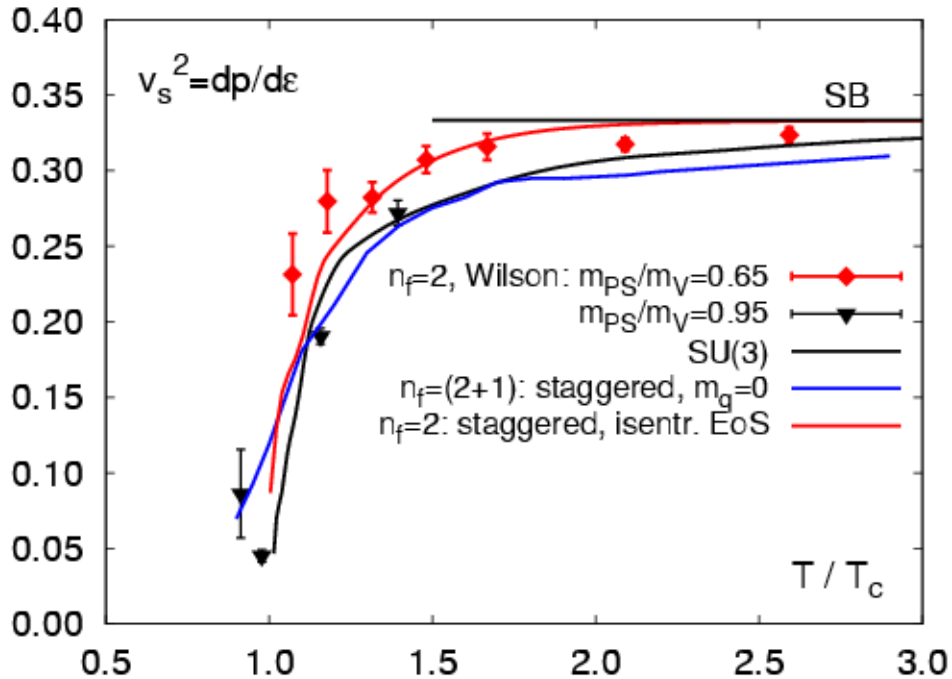
$$\frac{p}{\epsilon} = \frac{1}{3}\left(1 - \frac{1.2}{1 + 0.5\epsilon}\right)$$

S. Ejiri, F. Karsch, E. Laermann, CS, PRD 73 (2006) 054506.

Taylor coefficients with respect to the chemical potential

- the velocity of sound

- steep EoS:
  rapid change of energy density; slow change of pressure
  $\Rightarrow$ reduced velocity of sound $\Rightarrow$ more time for equilibration



pure gauge theory:
G. Boyd et al.,
NP B469 1996

$n_f = 2$:
A. Ali Khan et al.,
PR D64 2001

$n_f = (2+1)$:
Y. Aoki et al.,
hep-lat/0510084
$(TV^{1/3} = 2)$

F. Karsch, hep-lat/0601013 (PANIC 05)

Taylor coefficients with respect to the chemical potential

- the lattice EoS and hydro-expansion

- simple 1-d hydro: $\partial_\mu T^{\mu\nu} = 0 \;\Rightarrow\; \frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = -\frac{\epsilon + p}{\tau}$
  - ideal gas EoS: $p/\epsilon = 1/3$

$$\frac{\epsilon(\tau)}{\epsilon(\tau_0)} = \left(\frac{\tau_0}{\tau}\right)^{4/3} \quad \Rightarrow \quad \tau_f = \tau_0 \left(\frac{\epsilon(\tau_0)}{\epsilon(\tau_c)}\right)^{3/4}$$

  - lattice EoS: $p/\epsilon < 1/3$ $\Rightarrow$ slows down expansion;
    $\Rightarrow$ increases plasma lifetime

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = -\frac{4}{3}\frac{\epsilon}{\tau}\left(1 - \frac{0.3}{1 + 0.2\,\epsilon\ \mathrm{fm}^3/\mathrm{GeV}}\right)$$

$\epsilon(\tau_0 = 1\ \mathrm{fm}) \simeq 10\mathrm{GeV/fm}^3 \;\Rightarrow\; \tau_f \simeq 5.5\ \mathrm{fm}$ $(\epsilon = 3p)$
$\simeq 7\ \mathrm{fm}$ (LGT EoS)

F. Karsch, hep-lat/0601013 (PANIC 05)

Summary

- **A large scale calculation of the critical temperature and EoS is on-going (RBC-Bielefeld on QCDOC+APEnext):**

  -- we find the critical temperature of QCD, continuum extrapolated from Nt=4 and Nt=6 lattices to be

  $$T_c \simeq 192(7)(4)\ MeV$$

  -- the critical energy density shows a rapid crossover at this temperature
  -- the Nt=6 Eos is work in progress
  -- the calculation of Taylor coefficients to the 8th order has been started

- **The transition is most likely crossover**

- **Taylor coefficients are a powerful tool to analyze bulk thermodynamic quantities at non-zero density:**

  -- pressure, quark number density, quark number susceptibility energy and entropy density can be studied at non-zero chemical potential
  -- we find increasing hadronic fluctuations with decreasing quark mass
  -- the isentropic EoS, speed of sound has been calculated
  -- the critical point ?
  -- d.o.f. in the QGP can be analyzed