TASI Lectures: Cosmology II

Numbers and Definitions

1. Redshift z is defined as $1+z \equiv 1/a$.
2. Baryon/photon number density ratio:

$$\frac{n_b}{n_\gamma} = 5.5 \times 10^{-10} \left(\frac{\Omega_b h^2}{0.02} \right).$$

3. Equation of state for species i is $w_i \equiv P_i/\rho_i$.
4. Epoch of equality: $a_{\rm eq} = 4.15 \times 10^{-5}/(\Omega_m h^2) \simeq 3.27 \times 10^{-4}$.
5. Epoch of recombination: $a_* = 1088$.

Equations

1. Acceleration equation

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3} [\rho + 3P]$$

2. Evolution of the energy density of species i with w_i:

$$\rho_i(a) = \rho_0 \exp \left\{ 3 \int_a^1 \frac{da'}{a'} \left[1 + w(a') \right] \right\}.$$

3. Equilibrium condition for particles participating in a reaction $1 + 2 \leftrightarrow 3 + 4$ which proceeds more rapidly than the expansion rate:

$$\frac{n_3 n_4}{n_3^{\rm eq} n_4^{\rm eq}} = \frac{n_1 n_2}{n_1^{\rm eq} n_2^{\rm eq}}$$

The zero chemical potential number density $n^{\rm eq}$ can often be approximated using Boltzmann statsitics

$$n_i^{\rm eq} = g_i \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T} = \begin{cases} g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T} & m_i \gg T \\ g_i \frac{T^3}{\pi^2} & m_i \ll T \end{cases}.$$

4. Energy density of a thermally produced WIMP

$$\Omega_X h^2 \simeq 0.1 \left(\frac{m}{30 T_{\rm freeze-out}} \right) \frac{10^{-37}\,{\rm cm}^2}{\langle \sigma_{\rm annihilation} v \rangle}.$$

Exercises

1. Suppose that the dark energy is in the form of a cosmological constant. Determine the ratio of the energy density of the cosmological constant to the total energy density when the temperature of the cosmic plasma was equal to the Planck mass. This very small number quantifies the fine-tuning problem: the cosmological constant must be fine-tuned to a very small fraction of the total density at early times in order for it to just start dominating the energy budget today.
2. Determine the age of the universe at recombination ($a_* = 9.1 \times 10^{-4}$). To do this, rewrite the age as
$$t = \int_0^t dt' = \int_0^{a_*} \frac{da}{aH(a)}$$
and use the Friedmann equation to express $H(a)$ in terms of Ω_m (take it to be 0.26), h (set to 0.73) and Ω_R. The integral can be done analytically as long as you neglect the dark energy, which should be a good approximation at early times.
3. Apply the equilibrium condition above to the recombination process: $e^- + p \leftrightarrow H + \gamma$. Take the photons to be in equilibrium so $n_\gamma = n_\gamma^{\rm eq}$. Use the ensuing condition, which is called the Saha equation, to determine the epoch of recombination. In particular, find the temperature at which the free electron density drops to a tenth of the total electron density. Of course, use the fact that the universe is neutral so $n_e = n_p$.
4. What is the value of $a_{\rm eq}$ if there are 4 thermalized neutrinos in the universe (one sterile species in addition to the 3 active species)?
5. There is a fundamental limitation on the annihilation cross section of a particle with mass m. Because of unitarity, $\langle \sigma v \rangle$ must be less than or equal to $1/m^2$, give or take a factor of order unity. Determine Ω_X for a particle which saturates the bound, i.e. for a particle with $\langle \sigma v \rangle = 1/m^2$. For what value of m is Ω_X equal to 0.25? Note that if m is greater than this value, Ω_X is too large, so a stable particle with this large a mass is ruled out. See, e.g., `hep-ph/0206071`, which uses this argument to place an *upper bound* on the lightest Kaluza-Klein particle in models with universal extra dimensions.