Hydro with noise,
Hydrokinetics,
and fractional powers of gradients for a Bjorken expansion

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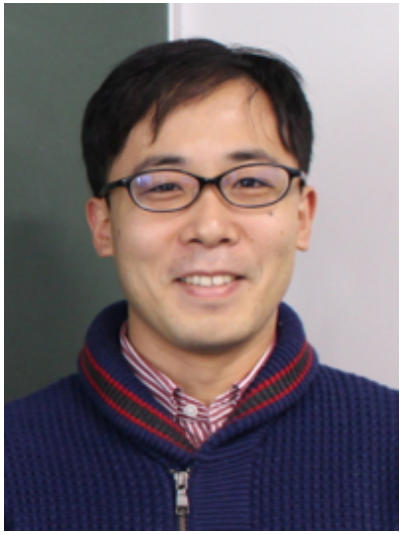
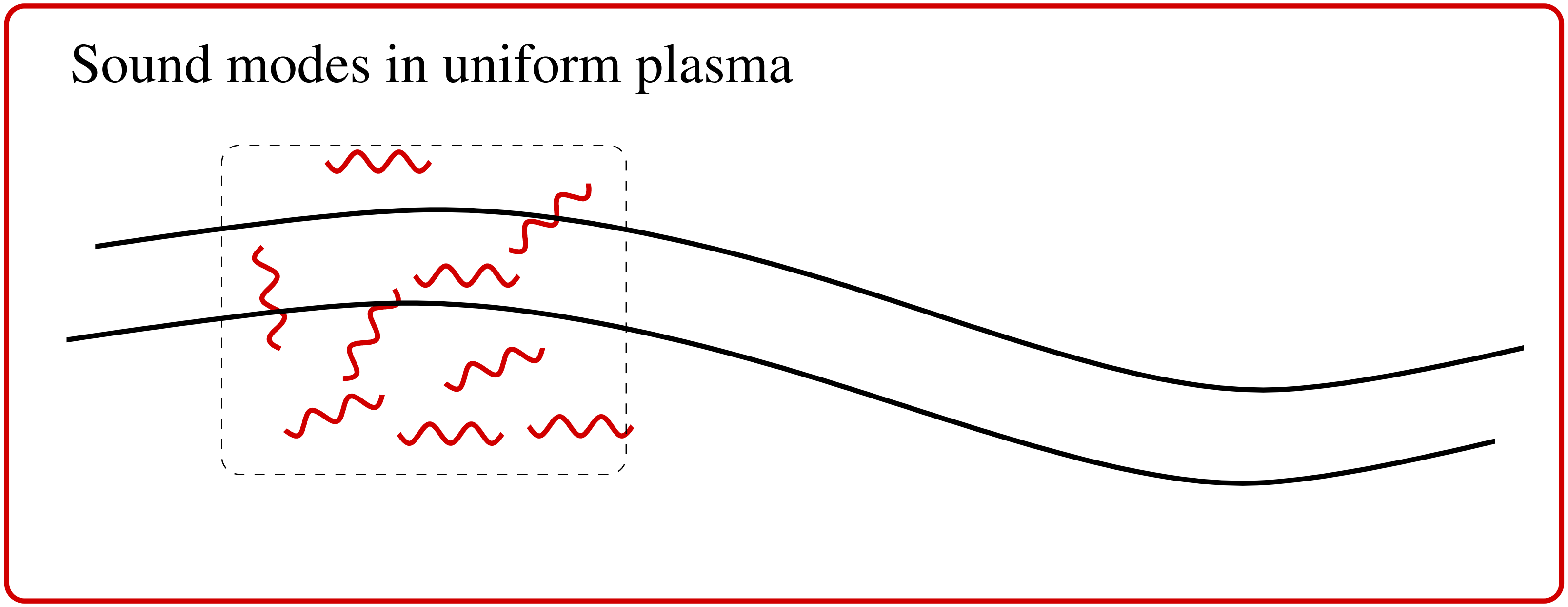
- Yukinao Akamatsu, Aleksas Mazeliauskas, DT, in progress



Thermal fluctuations:

Long time tails; Kovtun, Yaffe; Kovtun, Moore, Romatschke



These hard sound modes are part of the bath, giving to the pressure and shear viscosity

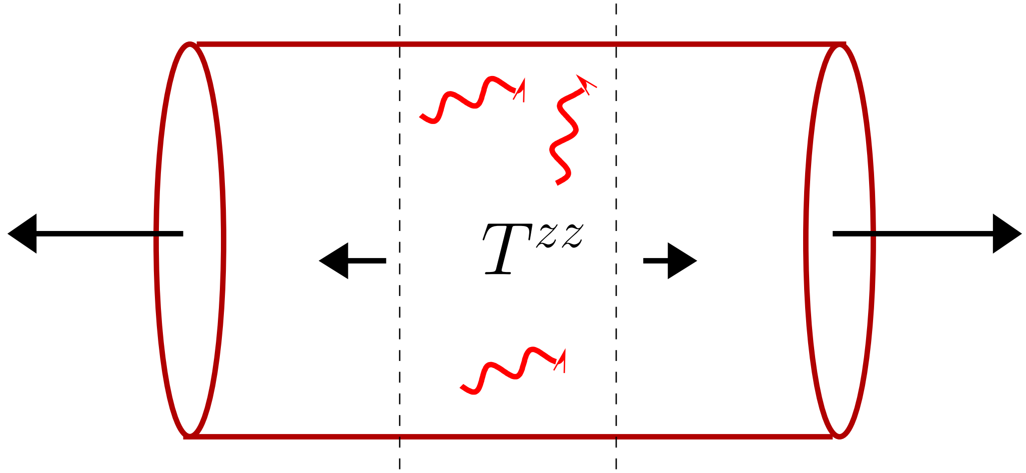
$$N_{ee}(\boldsymbol{k},t) \equiv \underbrace{\langle e^*(\boldsymbol{k},t)e(\boldsymbol{k},t)\rangle}_{\text{energy density flucts}} = (e+p)T/c_s^2$$

$$N_{gg}(\boldsymbol{k},t) \equiv \underbrace{\left\langle g^{*i}(\boldsymbol{k},t)g^j(\boldsymbol{k},t)\right\rangle}_{\text{momentum flucts}} = (e+p)T\delta^{ij}$$

In an expanding system these correlators will be driven out of equilibrium.

This changes the evolution of the slow modes.

A Bjorken expansion



1. The system has an expansion rate of $\partial_\mu u^\mu = 1/\tau$

2. The hydrodynamic expansion parameter is

$$\epsilon \equiv \frac{\eta}{(e+p)\tau} \ll 1$$

   and corrections to hydrodynamics are organized in powers of ϵ

$$T^{zz} = p\Big[1 + \underbrace{\mathcal{O}(\epsilon)}_{\text{1st order}} + \underbrace{\mathcal{O}(\epsilon^2)}_{\text{2nd order}} + \ldots\Big]$$

High k modes are brought to equilibrium by the dissipation and noise

The transition regime:

- There is a wave number where the damping rate competes with the expansion

$$\underbrace{\frac{\eta k^2}{e+p}}_{\text{damping rate}} \sim \underbrace{\frac{1}{\tau}}_{\text{expansion rate}}$$

and thus the transition happens for: $\epsilon \equiv \eta/(e+p)\tau$

$$k \sim \underbrace{\frac{1}{\tau}\frac{1}{\sqrt{\epsilon}}}_{\text{Large!}}$$

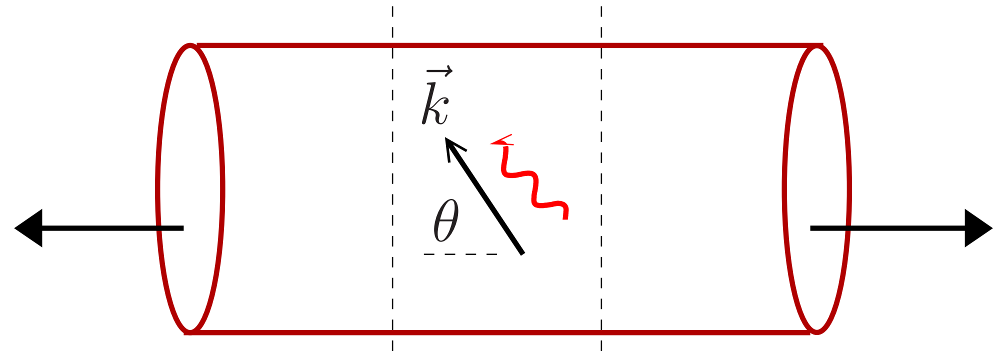
- Thus:

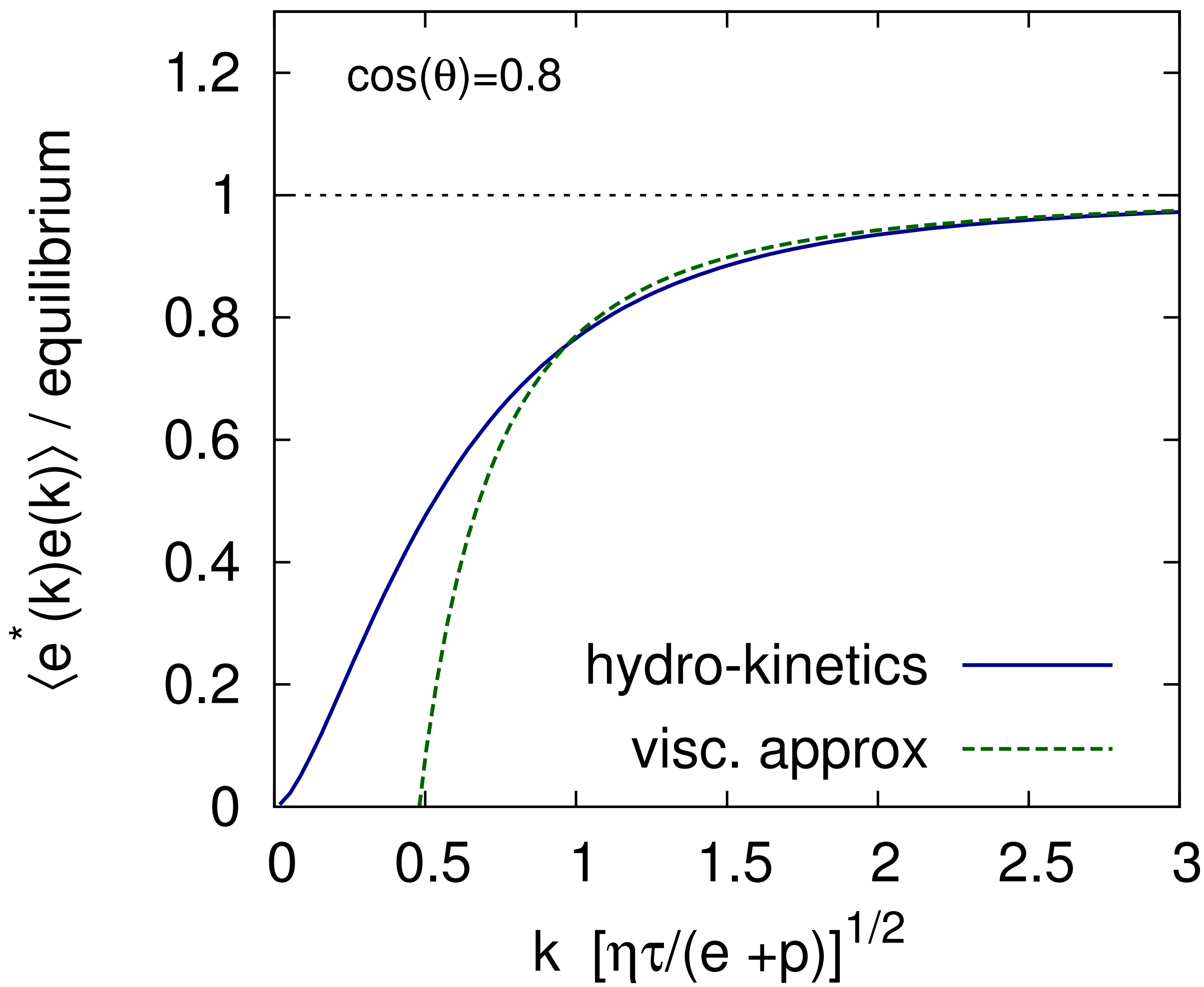
$$k \gg \frac{1}{\tau}\frac{1}{\sqrt{\epsilon}} \qquad \text{equilibrium} \qquad (1)$$

$$k \sim \frac{1}{\tau} \ll \frac{1}{\tau}\frac{1}{\sqrt{\epsilon}} \qquad \text{non-equilibrium, initial conditions} \qquad (2)$$

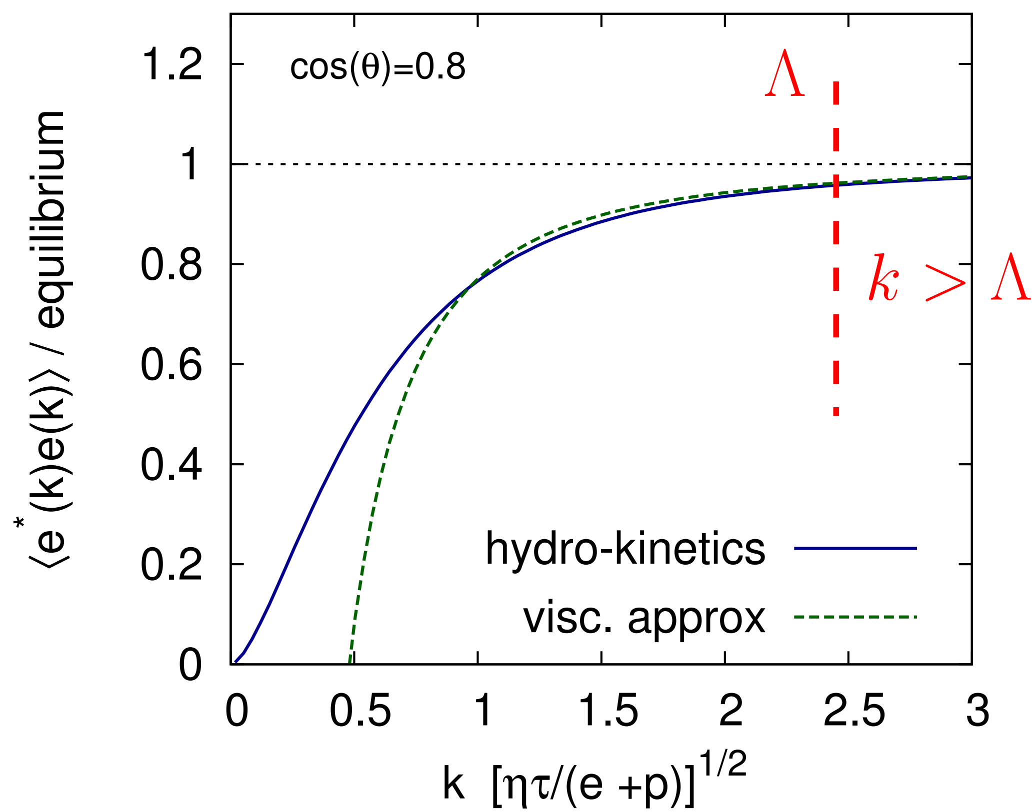
Want to develop a set of hydro-kinetic equations for $k \sim 1/(\tau\sqrt{\epsilon})$

Preview: $\langle e^*(k)e(k)\rangle$ for a Bjorken





A separation scale between the thermalized modes and the kinetic modes



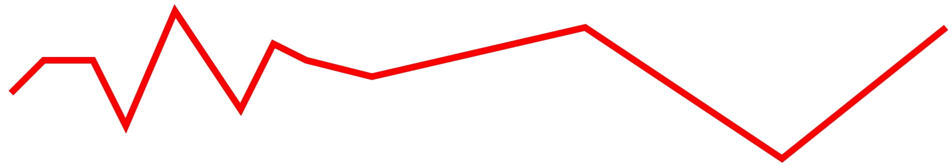
1. Fluctuations of modes $k > \Lambda$ are incorporated into the transport coefficients used for simulating modes with $k < \Lambda$, the pressure, shear etc all depend on Λ

$$p(\Lambda), \eta(\Lambda), \ldots$$

2. The final physics should not depend on this scale

Developing hydro-kinetics – Brownian motion

Random Walk



$$\frac{dp}{dt} = -\eta p + \xi \qquad \left\langle \xi(t)\xi(t') \right\rangle = 2TM\eta\, \delta(t-t')$$

1. Then we want to calculate

$$N(t) = \left\langle p^2(t) \right\rangle$$

2. Integrate the equation for short times

$$p(t+\Delta t) = -\eta\, p(t)\Delta t + \int_t^{t+\Delta t} \xi(t')dt'$$

3. Compute $\langle p(t+\Delta t)\, p(t+\Delta t)\rangle$ and find an equation

$$\frac{\Delta N}{\Delta t} = -2\eta\big[N - \underbrace{TM}_{\text{equilibrium}}\big]$$

Developing hydro-kinetics – linearized hydro in a uniform system

1. Evolve fields of linearized hydro with parameters $p(\Lambda)$, $\eta(\Lambda)$, $s(\Lambda)$ etc

$$\phi_a(\boldsymbol{k}) \equiv \Big(e(\boldsymbol{k}), g^x(\boldsymbol{k}), g^y(\boldsymbol{k}), g^z(\boldsymbol{k})\Big)$$

2. Then the equations are schematically exactly the same

$$\frac{d\phi_a(\boldsymbol{k})}{dt} = \mathcal{L}_{ab}(\boldsymbol{k})\phi_b(\boldsymbol{k}) + \xi_a \qquad \langle \xi_a \xi_b \rangle = 2T\mathcal{D}_{ab}(\boldsymbol{k})\delta(t-t')$$

3. Break up the equations into eigen modes of $\mathcal{L}_{ab}$, and analyze exactly same way:

$$\overbrace{\lambda_+ = +ic_s k - \tfrac{1}{2}\Gamma_s k^2}^{\text{right moving sound}} \qquad \overbrace{\lambda_- = -ic_s k - \tfrac{1}{2}\Gamma_s k^2}^{\text{left moving sound}} \qquad \overbrace{\lambda_T = -\eta k^2/(e+p)}^{\text{two diffusion modes}}$$

So for k in the z direction, work with the following linear combos (eigenvects)

$$\phi_A \equiv \left[\underbrace{c_s e(\boldsymbol{k}) \pm g^z(\boldsymbol{k})}_{\phi_+ \text{ and } \phi_-} \ , \ \underbrace{g^x(\boldsymbol{k})}_{\equiv \phi_{T_1}} \ , \ \underbrace{g^y(\boldsymbol{k})}_{\equiv \phi_{T_2}} \right]$$

The kinetic equations in flat space

1. The relevant correlators are e.g.

$$N_{++}(\boldsymbol{k},t) = \left\langle \phi^*_{+}(\boldsymbol{k})\phi_{+}(\boldsymbol{k}) \right\rangle \qquad N_{T_1T_1} = \left\langle \phi^*_{T_1}(\boldsymbol{k})\phi_{T_1}(\boldsymbol{k}) \right\rangle$$

2. Thus

$$\frac{dN_{++}}{dt} = -\frac{\frac{4}{3}\eta k^2}{e+p}\left[N_{++} - N^{\rm eq}_{++}\right]$$

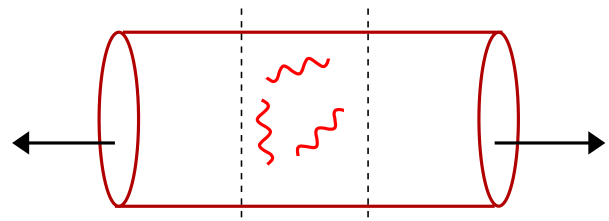
$$\frac{dN_{T_1T_1}}{dt} = -\frac{2\eta k^2}{e+p}\left[N_{T_1T_1} - N^{\rm eq}_{T_1T_1}\right]$$

and similar equations for N_{--} and $N_{T_2T_2}$. Here

$$N^{\rm eq}_{T_1T_1} \equiv (e+p)T \qquad \text{and} \qquad N^{\rm eq}_{++} \equiv (e+p)T$$

Now we will do the same for an expanding system

Kinetic equations for a Bjorken expansion



- The hydrodynamic field fields $\phi_a = (e, g^x, g^y, \tau g^\eta)$ are:

$$\phi_a(\tau, \boldsymbol{k}_\perp, \kappa) = \int d^2\boldsymbol{x} \int d\eta \; e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp + i\kappa\,\eta} \; \phi_a(\tau, \boldsymbol{x}_\perp, \eta)$$

- The equations take the form:

$$\frac{d}{d\tau}\phi_a(\boldsymbol{k}_\perp, \kappa) = \mathcal{L}_{ab}(\tau, \boldsymbol{k}_\perp, \kappa)\,\phi_b(\boldsymbol{k}_\perp, \kappa) + \xi_a$$

The previous analysis goes through with a few complications

1. The eigenvalues depend on the expansion rate and the damping rates

$$\lambda_\pm = \pm i c_s k - \tfrac{1}{2}\Gamma_s k^2 - \frac{2 + c_s^2}{2\tau} \quad \text{with} \quad k^2 \equiv k_\perp^2 + \frac{\kappa^2}{\tau^2}$$

2. The eigen vectors slowly change in time – use an adiabatic approximation

The kinetic equations and approach to equilibrium:

- The kinetic equations and approach to equilibrium

$$\frac{\partial}{\partial\tau}N_{++} = -\frac{1}{\tau}\Big[\underbrace{2+c_{s0}^2}_{\text{expansion}} + \underbrace{\frac{\kappa^2/\tau^2}{k_\perp^2+\kappa^2/\tau^2}}_{\text{rotation}}\Big]N_{++} \underbrace{- \frac{\frac{4}{3}\eta_0}{s_0T_0}\left(k_\perp^2+\frac{\kappa^2}{\tau^2}\right)\left[N_{++}-\frac{s_0T_0^2}{2c_{s0}^2\tau}\right]}_{\text{damping to equilibrium}},$$

$$\frac{\partial}{\partial\tau}N_{T_2T_2} = -\frac{2}{\tau}\Big[\underbrace{1}_{\text{expansion}} + \underbrace{\frac{k_\perp^2}{k_\perp^2+\kappa^2/\tau^2}}_{\text{rotation}}\Big]N_{T_2T_2} \underbrace{- \frac{2\eta_0}{s_0T_0}\left(k_\perp^2+\frac{\kappa^2}{\tau^2}\right)\left[N_{T_2T_2}-\frac{s_0T_0^2}{\tau}\right]}_{\text{damping to equilibrium}}.$$

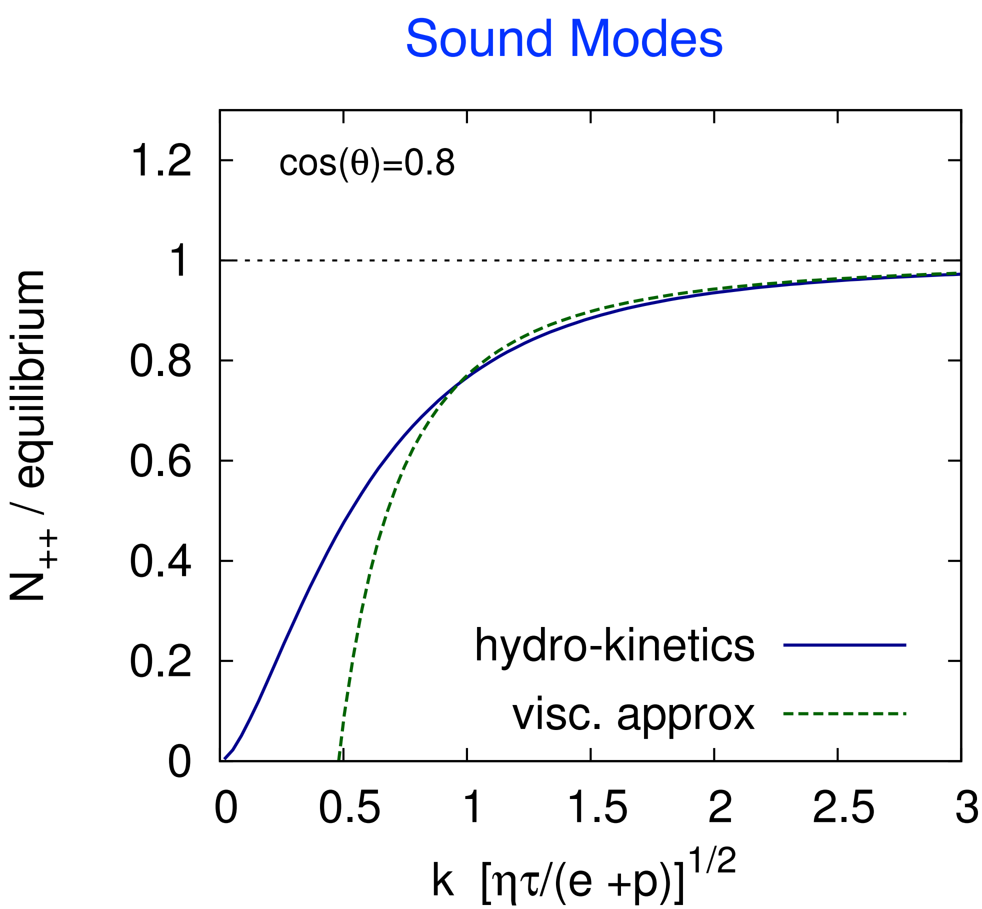
  and similar equations for the other modes

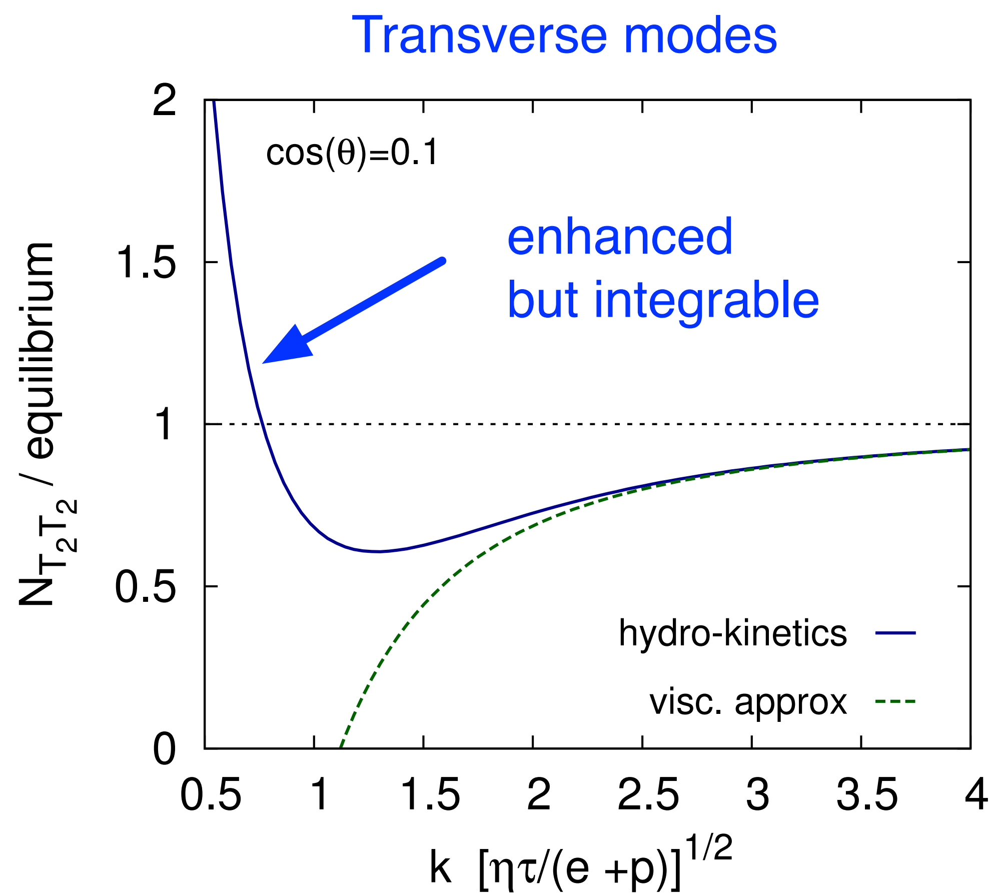
- For large k, we solve, and the modes approximately equilibrate:

$$N_{++} \simeq \frac{s_0T_0^2}{2c_{s0}^2\tau}\Big[\underbrace{1}_{\text{equilibrium}} + \underbrace{\frac{s_0T_0}{\left(\zeta_0+\frac{4}{3}\eta_0\right)\left(k_\perp^2+\kappa^2/\tau^2\right)}\left(c_{s0}^2-\frac{\kappa^2/\tau^2}{k_\perp^2+\kappa^2/\tau^2}\right)}_{\text{first viscous correction analogous to } \delta f} + \ldots\Big]$$

Now we solved these kinetic equations numerically

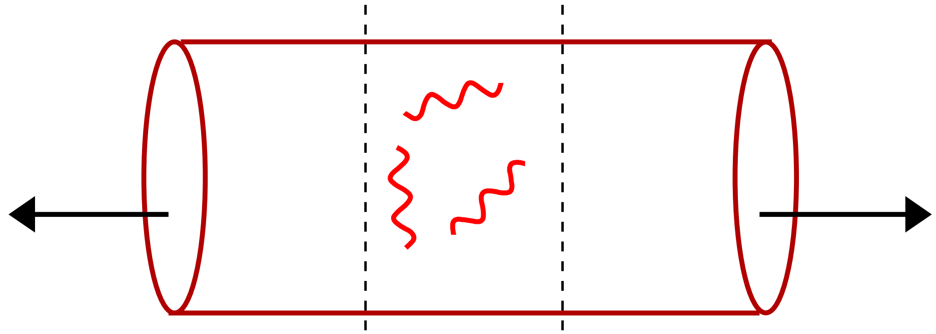
The non-equilibrium steady state at late times:





What are the consequences of this distribution of sound and diffusion modes??

The evolution of the background



$$\frac{de}{d\tau} = -\frac{e + T^{zz}}{\tau}$$

where

$$T^{zz}_{\rm hydro} = \underbrace{p(\Lambda)}_{\text{Ideal}} - \underbrace{\frac{\frac{4}{3}\eta(\Lambda)}{\tau}}_{\text{first order}} + \underbrace{(\lambda_1 - \eta\tau_\pi)\frac{8}{9\tau^2}}_{\text{second order}} + \ldots$$

In addition the fluctuations give another contribution:

$$T^{zz}_{\rm flucts} = (e+p)\left\langle v^z v^z \right\rangle$$

Evaluating the fluctuation contribution:

$$\frac{T^{zz}_{\rm flucts}}{e+p} = \langle v^z v^z \rangle$$
$$= \int \frac{d^2k_\perp d\kappa}{(2\pi)^3} \frac{1}{(e_o+p_o)^2} \left[(N_{++} + N_{--}) c_s^2 \cos^2\theta + N_{T_2T_2} \sin^2\theta \right]$$
$$= \frac{T_o\Lambda^3}{6\pi^2} - \left(\frac{17\Lambda}{120\pi^2} \frac{s_o T_o^2}{\eta_o(\Lambda)} \right) \frac{\frac{4}{3}}{\tau} + \text{finite}$$

Thus the full stress is then:

compare Kovtun, Moore, Romatschke

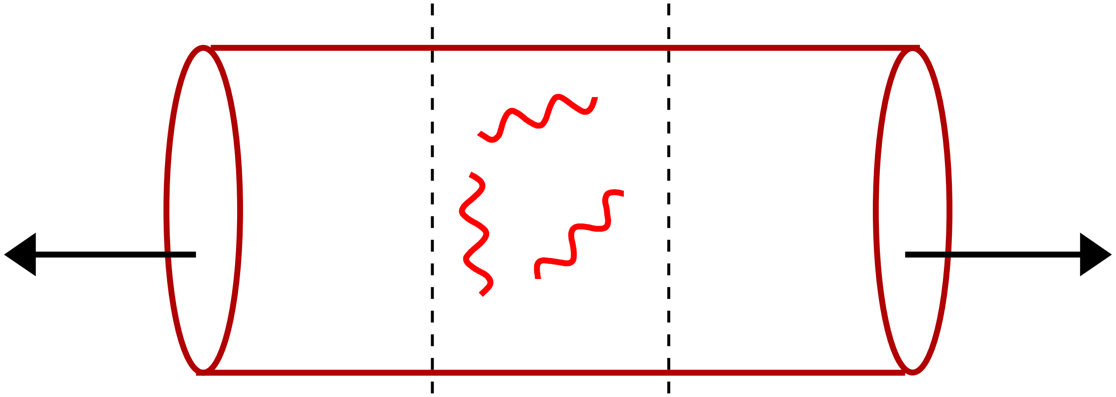
$$T^{zz} = T^{zz}_{\rm hydro} + T^{zz}_{\rm flucts}$$
$$= p_{\rm phys} - \frac{\frac{4}{3}\eta_{\rm phys}}{\tau} + \text{finite}$$

$$p_{\rm phys} \equiv p_0(\Lambda) + \frac{T_o\Lambda^3}{6\pi^2}$$
$$\eta_{\rm phys} \equiv \eta_0(\Lambda) + \left(\frac{17\Lambda}{120\pi^2} \frac{s_o T_o^2}{\eta_o(\Lambda)} \right)$$

where the physical quantities, $p_{\rm phys}$ and $\eta_{\rm phys}$, are independent of Λ

What's the finite correction?

Final result for a Bjorken expansion:



$$\frac{de}{d\tau} = -\frac{e + T^{zz}}{\tau}$$

$$\frac{T^{zz}}{e+p} = \Bigg[\frac{p}{e+p} - \underbrace{\frac{\frac{4}{3}\eta}{(e+p)\tau}}_{\text{first order}} + \underbrace{48.252\left(\frac{T^3}{s}\right)\frac{1}{(4\pi\eta/s)^3}\left(\frac{\eta}{(e+p)\tau}\right)^{3/2}}_{\text{3/2 order from flucts}}$$

$$+ \underbrace{\frac{(\lambda_1 - \eta\tau_\pi)}{e+p}\frac{8}{9\tau^2}}_{\text{second order}} + \ldots\Bigg]$$

- The correction is inversely proportional to entropy
- The $3/2$ power follows from $k \sim \sqrt{(e+p)/\eta\tau}$ and phase space:

$$\int d^3k \sim k^3 \sim \left(\frac{e+p}{\eta\tau}\right)^{3/2}$$

Numerical results:

Take representative numbers

$$\frac{(\lambda_1 - \eta\tau\pi)}{e+p} \simeq -1 \left(\frac{\eta}{e+p}\right)^2 \qquad \frac{T^3}{s} \simeq \frac{1}{16}$$

For $\eta/s = 1/4\pi$ find:

$$\frac{T^{zz}}{e+p} = \frac{1}{4}\left[1. - \underbrace{0.141}_{\text{first}}\left(\frac{3}{\tau T}\right) + \underbrace{0.0521}_{\text{3/2 order}}\left(\frac{3}{\tau T}\right)^{3/2} - \underbrace{0.0025}_{\text{second}}\left(\frac{3}{\tau T}\right)^2\right]$$

while for $\eta/s = 2/4\pi$ we have:

$$\frac{T^{zz}}{e+p} = \frac{1}{4}\left[1. - \underbrace{0.283}_{\text{first}}\left(\frac{3}{\tau T}\right) + \underbrace{0.0184}_{\text{3/2 order}}\left(\frac{3}{\tau T}\right)^{3/2} - \underbrace{0.010}_{\text{second}}\left(\frac{3}{\tau T}\right)^2\right]$$

Fluctuation contribution is a correction to first order hydro

but larger than second order in practice

Summary

1. For wavenumbers of order

$$k \sim \sqrt{\frac{e+p}{\eta\tau}}$$

   the system transitions to equilibrium

2. Worked out an alternate description of hydro with noise:

   - Hydro + hydro-kinetics

$$\begin{aligned}\partial_\mu(T^{\mu\nu}_{\rm hydro} + T^{\mu\nu}_{\rm flucts}) =& 0 \\ \partial_\tau N_{\rm flucts}(\boldsymbol{k},\tau) =& \ldots\end{aligned}$$

   This should be generalized to a general flows.

3. How is the non-linear $T^{\mu\nu}_{\rm flucts}$ imprinted on the particles?

$$\delta f_{\rm flucts} = ????$$

4. Fluctuating hydro corrections are as important as second order hydro in practice!