QCD EoS from lattice calculations with p4-improved staggered fermions

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Hydrodynamics in Heavy Ion Collisions and QCD Equation of State

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RBC-Bielefeld Collaboration

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Other: K. Petrov (DESY), T. Umeda (Tsukuba), S. Datta (Tata)

HotQCD = **RBC-Bielefeld** + **MILC** + **LLNL** + **LANL**

(talk by **R. Soltz**)

Contents

- Cut-off effects in lattice thermodynamics.
- Equation of state calculations with **p4** fermions.
- Interaction measure **(ε – 3p)** at low and high T.
- Pressure, energy density, entropy density.
- Comparison of **p4** and **Asqtad** Results.
- **p/ε** and speed of sound.
- EoS at finite chemical potential.

QCD Thermodynamics

- **Hadron resonance model**: At low T, QCD dynamics dominated by hadron resonances. As T is increased, we get thermal production of large number of heavier resonances → **ε** increases until Hagedorn temperature is reached → new phase of matter
- Manifested in QCD by a deconfinement transition where quark degrees of freedom are liberated, producing quark-gluon plasma (QGP)
- QGP is strongly interacting (esp. near transition temperature $\mathbf{T_c}$). Lattice QCD is tool to study non-perturbative aspects of QCD at finite temperature
- Goals: a) Study QCD transition region, $\mathbf{T_c}$ (R. Soltz)

  b) Equation of State (**EoS)** gives insight into dynamics of QGP, useful for understanding heavy ion collisions

Staggered Fermions

Attempt to alleviate fermion doubling problem by staggering fermion degrees of freedom on neighboring lattice sites.

Advantages: 16 fermion flavors $\rightarrow$ 4 fermion flavors, Numerically inexpensive, preserves U(1) remnant of chiral symmetry.

Disadvantages: Spin and flavor degrees of freedom become mixed up $\rightarrow$ $O(a^2)$ splittings for normally degenerate hadrons.

Improved staggered fermions (p4, Naik) improve dispersion relation.

"Smeared" gauge links help improve flavor symmetry by suppressing flavor-mixing terms in the action.

Dispersion Relation

Pressure of free fermions on the lattice (at fixed $\mathbf{N_t}$):

$$\frac{P}{T^4} = \frac{N_c N_f N_t^3}{8} \int_0^{2\pi} \frac{d^3(pa)}{(2\pi)^3} \left[\sum_{p_4 a} \ln\left(E(\mathbf{p}, p_4, m)^2\right) \right]$$

Standard staggered dispersion relation: $E^2 a^2 = \sum_\mu \sin^2(p_\mu a) + m^2 a^2$

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$$\frac{p_{lattice}}{p_{continuum}} = 1.75 \quad (N_t = 4)$$

$O(a^2)$ correction

Standard staggered fermions give result with large $O(a^2)$ corrections:

Improved Staggered Fermions

p4, **Naik** actions use higher-order approximation for fermion derivative, leading to a more accurate disperison relation → smaller cut-off effects compared to standard staggered fermions.

$$\frac{p_{lattice}}{p_{continuum}} = 1 + 0 - \frac{1143}{980}\left(\frac{\pi}{N_t}\right)^4 + \frac{73}{2079}\left(\frac{\pi}{N_t}\right)^6 + \ldots \text{ (p4)}$$

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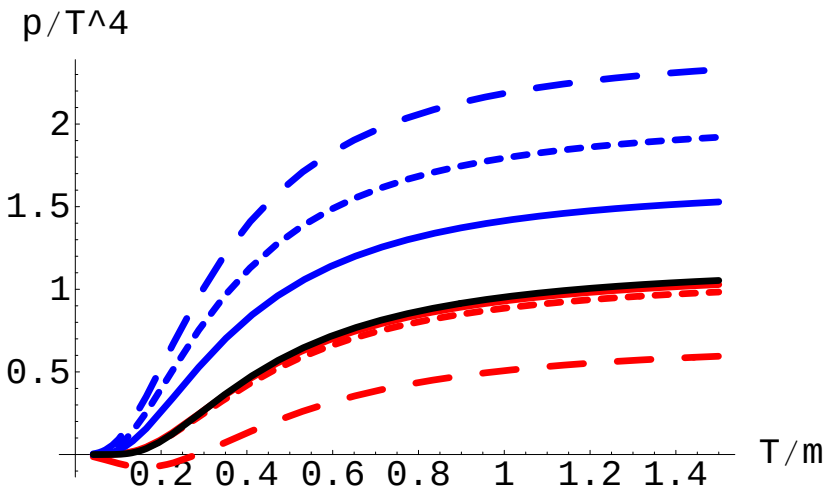
No $O(a^2)$ corrections

Reduced higher-order corrections

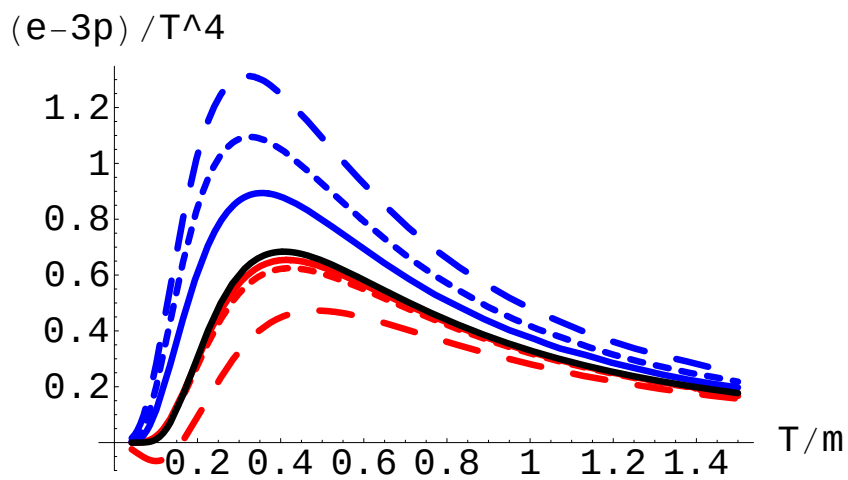
Free field Equation of State



Staggered: Nt = 4, 6, 8

p4: Nt = 4, 6, 8

- For p4 fermions, Nt = 6 is already close to the continuum result.
- For standard staggered, Nt = 8 is still far away.



EoS calculations with p4 fermions

- Calculations at **N_t = 4, 6** with RBC-Bielefeld Collaboration
- Use Rational Hybrid Monte Carlo **(RHMC)** to simulate two light dynamical flavors and 1 strange flavor **(2+1 flavor)**
- Light quark mass 2x heavier than physical mass **(m_l = 0.1 m_s)**
- $\mathbf{m_\pi \approx 220 MeV}$ Kaon mass tuned to physical value $\mathbf{m_K \approx 490 MeV}$

- Large temperature range:
  140 MeV < T < 800 MeV → **0.7 Tc < T < 4 Tc**

- Calculations at **Nt = 8** with **HotQCD**
- Compare with **Asqtad** (talk by **L. Levkova**)

Calculating EoS on the lattice

The pressure:

$$\frac{p}{T^4}(\beta) = N_t^4 \int_{\beta_0}^{\beta} d\beta [\frac{1}{N_s^3 N_t} \left(\langle S_g \rangle_0 - \langle S_g \rangle_T \right) - \sum_f \left(\frac{\partial m_f}{\partial \beta} \right) \left(\langle \bar{\psi} \psi_q \rangle_0 - \langle \bar{\psi} \psi_q \rangle_T \right)]$$

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The interaction measure:

$$\frac{\epsilon - 3p}{T^4} = T\frac{d}{dT}\left(\frac{p}{T^4}\right) = a\frac{d\beta}{da}\frac{\partial p/T^4}{\partial \beta} = \left(\frac{\epsilon - 3p}{T^4}\right)_{gauge} + \left(\frac{\epsilon - 3p}{T^4}\right)_{fermion}$$

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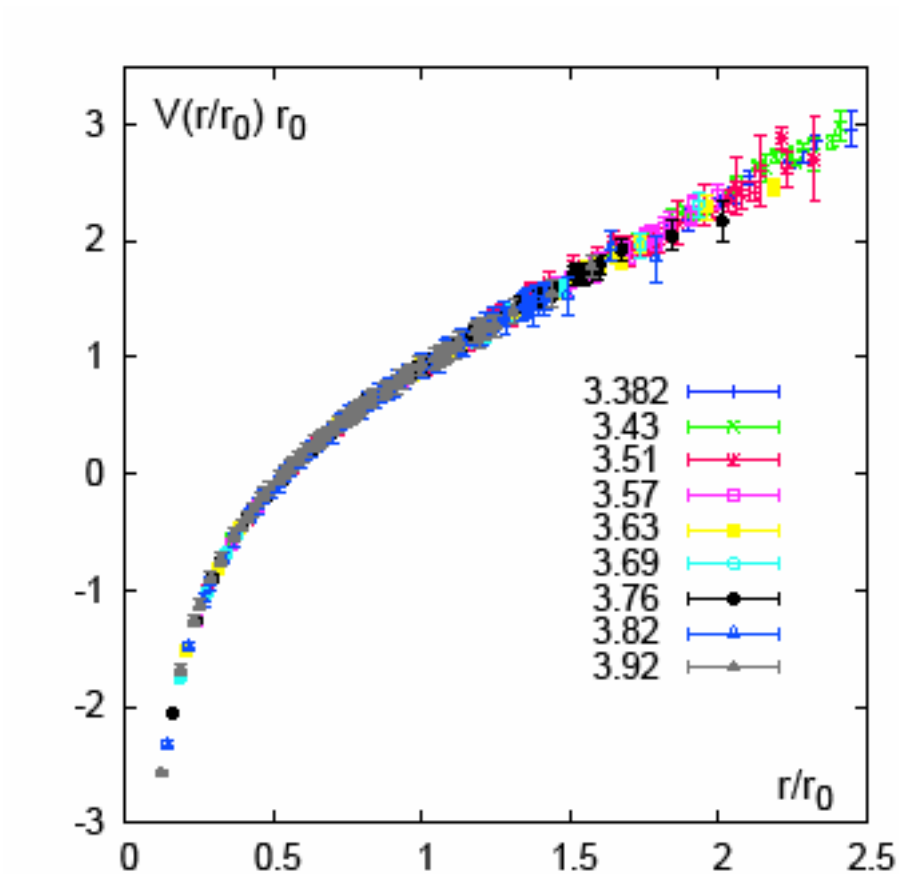
Interaction measure is the basic quantity measured on lattice. Pressure, energy density, entropy can be reconstructed via thermodynamic relations from **ε-3p**.

Lattice scale

- Temperature on lattice given by **T = 1/(Nt a)**
- Lattice spacing, **a**, controlled by gauge coupling **β**
- Need to determine lattice scale at each **β** in order to extract physical temperature.
- Also need good control of **a(β)** in order to extract $\mathbf{a}\frac{\mathbf{d}\beta}{\mathbf{da}}$
- Lattice scale determined by calculating quantity on lattice and matching to actual physical value.
- Bare quark masses **(m_l, m_s)** are adjusted along with **β** to keep physical hadron masses $(\mathbf{m}_\pi, \mathbf{m}_\mathbf{K}, \mathbf{m}_{\eta_s})$ fixed **(Line of Constant Physics)**.

Heavy quark potential

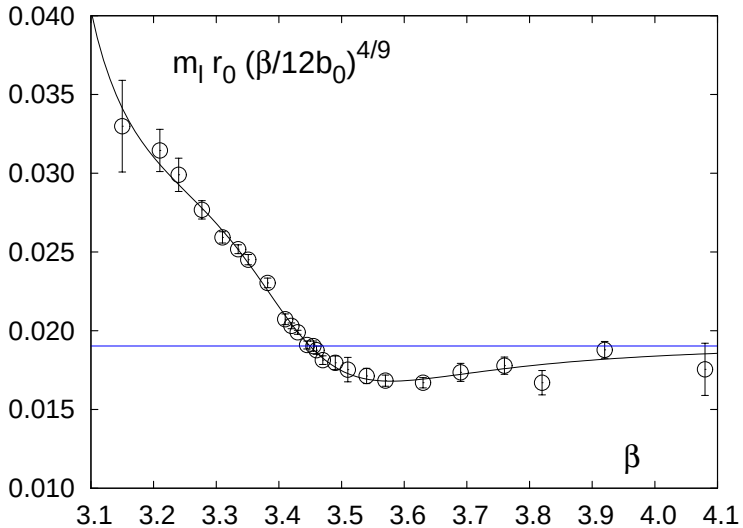


- **V(r)** – Potential between two heavy quarks.
- No visible cut-off effects over large range of gauge couplings
- Can use string tension **(σ)** or Sommer parameter **(r_0)** in order to set lattice scale.
- **r_0= 0.469(7) fm.**

$$V(r) = -\frac{\alpha}{r} + \sigma r$$

$$r^2 \frac{dV(r)}{dr}\Big|_{r=r_0} = 1.65$$

Line of constant physics



- **$m_l a$** adjusted with **β** to keep meson masses fixed. Ratio of light quark mass to strange quark **(m_l/m_s = 0.1)** also kept fixed.
- Fit to renormalization group-inspired ansatz used to extract **$dm_q/d\beta$**

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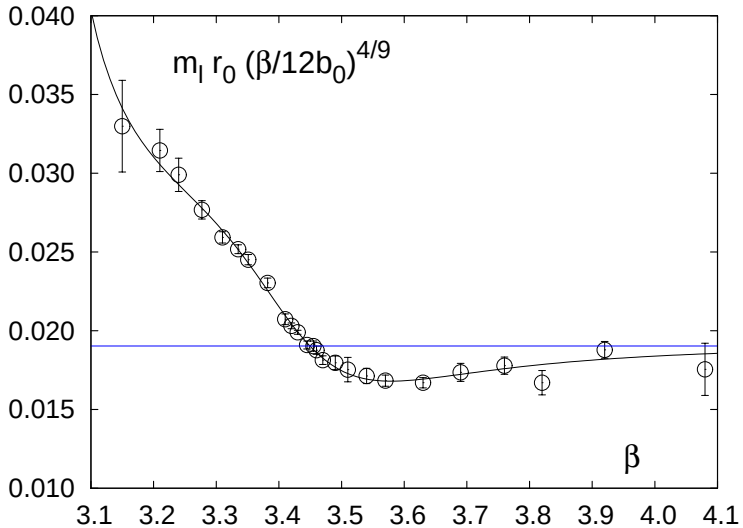

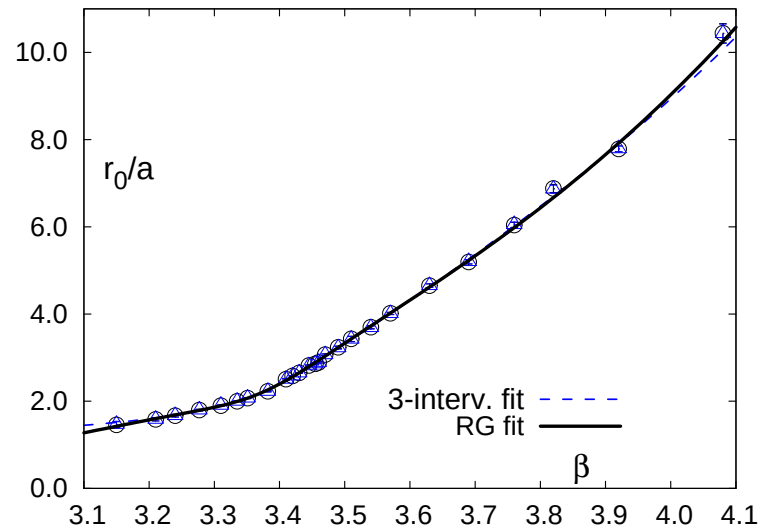

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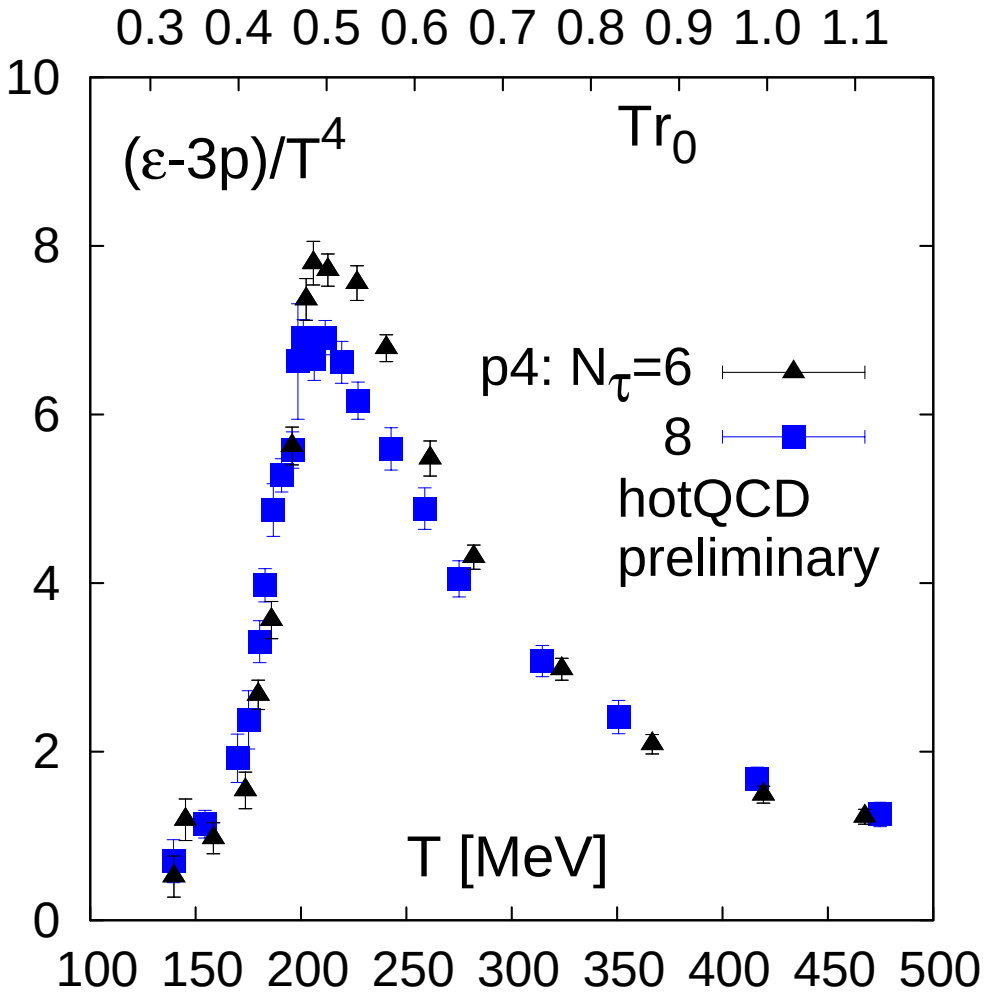
- Fit to renormalization group-inspired ansatz used to extract **$dm_q/d\beta$**
- Piecewise fit of r_0/a used to parameterize scale.
- $\mathbf{a}\dfrac{\mathbf{d}\beta}{\mathbf{da}}$ determined from this fit.

Interaction measure



- Liberation of quark degrees of freedom passing through the transition → rapid rise in **$(\varepsilon-3p)/T^4$**
- Peak occurs at **$T>T_c$**
- Cut-off effects still most evident in vicinity of peak and just above.

Hadron Resonance Gas

At low temperatures, hadron resonances are operative degrees of freedom. QCD thermodynamics is well-described by a non-interacting hadron gas.

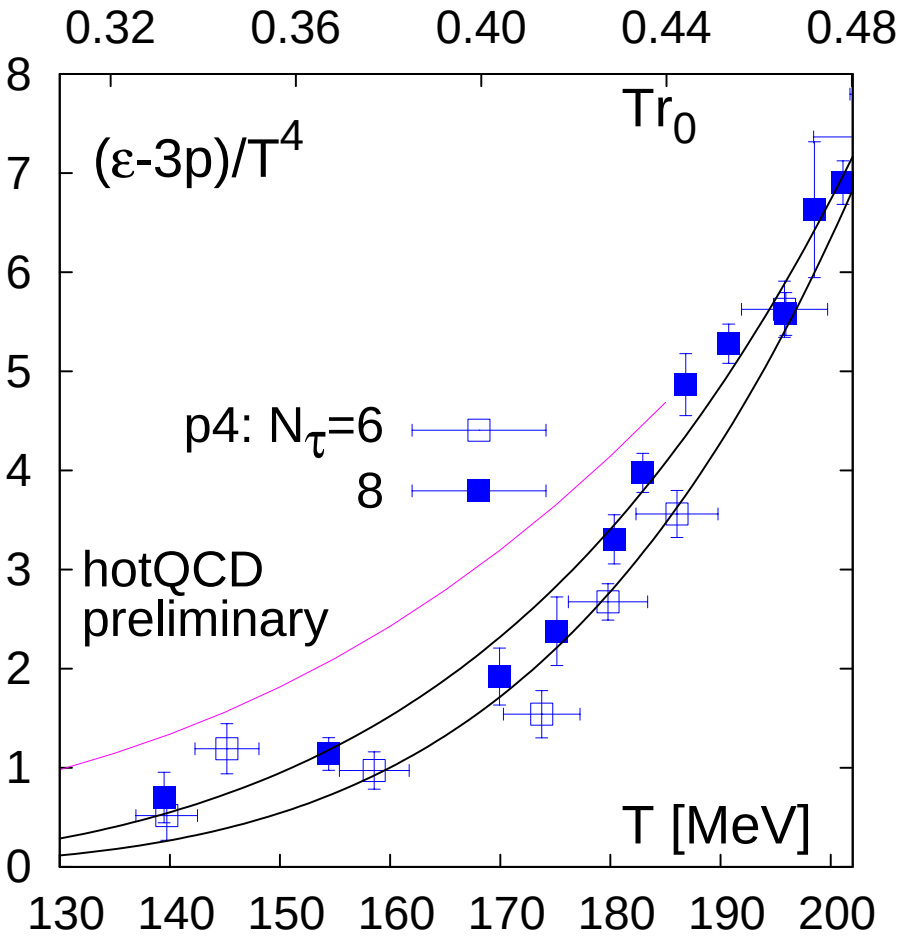
$$\left(\frac{\epsilon - 3p}{T^4}\right)_{HRG} = \sum_{m_i \in hadrons} \frac{d_i}{2\pi} \sum_{k=1}^{\infty} (-\eta_i)^{k+1} \frac{1}{k} \left(\frac{m_i}{T}\right)^3 K_1(km_i/T)$$

Heavier modes, though exponentially suppressed at low T, still contribute because of sheer numbers.

Models based on the **hadron resonance gas (HRG)** explain the relative particle abundances in heavy ion collisions quite well.

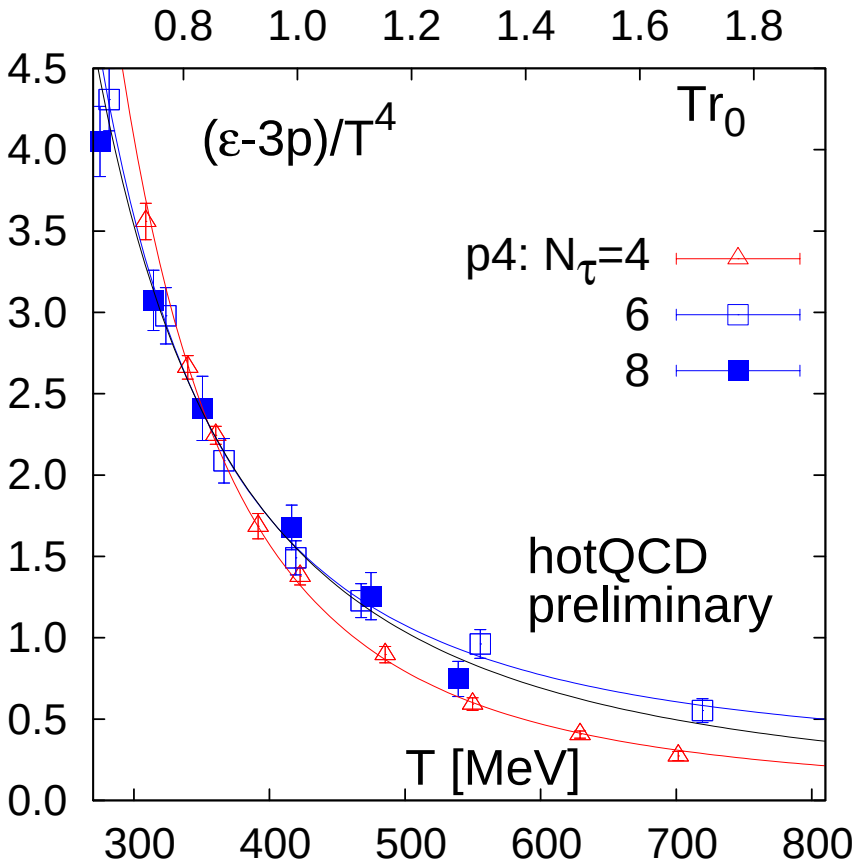
Compare lattice data at low T with results from **HRG**.

Low temperature EoS



- Still some discrepancy between **hadron resonance gas** and EoS at low T.
- Some difference explained by unrealistic mass spectrum – hadron states are heavier than in nature → oversuppression
- Approximately 5 Mev shift in T-scale to adjust for unrealistic hadron spectrum.

High temperature EoS



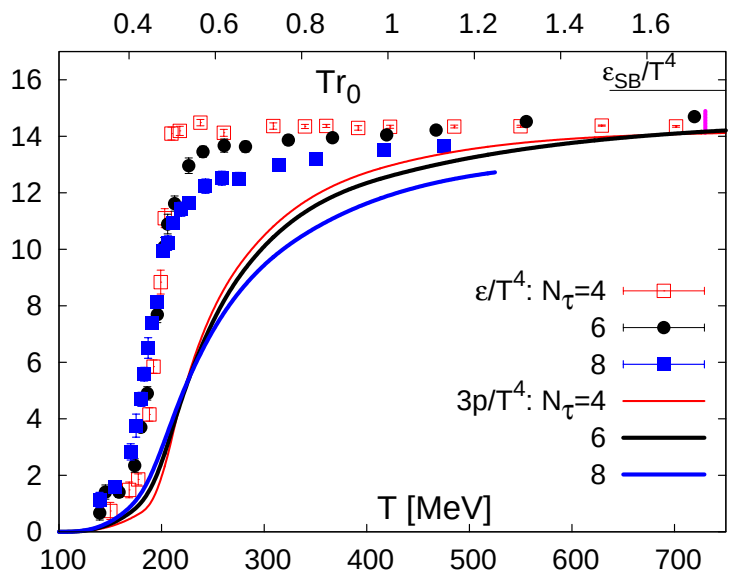
- **Nt = 4** decreases more quickly than **Nt = 6,8** → **p, ε** approach SB limit more quickly.
- Phenomenological parameterization:

$$\left(\frac{\epsilon - 3p}{T^4}\right)_{highT} = \frac{a}{T^2} + \frac{b}{T^4} + c$$

- Match to perturbation theory?

Pressure, Energy, Entropy

Pressure is obtained by integrating interaction measure. Then, can reconstruct energy density. Assume **P(T = 100 MeV) = 0** → small systematic error.



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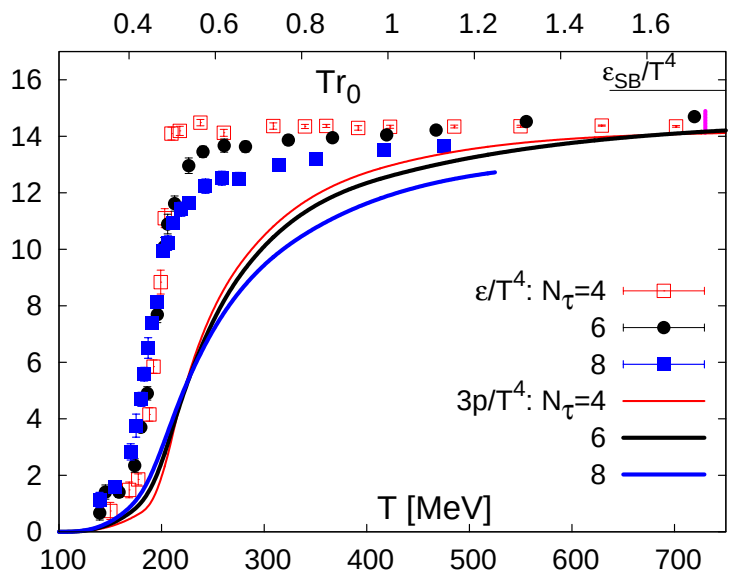
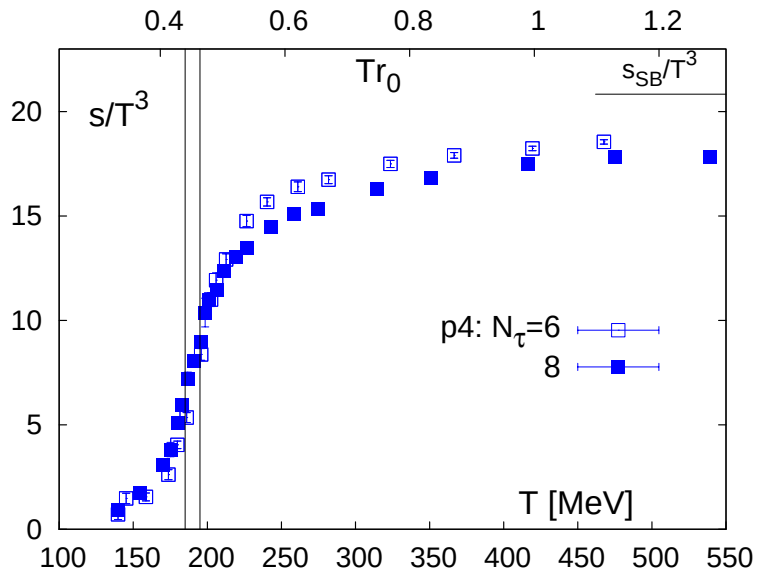
$$\frac{\mathbf{s}}{\mathbf{T^3}} = \frac{\epsilon + \mathbf{p}}{\mathbf{T^4}}$$





Differences between p4 and Asqtad

- HotQCD collaboration: **Nt = 8** with **p4** and **Asqtad** formulations.
- **Independent formulations! Different O(a^n) corrections!**
- **p4**: Adds "knight's move" term to fermion derivative

  Slightly better dispersion relation, free-field results.
- **Asqtad**: "Naik" term in fermion derivative, more complicated gauge-link smearing and tadpole-improvement to improve flavor symmetry.

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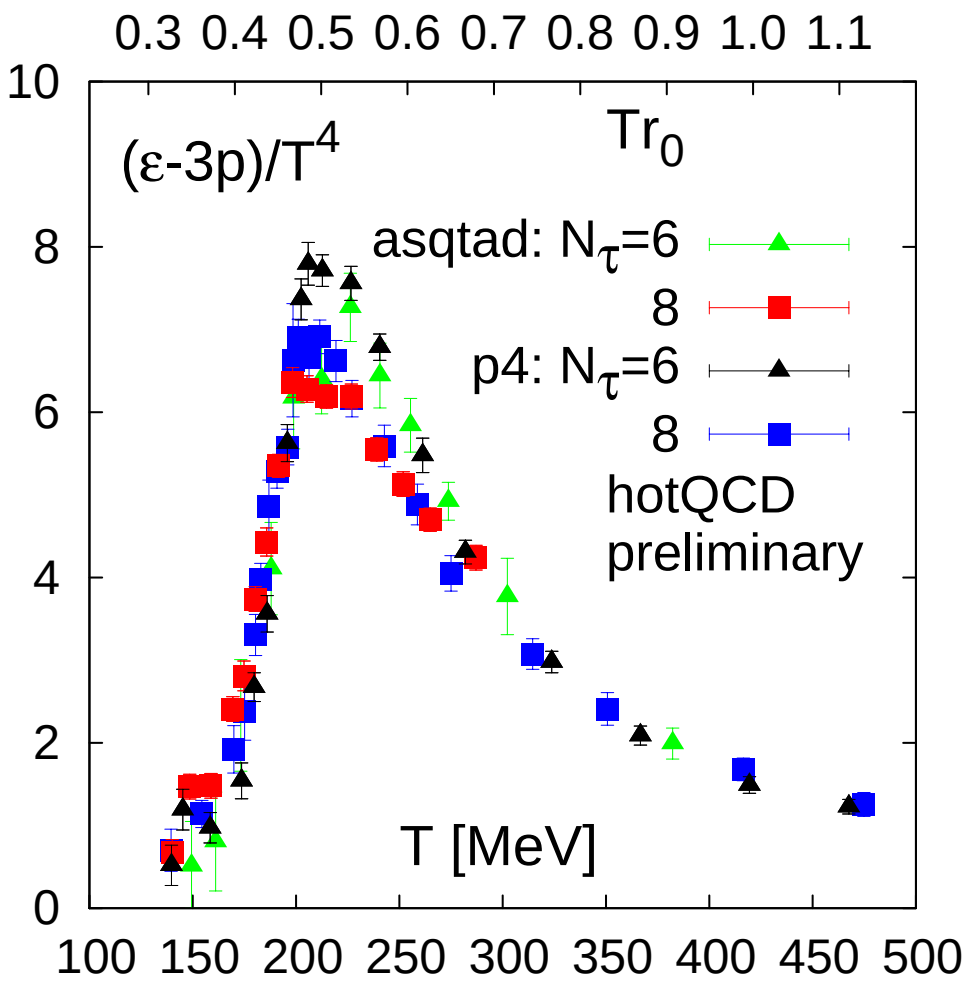
- **p4**: Scale setting with Sommer parameter $\mathbf{r_0}$, string tension σ.
- **Asqtad**: Scale setting with $\mathbf{r_1}$ parameter.
- Heavy quark potential well-known from hadron spectrum studies.

Interaction Measure



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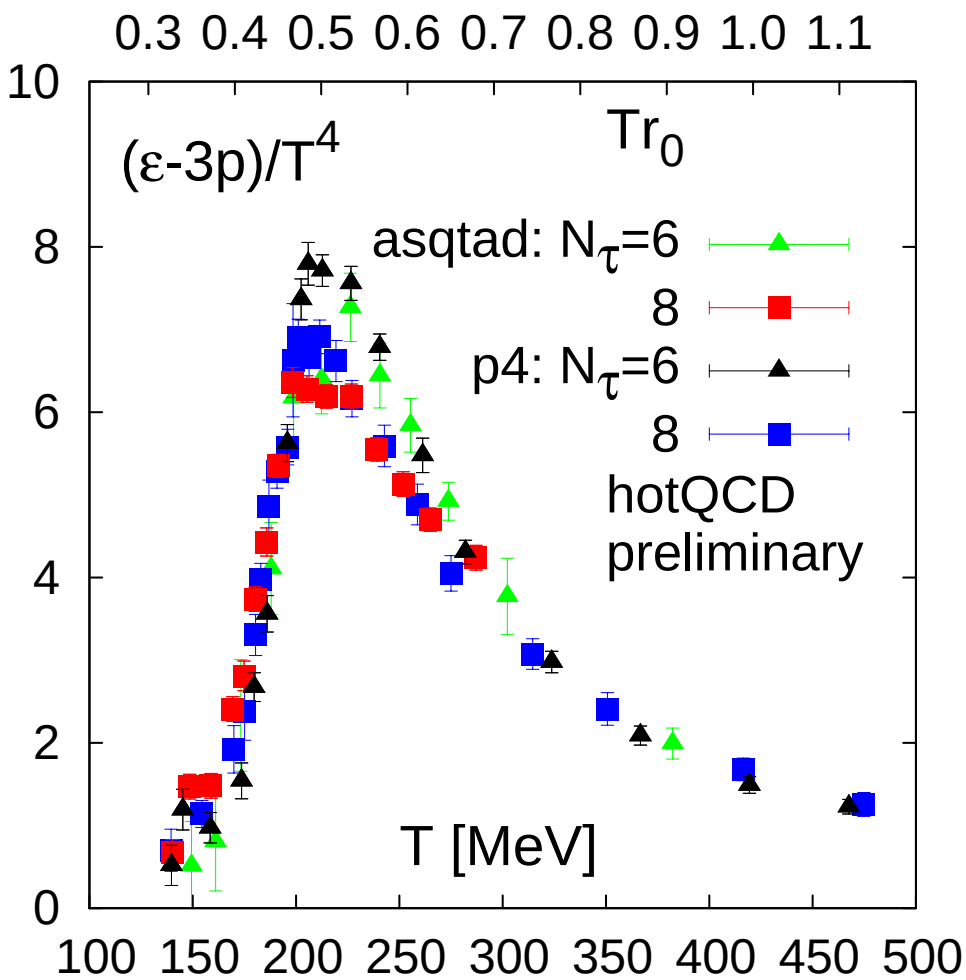

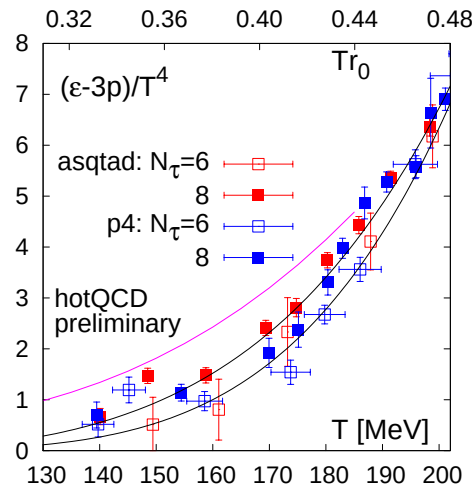

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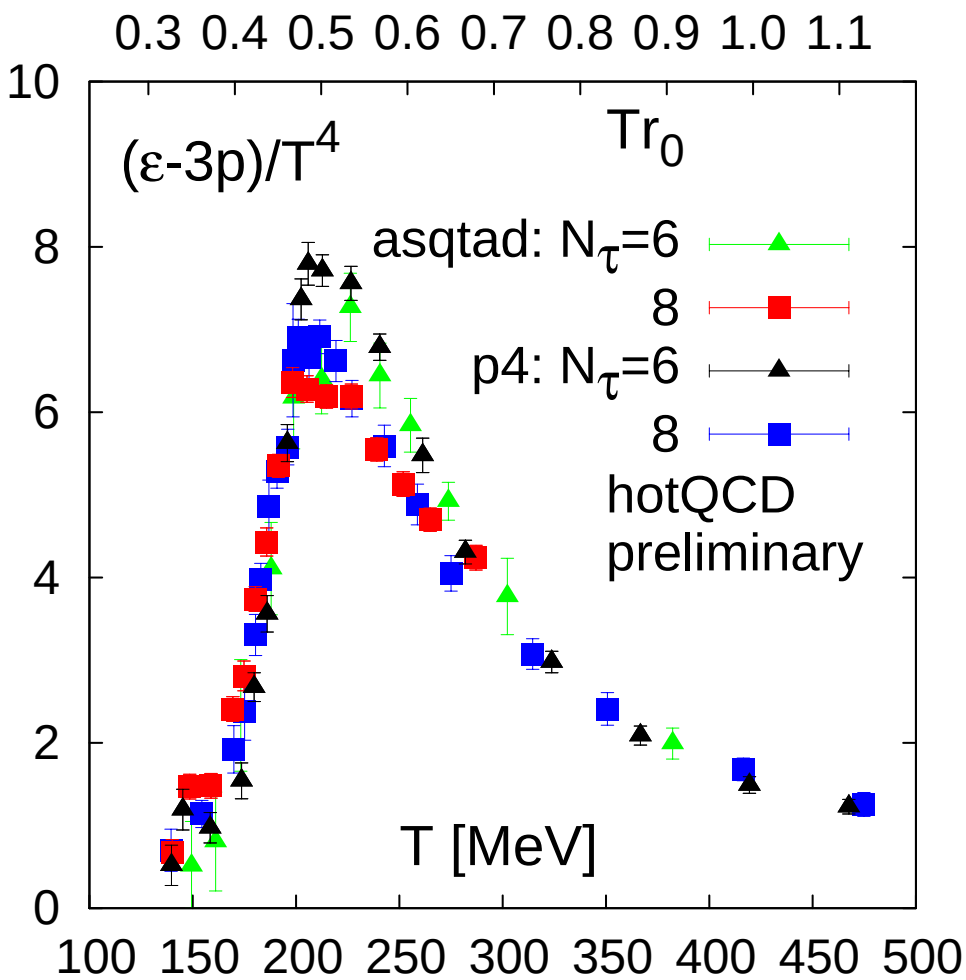
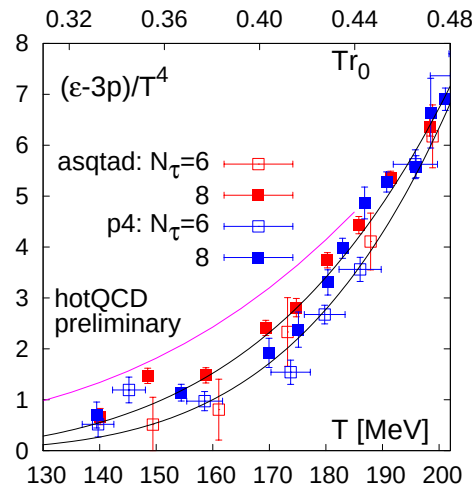
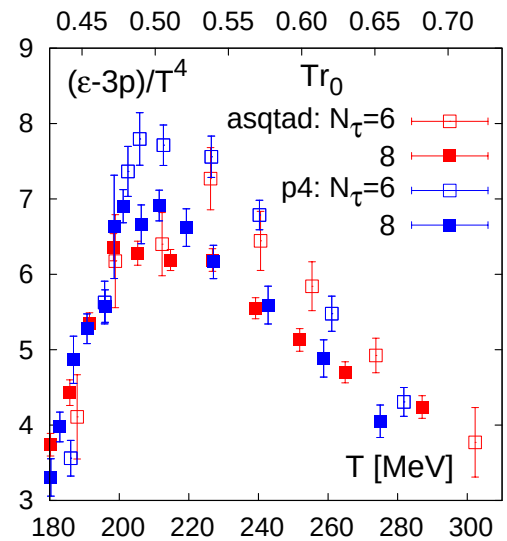
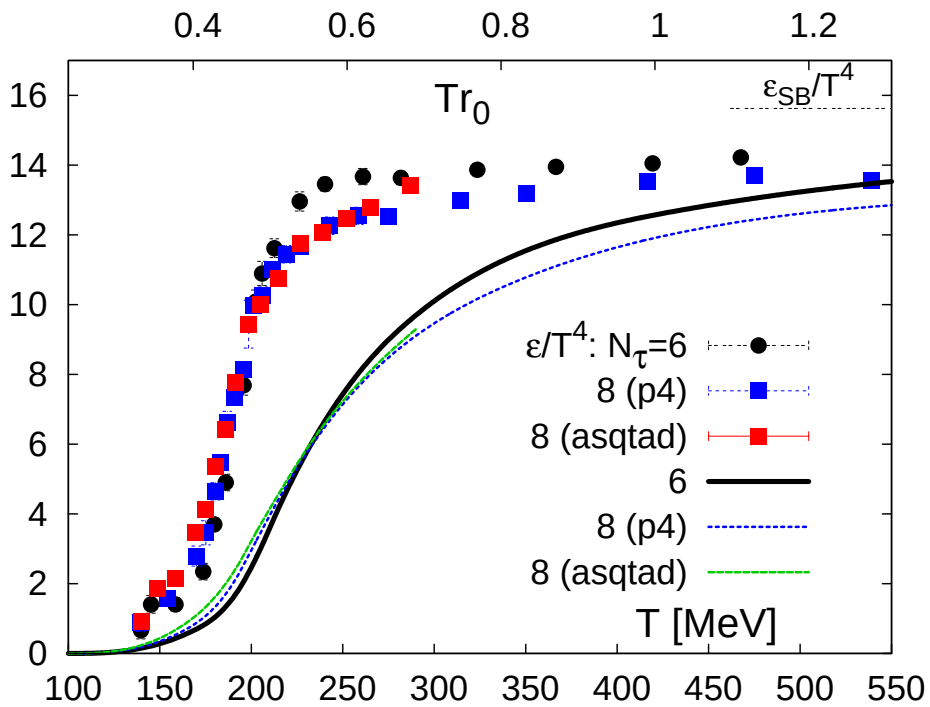
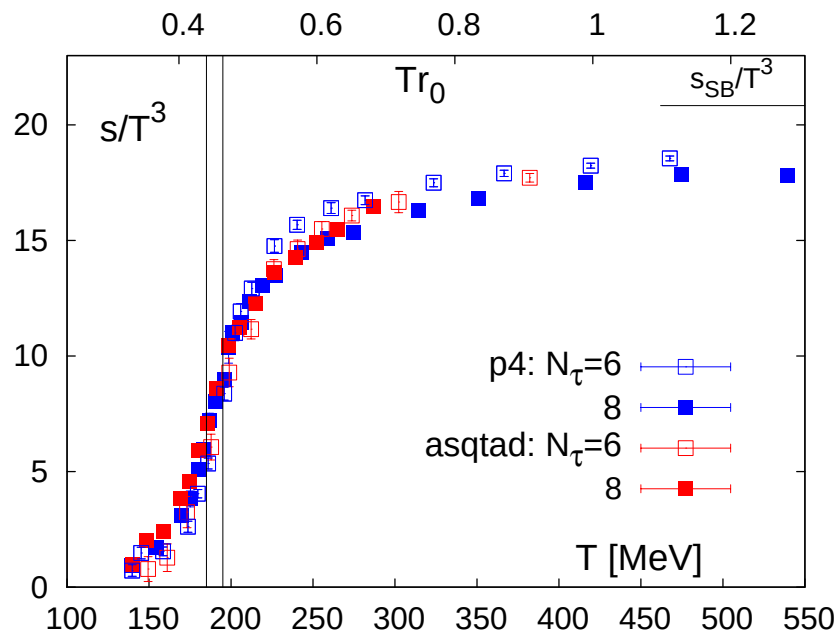




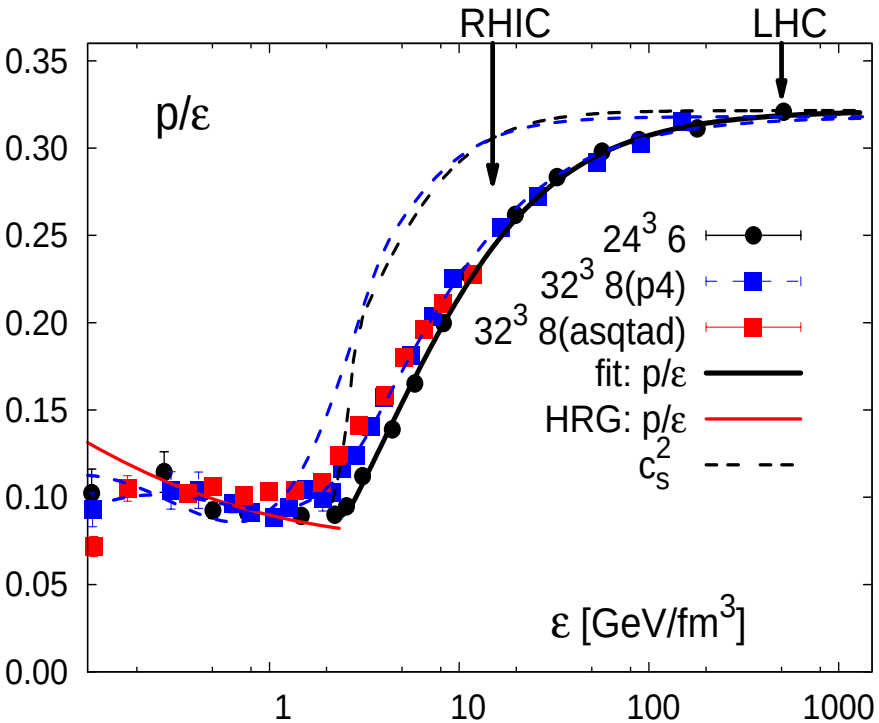


Pressure, Energy, Entropy





p/ε and Speed of Sound



$$c_s^2 = \frac{dp}{d\epsilon} = \epsilon \frac{dp/\epsilon}{d\epsilon} + \frac{p}{\epsilon}$$

- Convenient to eliminate T in favor of **ε** for **p/ε**.
- Good agreement between **Nt = 6, 8 p4** and **Asqtad** calculations.
- Deviations at lowest temperatures when compared with HRG.
- SB limit:

$$c_s^2 \approx \frac{p}{\epsilon} \approx \frac{1}{3}$$

EoS at μ≠0

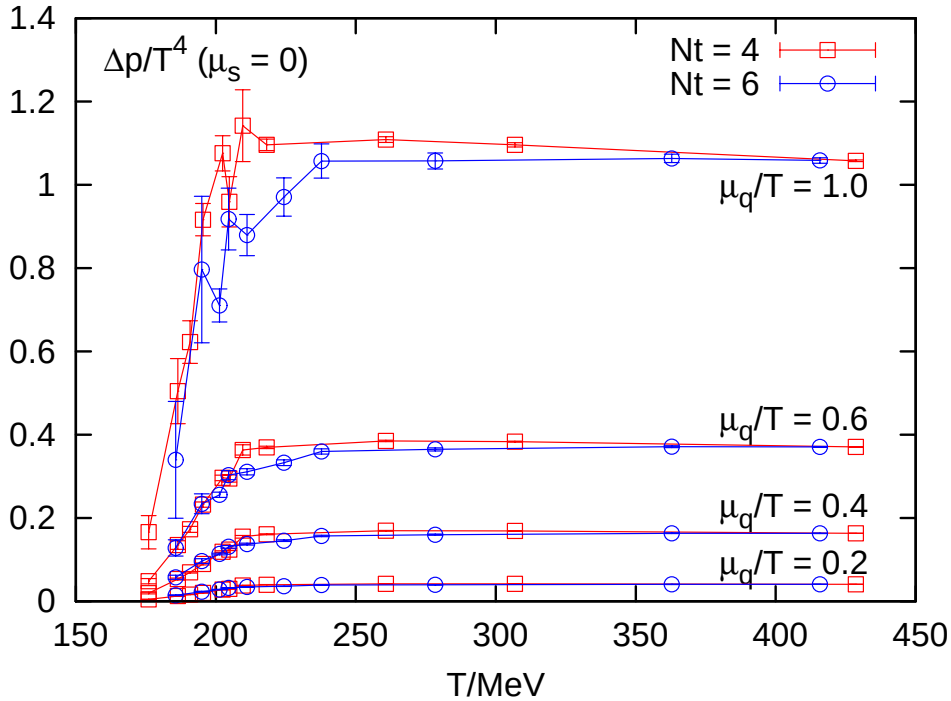
- Dynamical lattice calculations done with zero chemical potential, i.e. **μ=0** in fermion action. Incorporating finite chemical potential directly is numerically challenging.
- Study effects of finite density via Taylor expansion about **μ=0**.

$$\frac{p}{T^4}(T,\mu_q,\mu_s) = \frac{p}{T^4}(T,\mu_q=0,\mu_s=0) + \sum_{i,j} c_{i,j}(T)\left(\frac{\mu_q}{T}\right)^i\left(\frac{\mu_s}{T}\right)^j$$

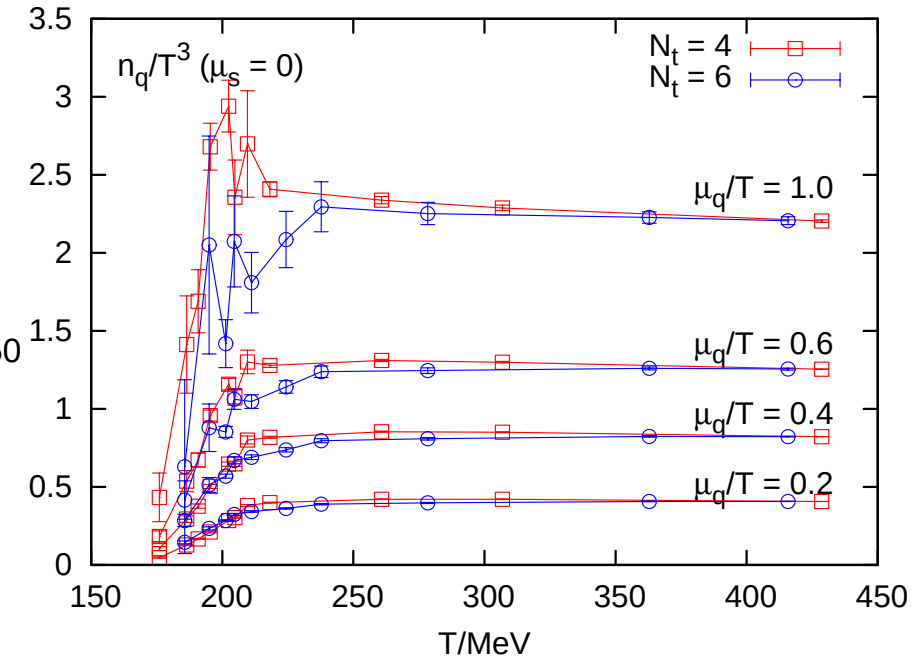
- Expansion coefficients $c_{i,j}(T)$ calculable on the lattice
- **Nt = 4:** $O(\mu^8)$ **Nt = 6:** $O(\mu^4)$
- Quark number densities:

$$\frac{\langle n_{q,s}\rangle}{T^3} = \frac{\partial}{\partial\mu_{q,s}}\frac{p}{T^4}(T,\mu_q,\mu_s)$$

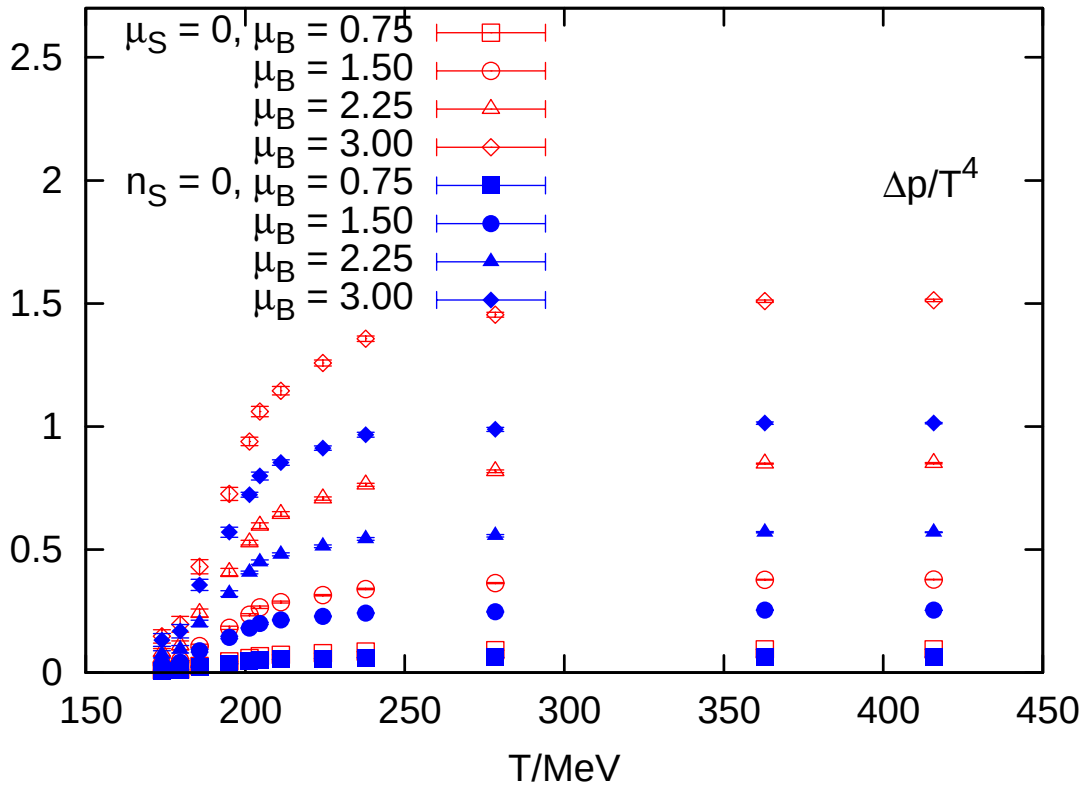
Pressure at finite μ_B



$\mu_S = 0$



Pressure at $\langle n_s \rangle = 0$



$$\langle n_s \rangle = 0$$

- Setting $\mu_s = 0$ does not give $\langle n_s \rangle = 0$
- Heavy ion collisions have zero net strangeness.
- Tune μ_s to constrain net strangeness to zero.
- Pressure expansion in chemical potential changes → corrections to pressure reduced compared to $\mu_s = 0$

Conclusions

- Improved staggered fermions necessary to reduce cut-off effects in thermodynamic quantities.
- RBC-Bielefeld lattice calculations of EoS with **p4** action complete with **Nt = 4,6**, under investigation at **Nt = 8** with **HotQCD** collaboration.
- Preliminary results indicate that errors are under fairly good control between **Nt = 6, 8**. Allows reasonable comparison at low T with HRG model.
- Excellent agreement between improved actions (**p4**, Asqtad) give further evidence that systematic errors are well-controlled.
- Calculations at **$\mu \neq 0$** hopefully can give results that better approximate conditions in heavy ion collisions.