# Non-Newtonian nature of Causal Hydrodynamics

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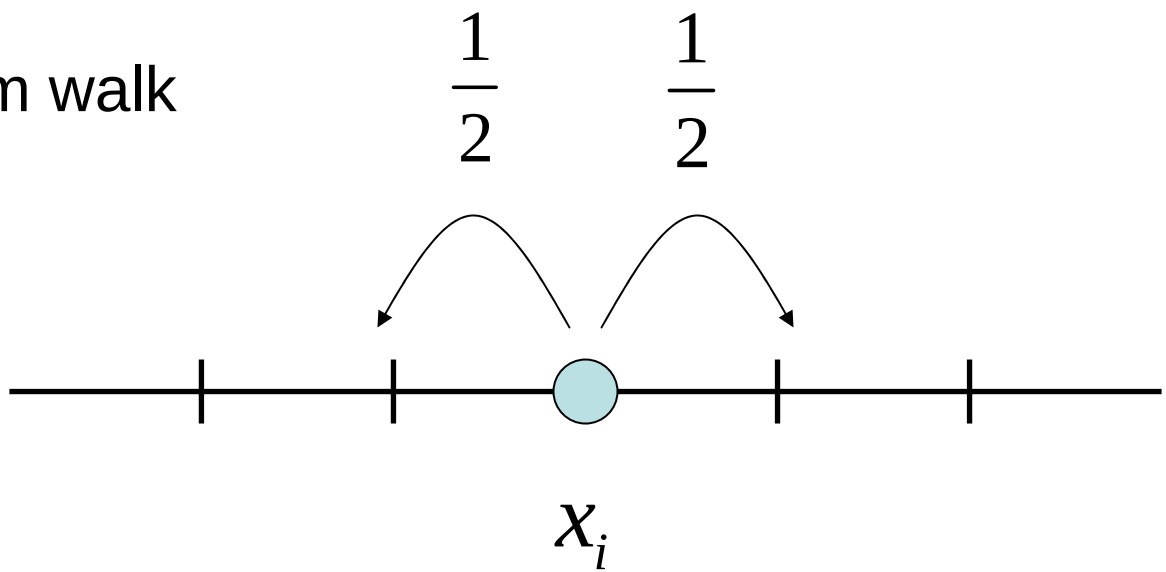
Because of causality, the relativistic dissipative fluid will be a non-Newtonian fluid. Thus

1) The GKN formula should be modified.
2) $1/4\pi$ can be a lower bound of the shear of Newtonian fluids.
3) The fluid expands to vacuum by forming a stationary wave.
4) The additional viscosity is still necessary stabilize solutions.

# Infinite speed in diffusion process

Random walk




$$x_{i+1} = x_i \pm \Delta x \qquad t_{i+1} - t_i = \Delta t$$

# Infinite speed in diffusion process

$$P(x_n, t+\Delta t) = \frac{1}{2}P(x_{n-1}, t) + \frac{1}{2}P(x_{n+1}, t)$$

$$\Delta x \to 0, \Delta t \to 0$$

$$\lim_{\Delta x, \Delta t \to 0} \frac{(\Delta x)^2}{\Delta t} \equiv D$$

$$v = \frac{\Delta x}{\Delta t} \to \infty$$

$$\frac{\partial}{\partial t} P(x,t) = D\nabla^2 P(x,t)$$

Dispersion relation $\omega = ik^2$

# Relativistic NR equation(1+1)

Hiscock&Lindblom(‘85)

Linear perturbation analysis

$$\omega = i\frac{\varsigma k^2}{2w_0} \pm k\sqrt{\alpha - \left(\frac{\varsigma k}{2w_0}\right)^2} \qquad k_{cri} = 2w_0\sqrt{\alpha}/\varsigma$$

>0. stable

At $k \geq k_{cri}$, the dispersion relation is reduced to the diffusion type.

# Introduction of memory effect

Equation of continuity

$$\frac{\partial}{\partial t} n = -\nabla J$$

Fick's law

$$J = D \nabla n$$

$$\frac{\partial}{\partial t} n = D \nabla^2 n$$

$$J = \frac{D}{\tau_R} \int_{-\infty}^{t} ds e^{-(t-s)/\tau_R} \nabla n(s)$$

$$\tau_R \frac{\partial^2}{\partial t^2} n + \frac{\partial}{\partial t} n - D \nabla^2 n = 0$$

$$V = \sqrt{\frac{D}{\tau_R}}$$

Acausal (Cattaneo)
Breaking sum rules (Kadanoff&Martin, T.K.)

# Landau-Lifshitz theory

Equation of Continuity

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial^\mu N_\mu = 0$$

bulk viscosity shear viscosity

$$T^{\mu\nu} = (\varepsilon + P + \Pi)u^\mu u^\nu - g^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}$$

$$N^\mu = nu^\mu + \nu^\mu$$

heat current

$$u_\mu \nu^\mu = 0 \qquad u_\mu \pi^{\mu\nu} = 0$$

# Introduction of Memory Effect

Entropy four current

T.K.,Denicol,Mota&Kodama(2007)

$$\partial_\mu \sigma^\mu = \frac{1}{T}(-P^{\mu\nu}\Pi + \pi^{\mu\nu})\partial_\mu u_\nu - \nu^\mu \partial_\mu \frac{\mu}{T} \geq 0$$

$$\Pi = -\varsigma \partial_\mu u^\mu$$

$$\pi^{\mu\nu} = \eta P^{\mu\alpha\nu\beta}\partial_\alpha u_\beta$$

$$\nu^\mu = -\kappa P^{\mu\nu}\partial_\nu \frac{\mu}{T}$$

**Landau-Lifshitz**

**Memory effect**

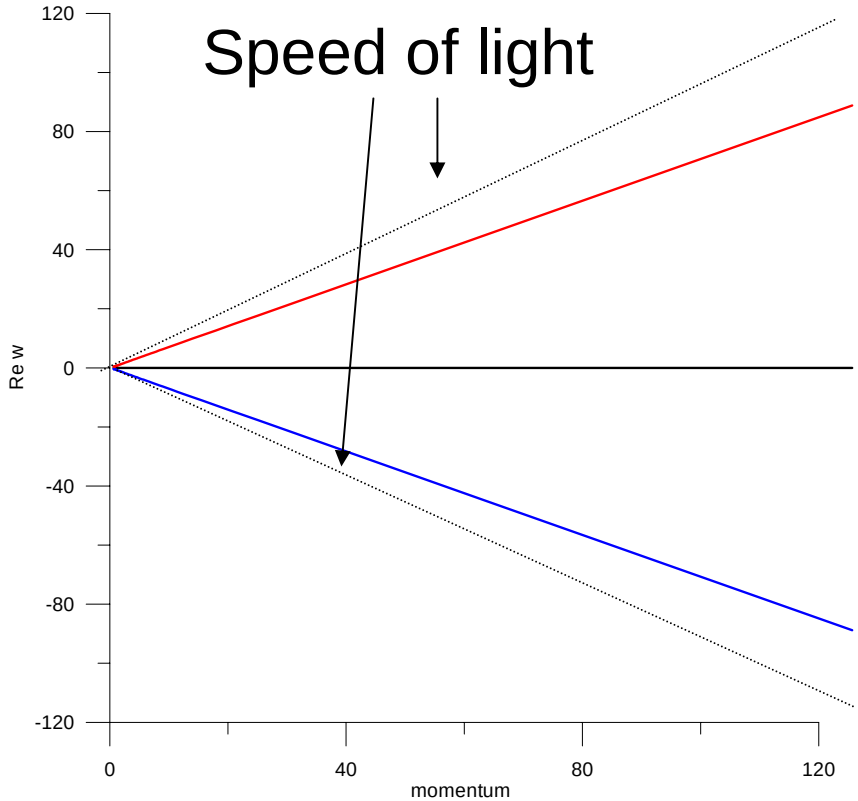
$$\Pi = -\int^\tau ds G(\tau, s)\varsigma \partial_\mu u^\mu$$

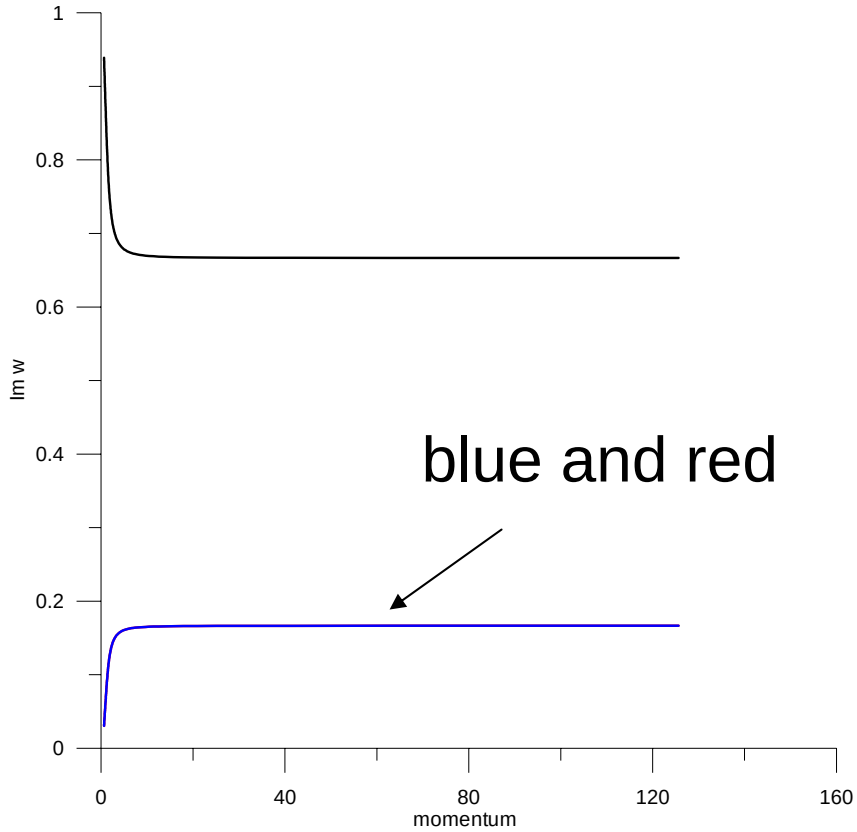
$$\pi^{\mu\nu} = P^{\mu\alpha\nu\beta}\int^\tau ds G(\tau, s)\eta \partial_\alpha u_\beta$$

$$\nu^\mu = -P^{\mu\nu}\int^\tau ds G(\tau, s)\kappa \partial_\nu \frac{\mu}{T}$$

**Causal Hydrodynamics**

# Dispersion relation(1+1)




Stable in the sense of the linear analysis.

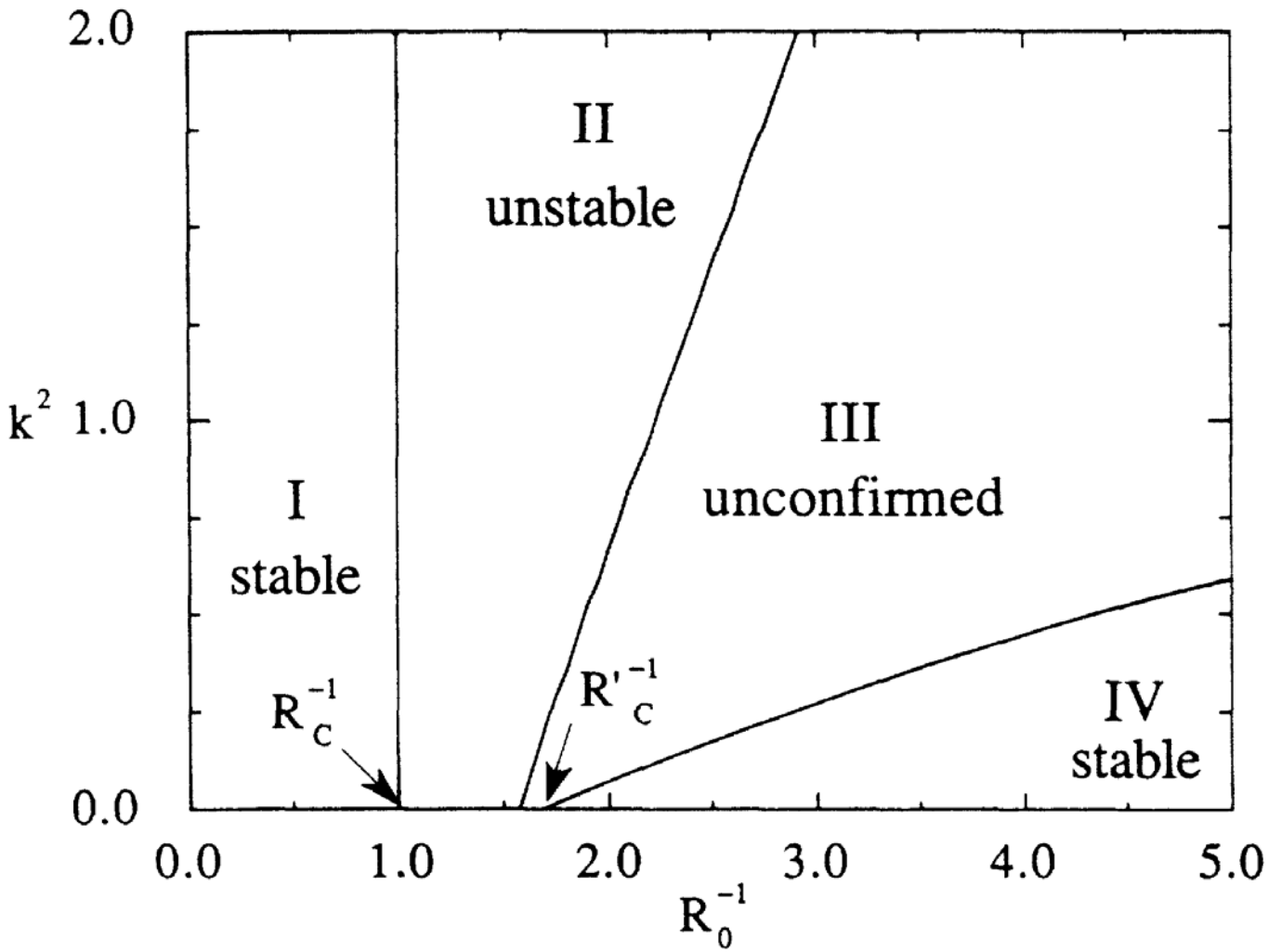
# Linear analysis of stability

Hiscock& Lindblam (‘85), Kouno,Maruyama,Takagi&Saito (‘90)

| | LL equation | CDhydro |
|---|---|---|
| Equilibrium | Stable | Stable |
| Lorentz boost | Unstable | Stable |
| Scaling solution | Stable / Unstable | ? |

☆ The Eckert theory is more unstable than the LL theory.

# Stability around scaling solution




$$R_0^{-1} = \frac{\varsigma}{Ts\tau}$$

$$\Pi = -\varsigma \partial_\mu u^\mu$$

$$\pi^{\mu\nu} = \eta P^{\mu\alpha\nu\beta} \partial_\alpha u_\beta$$

$$\nu^\mu = -\kappa P^{\mu\nu} \partial_\nu \frac{\mu}{T}$$

**Landau-Lifshitz**

**Memory effect**

$$\Pi = -\int^\tau ds G(\tau, s) \varsigma \partial_\mu u^\mu$$

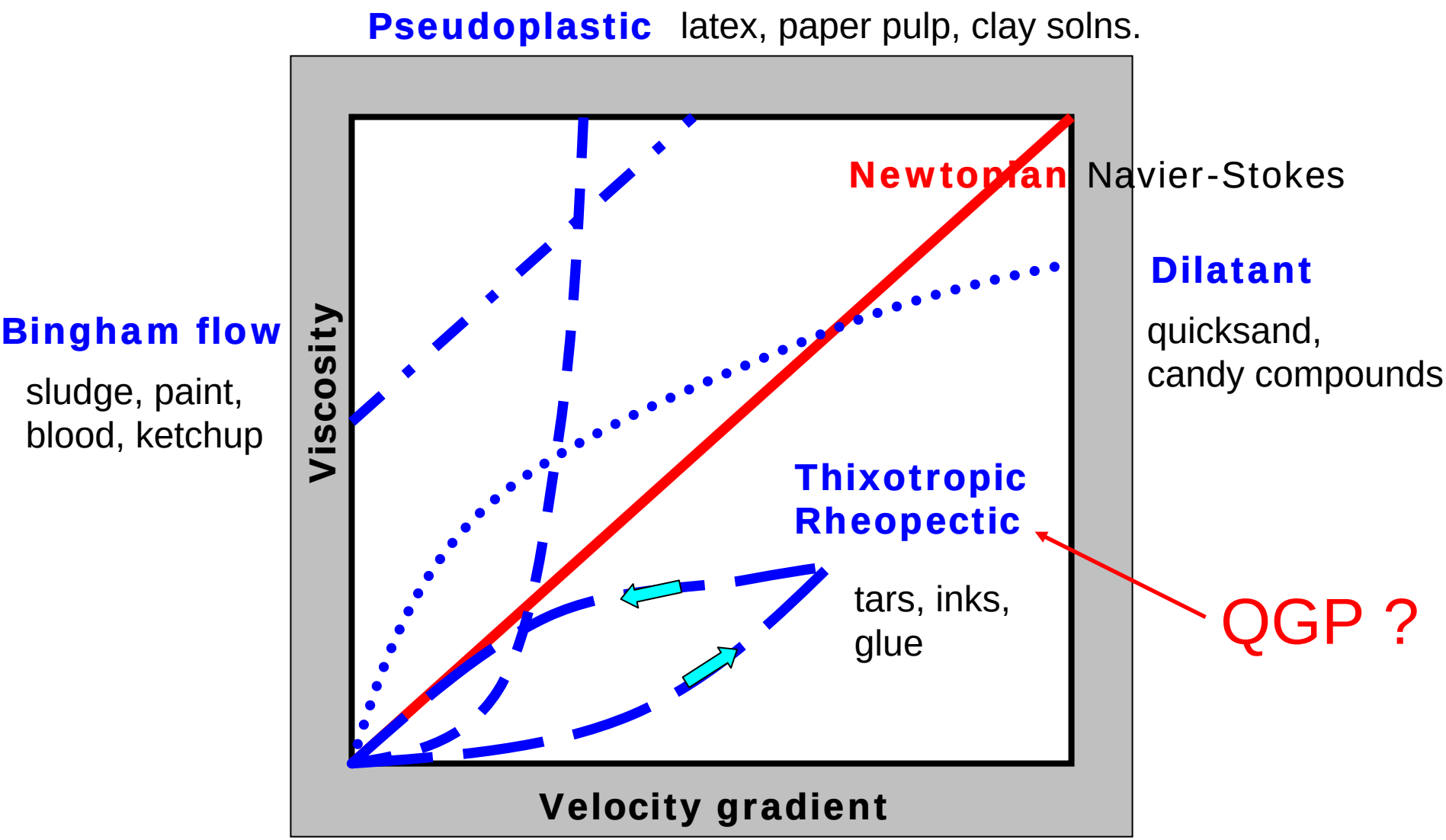
$$\pi^{\mu\nu} = P^{\mu\alpha\nu\beta} \int^\tau ds G(\tau, s) \eta \partial_\alpha u_\beta$$

$$\nu^\mu = -P^{\mu\nu} \int^\tau ds G(\tau, s) \kappa \partial_\nu \frac{\mu}{T}$$

**Causal Hydrodynamics**

**Newtonian fluid**

# Anomalous viscosity

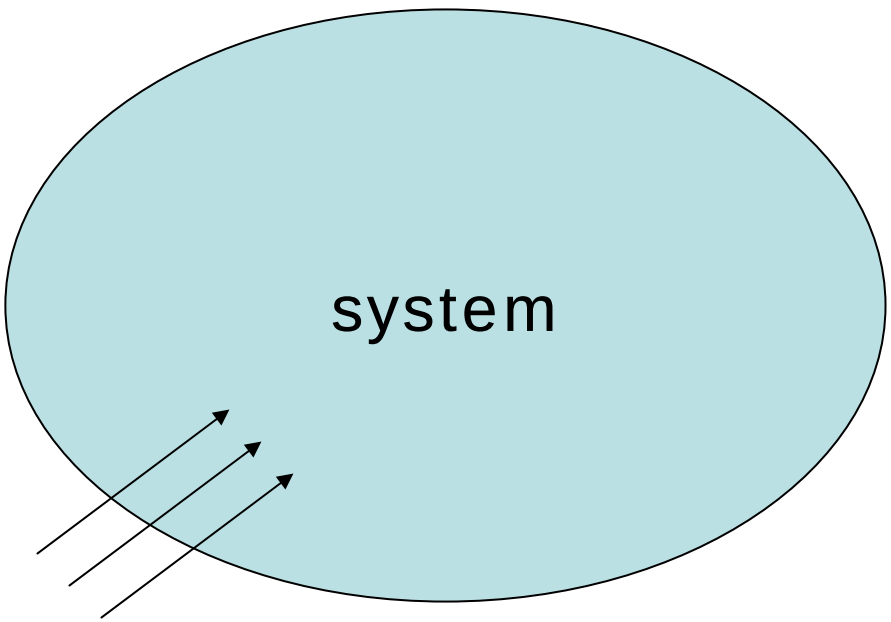


# Common sense?

- The transport coefficients are given by the Green-Kubo-Nakano formula.
- The lower bound of the shear viscosity of the CD hydro. is given by $1/(4\pi)$.
- The viscosity is small correction.
- We do not need any additional viscosity in the CD hydro.

Not trivial !!

# Linear response theory




Hamiltonian

$$H$$

$$\downarrow$$

External perturbation $F$

$$H - AF(t)$$

# Green-Kubo-Nakano formula

Exact!

$$\langle J \rangle = \int_{-\infty}^{t} ds \Phi(t-s) F(s)$$

$$\Phi(t) = \int_{0}^{\beta} d\lambda \left\langle \dot{A}(-i\lambda) J(t) \right\rangle_{eq}$$

In Newtonian fluid, transport coefficients are defined by $D = \dfrac{J(t)}{F(t)}$

We ignore the memory effect $\langle J \rangle \approx \int_{0}^{\infty} ds \Phi(s) F(t)$

GKN formula $D = \int_{0}^{\infty} ds \Phi(s)$

$$\langle J \rangle = \int_{-\infty}^{t} ds \Phi(t-s) F(s)$$

Newtonian

Non-Newtonian

$$J(t) = DF(t)$$

$$J(t) = \frac{D}{\tau_R} \int_{-\infty}^{t} ds e^{-(t-s)/\tau_R} F(s)$$

$$D = \int_{0}^{\infty} ds \Phi(s)$$

D=?

$$J(t) = \frac{D}{\tau_R}\int_{-\infty}^{t} ds e^{-(t-s)/\tau_R} F(s) \Longrightarrow \partial_t J(t) = -\frac{1}{\tau_R} J(t) + \frac{D}{\tau_R} F(t)$$

$$\Updownarrow$$

$$J(t) = \int_{-\infty}^{t} ds \Phi(t-s) F(s) \Longrightarrow \partial_t J(t) = \phi(0) F(t) + \int_{-\infty}^{t} ds \partial_t \phi(t-s) F(s)$$

Markov app. $\longrightarrow$ $\Downarrow$

$$\partial_t J(t) = \phi(0) F(t) + \int_{0}^{\infty} ds \partial_s \phi(s) F(t)$$

$$J(t) = D_{GKN} F(t) \longrightarrow \Downarrow$$

$$\partial_t J(t) = \phi(0) \frac{J(t)}{D_{GKN}} + \int_{0}^{\infty} ds \partial_s \phi(s) F(t)$$

# Generalization of GKN formula

T.K.&Maruyama (2004), T.K.(2005), T.K.(2007), T.K.&Kodama (2008)

| | |
|---|---|
| GKN formula | $\chi_1 = \langle J(t); J \rangle$ |
| New formula | $\chi_2 = \langle \rho(t); J \rangle$ |

1. In the GKN formula, we need $\chi_1$
   In the generalized formula, we need $\chi_1$ and $\chi_2$
2. $\chi_2$ characterizes the deviation from the GKN formula.
3. When $\chi_2$ vanishes in the low momentum limit,
   the new formula reproduces the GKN formula.
4. The result obtained in the new formula is consistent with
   sum rules.

# Lower limit of shear viscosity?

Kovtun,Son&Starinets(“05)

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

1) GKN formula

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt d^3x e^{i\omega t} \left\langle [T_{xy}(\vec{x},t), T_{xy}(0,0)] \right\rangle$$

2) Absorption cross section

$$\sigma_{abs}(\omega) = \frac{8\pi G}{\omega} \int dt d^3x e^{i\omega t} \left\langle [T_{xy}(\vec{x},t), T_{xy}(0,0)] \right\rangle$$

$$s = \frac{\sigma_{abs}(0)}{4G}$$

$$\eta = \frac{\sigma_{abs}(0)}{16\pi G}$$

G: Newton’s constant

# Lower limit of shear viscosity?

Kovtun,Son&Starinets(“05)

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

**Only for Newtonian!!**

1) GKN formula

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt d^3x e^{i\omega t} \left\langle [T_{xy}(\vec{x},t), T_{xy}(0,0)] \right\rangle$$

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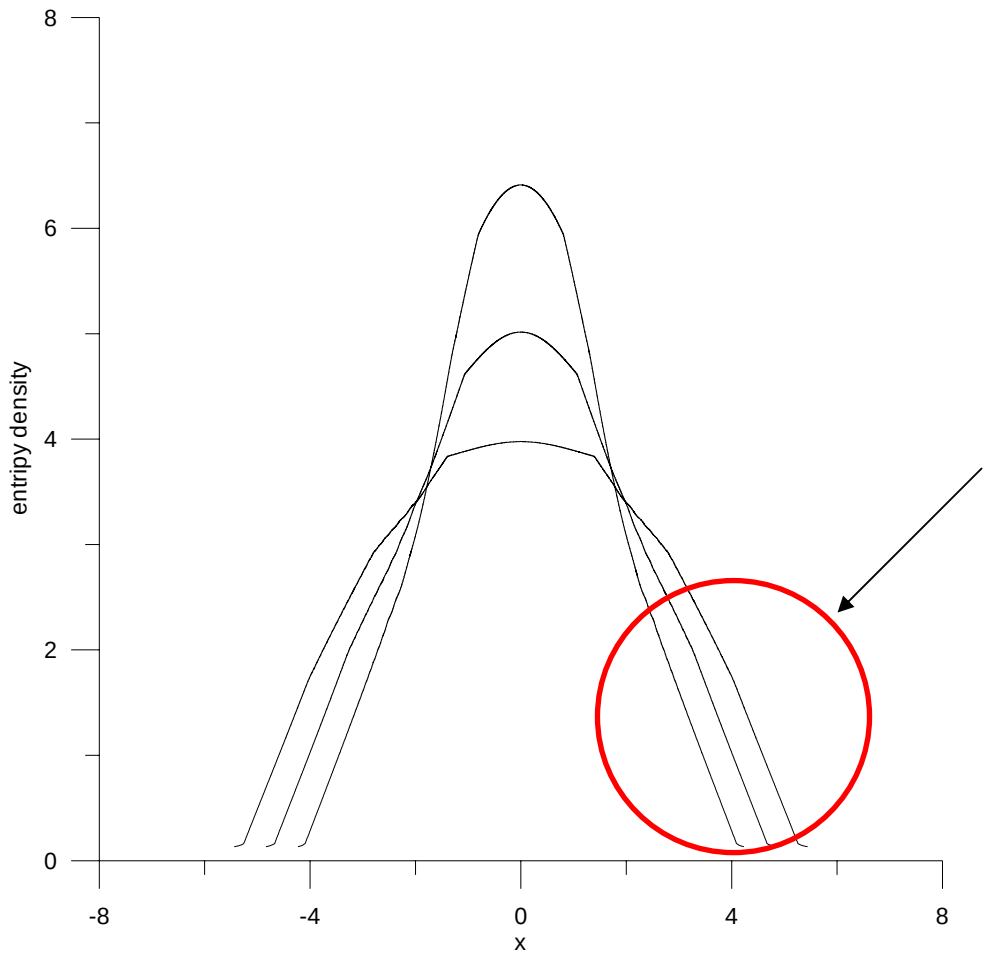
# Common sense?

- The transport coefficients are given by the Green-Kubo-Nakano formula.
- The lower bound of the shear viscosity of the CD hydro. is given by $1/(4\pi)$.
- The viscosity is small correction.

In the following calculations,
we consider only the 1+1 dimensional fluid.

# Expansion of fluid




It seems to be
a stationary wave.

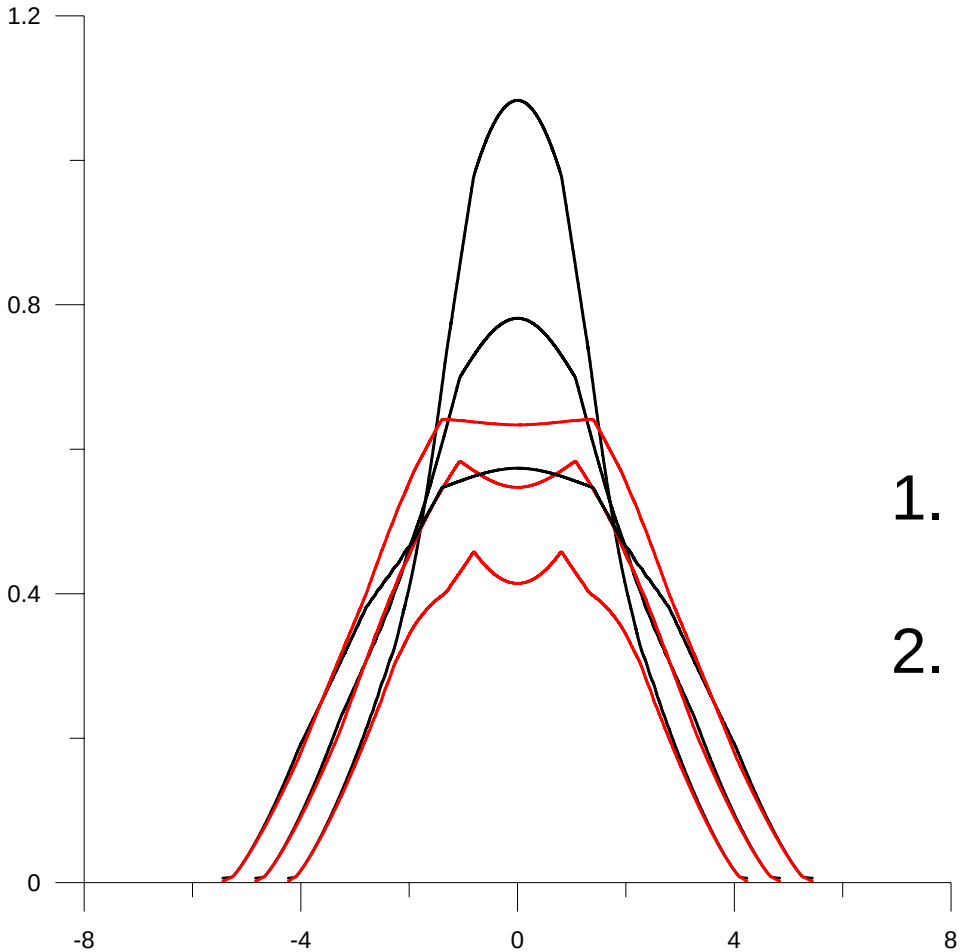
# Universal relation between pressure and viscosity

We assume that the fluid forms a stationary wave at the boundary to vacuum. Then at the boundary,

$$\alpha\left(T^{00}+T^{11}\right)=vT^{01}(1+v^2)$$

$$T^{11}=v^2T^{00}$$

$$\Rightarrow \quad P=-\Pi$$




pressure

(-1) * viscosity

1. The causal fluid expands as a stationary wave.
2. Viscosity can be the same order as energy density and pressure.

# Common sense?

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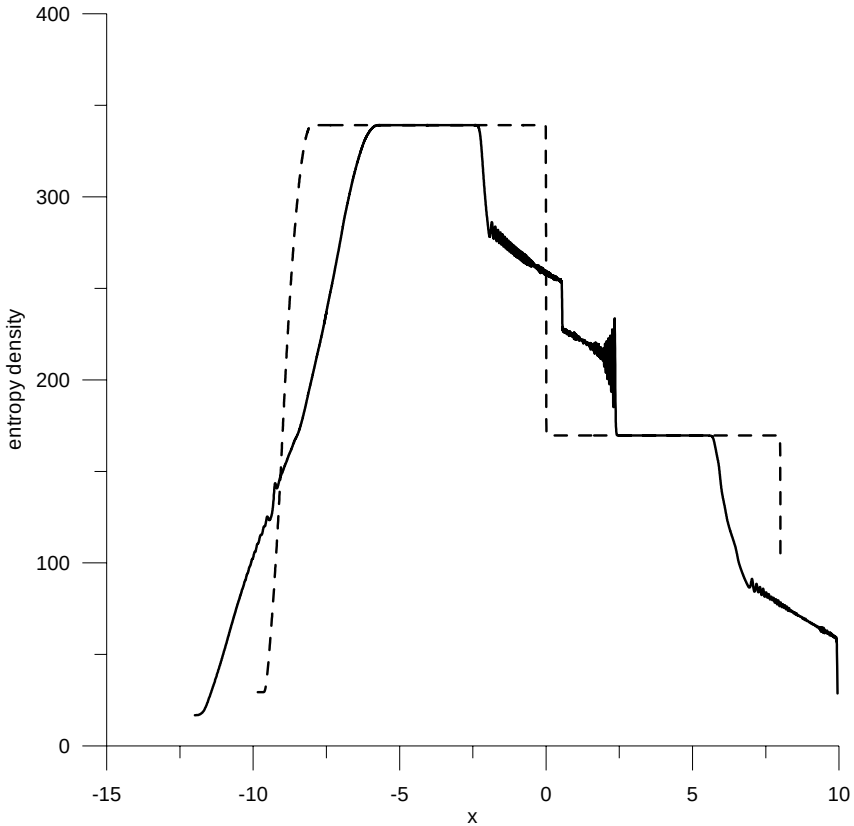
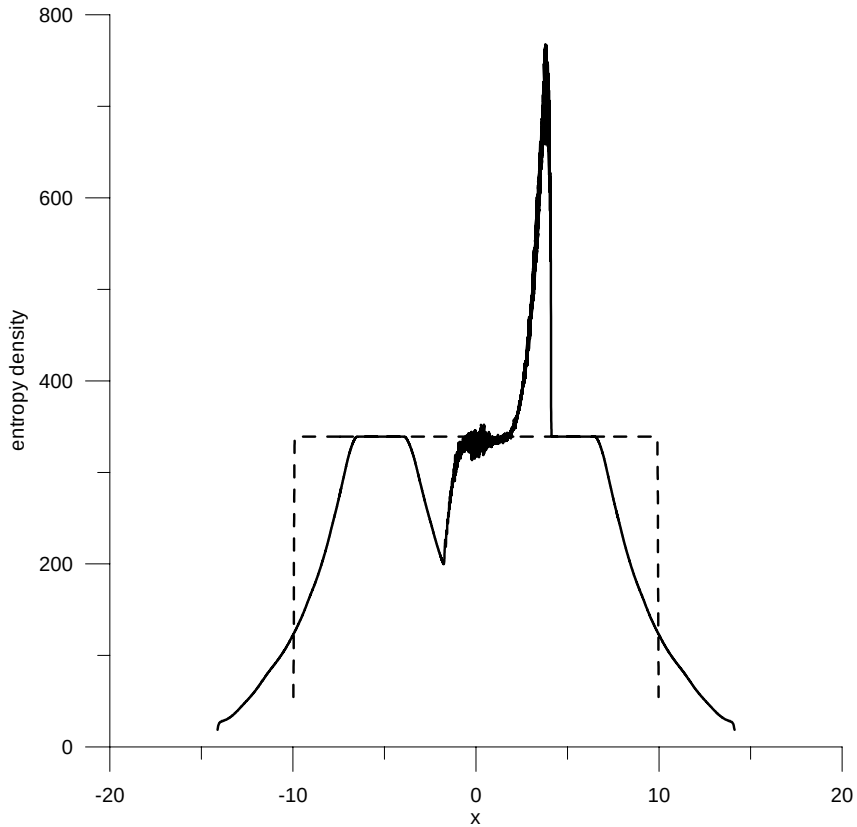
- The lower bound of the shear viscosity of the CD hydro. is given by $1/(4\pi)$.
- The viscosity is small correction.
- We do not need any additional viscosity in the CD hydro.

# Numerical oscillation




The frequency is the size of the grid.

unstable

# Usual “artificial” viscosity

Navier-Stokes type: $\partial^2 u / \partial x^2$

Burnett type: $\left(\partial u / \partial x\right)^2$

Neumann-Richtmyer type: $\left(\partial u / \partial x\right)\theta(-\partial u / \partial x)$

Generalization to relativistic cases ?

Causal?

# Additional causal viscosity

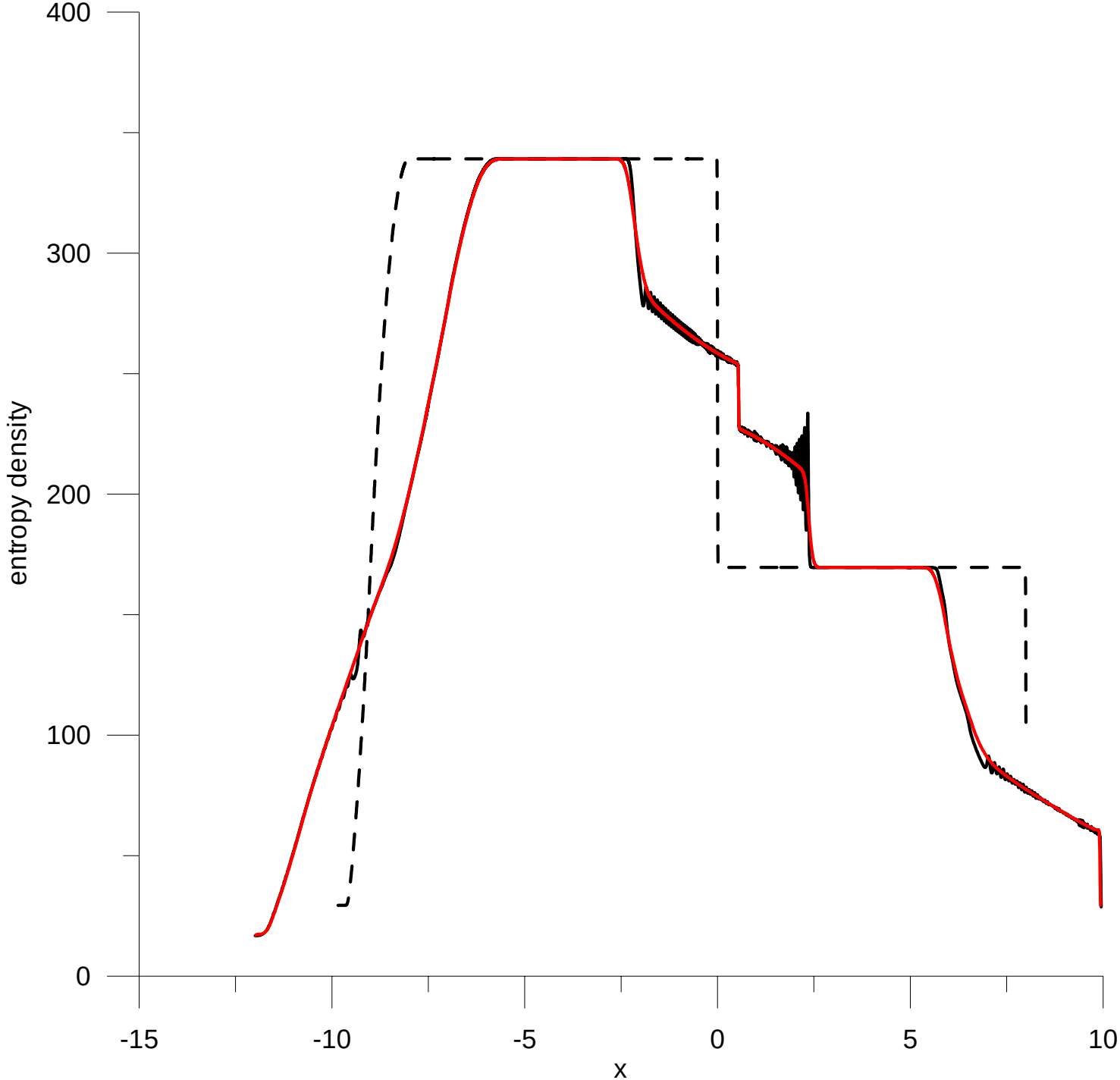
We need to introduce the artificial viscosity consistent with causality.

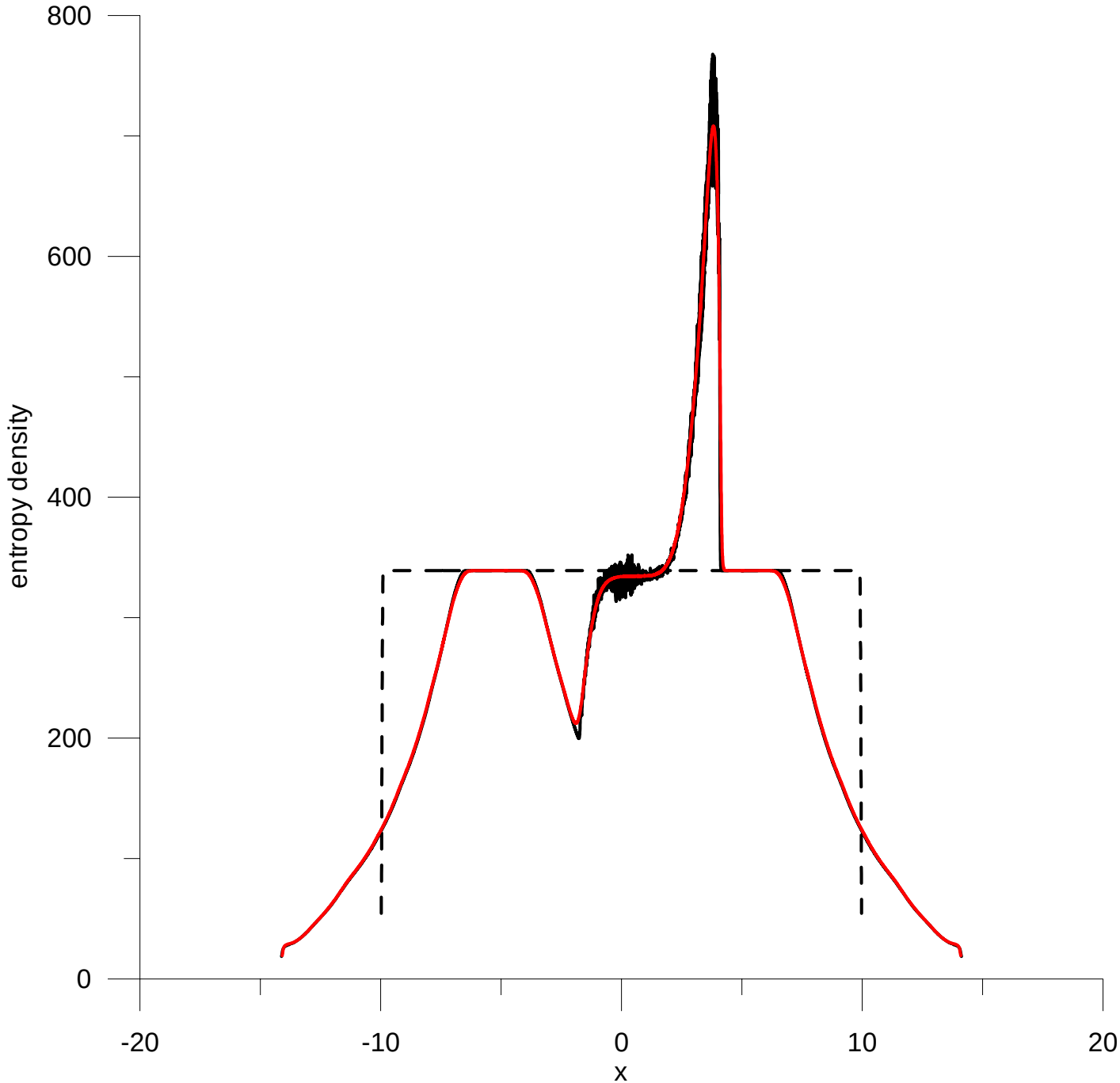
$$\Pi_{total} = \Pi + \Pi_{av}$$

$$\tau_{av} \frac{d}{d\tau} \Pi_{av} + \Pi_{av} = -\eta_{av} \partial_{\mu} u^{\mu}$$

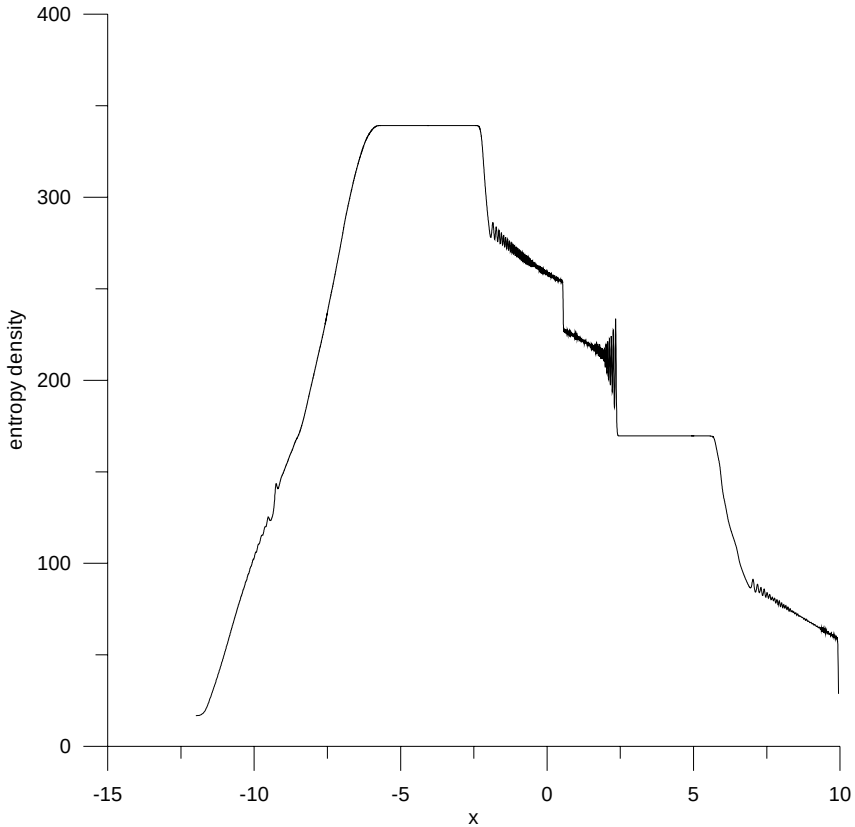
$$\eta_{av} \propto hP$$

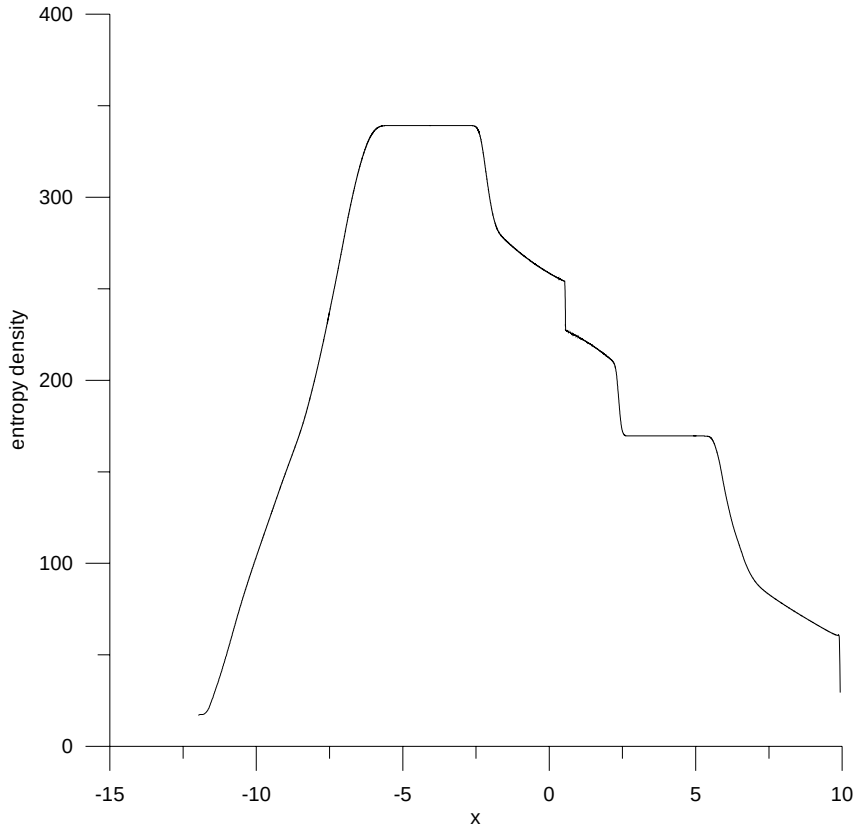
$h$ :the size of the grid (0.01 fm)


400
300
200
100
0
entropy density
-15
-10
-5
0
5
10
x

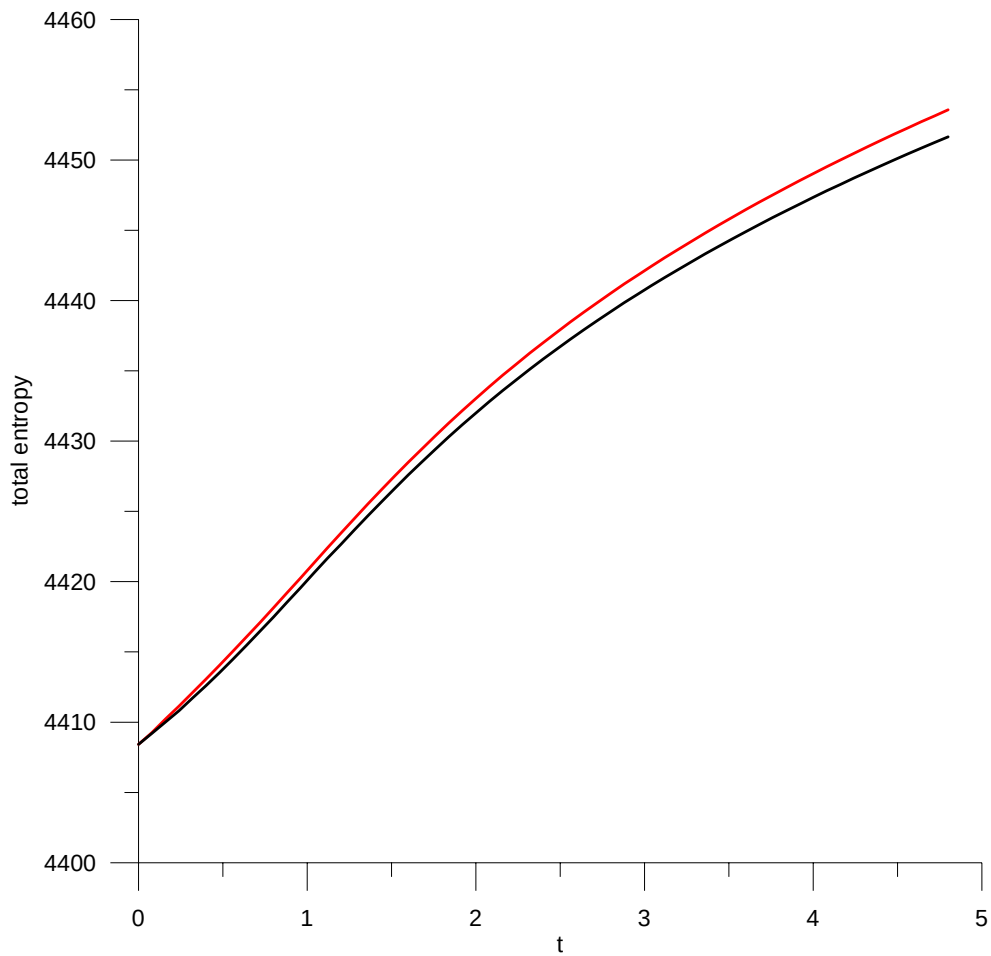

800
600
400
200
0
entropy density
-20
-10
0
10
20
x

# Additional viscosity



# Effect to entropy production




The effect of
the additional viscosity is

0.05%

This is the same order of
the breaking of
the energy conservation
in our code.

# Summary

- The transport coefficients are given by the Green-Kubo-Nakano formula.

  No. We need a new formula for no-Newtonian fluids

- The lower bound of the shear viscosity of the CD hydro. is given by $1/(4\pi)$ .

  Not yet clear for causal dissipative hydro.

- The viscosity is small correction.

  No. We have a relation $P = -\Pi$ and forms a stationary wave.

- We do not need any additional viscosity in the CD hydro.

  We need.

# We have to check

- Is the new formula really useful or not.
- What is the lower bound of the causal shear viscosity?
- CD hydro is a stable theory?
- Are numerical oscillations we found due to the problem of numerical calculation or the problem of the theory (turbulence)?

# That is, the causal hydrodynamics will be one of non-Newtonian fluids!!

Newtonian

Non-Newtonian

# Stability and Causality(1+1)

T.K. Denicol,Kodama&Mota ('07)

Linear perturbation analysis of the causal hydro.

$$\omega^3 - \frac{i}{\tau_R}\omega^2 - \left(\frac{\varsigma}{\tau_R}\frac{1}{w_0} + \alpha\right)k^2\omega + i\frac{\alpha}{\tau_R}k^2 = 0 \qquad w_0 = \varepsilon + P$$

$$\Rightarrow \quad \omega = k\sqrt{\frac{\varsigma}{\tau_R}\frac{1}{w_0} + \alpha} + \frac{i}{3\tau_R} \qquad \alpha = \frac{\partial P}{\partial \varepsilon}$$

$$v = \lim_{k\to\infty}\frac{\mathrm{Re}[\omega]}{k} = \sqrt{\frac{\varsigma}{\tau_R}\frac{1}{w_0} + \alpha}$$

Sound velocity is faster than ideal.

$$\mathrm{Im}\,\omega = \frac{1}{3\tau_R} > 0$$

Stable, Independent of momentum

# It is difficult to cut high momentum modes to avoid unstable modes.

Model equation of hydro

Non-linear effect

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \ldots$$

assumption

$$u = e^{ik(x-ut)}$$

$$\frac{\partial u}{\partial t} = -ike^{i2k(x-ut)}$$

$$k \approx k_{cri}$$

$$2k \gg k_{cri}$$

We need to find a new theory to satisfy causality.