Lecture 2: Basics of Field Theory at non-zero temperature

1) Quantum statistical mechanics

2) Scalar field theory at $T>0$ diagrams and IR divergences

3) Fermion and gauge fields at non-zero temperature and density

Quantum Statistical mechanics

Transition amplitude in QM and its path integral represenation

$$F(q', t'; q, t) = \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle$$

$t \to -i\tau,\ t' \to -i\tau$ (imaginary time)

$$F(q' - i\tau'; q, -i\tau) = \langle q'|e^{-\hat{H}(\tau'-\tau)}|q\rangle$$

$$\hat{H} = \frac{1}{2}p^2 + V(q)$$

$$F(q', -i\tau'; q, -i\tau) = \int \mathcal{D}q \exp\left[-\int_{\tau}^{\tau'} d\tau'' \left(\frac{1}{2}\dot{q}^2(\tau'') + V(q(\tau''))\right)\right]$$

$$q(\tau) = q,\ q(\tau') = q'$$

Partition function in statistical mechanics:

$$Z(\beta) = \mathrm{Tr} e^{-\beta\hat{H}},\ \beta = 1/T$$

$$Z(\beta) = \sum e^{-\beta E_n},\quad \hat{H}|n\rangle = E_n|n\rangle$$

$$Z(\beta) = \int dq \langle q | e^{-\beta \hat{H}} | q \rangle$$

$$Z(\beta) = \int dq F(q, -i\beta; q, 0)$$

$$\Downarrow$$

$$Z(\beta) = \int \mathcal{D}q(\tau) \exp\left[-\int_0^\beta d\tau \left(\frac{1}{2}\dot{q}^2(\tau) + V(q(\tau)) \right) \right],$$

$$q(\beta) = q(0)$$

Euclidean action $S_E(\beta) = \int_0^\beta d\tau \left(\frac{1}{2}\dot{q}^2(\tau) + V(q(\tau)) \right)$

We can also calculate the generating functional

$$Z(\beta; j) = \int \mathcal{D}q \exp\left[-S_E(\beta) + \int_0^\beta j(\tau) q(\tau) d\tau) \right]$$

$$\Delta(\tau_1, \tau_2) = \frac{1}{Z(\beta)} \frac{\delta^2 Z(\beta; j)}{\delta j(\tau_1) \delta j(\tau_2)} |_{j=0} = \frac{1}{Z(\beta)} \int \mathcal{D}q q(\tau_1) q(\tau_2) e^{-S_E(\beta)}$$

Correlation function in real and imaginary time in the operator formalism:

$$\hat{q}(-i\tau) = e^{\hat{H}\tau}\hat{q}e^{-\hat{H}\tau}$$

$$\hat{q}(t) = e^{i\hat{H}t}\hat{q}e^{-i\hat{H}t}$$

$$\Delta(\tau_1,\tau_2) = \langle T\hat{q}(-i\tau_1)\hat{q}(-i\tau_2)\rangle_\beta = \frac{1}{Z(\beta)}\mathrm{Tr}\left[T\hat{q}(-i\tau_1)q(-i\tau_2)e^{-\beta H}\right]$$

$$\Delta(\tau) = \Delta(\tau,0) = \Delta(\tau-\beta)$$

$$D^{>}(t,t') = \langle\hat{q}(t)\hat{q}(t')\rangle_\beta$$

$$D^{<}(t,t') = \langle\hat{q}(t')\hat{q}(t)\rangle_\beta$$

$$D_R(t,t') = \langle\theta(t-t')[\hat{q}(t),\hat{q}(t')]\rangle_\beta$$

$$e^{-\beta\hat{H}}\hat{q}(t)e^{\beta\hat{H}} = \hat{q}(t+i\beta) \Rightarrow D^{>}(t,t') = D^{<}(t+i\beta,t')$$

Kubo-Martin-Schwinger (KMS) condition

$$\boxed{\Delta(\tau) = D^{>}(-i\tau,0)}$$

Different correlation functions $\Delta(\tau)$, $D^{>}(t)$, $D^{<}(t)$ and $D_R(t)$ are related to the spectral function $\sigma(k_0)$

$$D^{>}(k_0) = \int_{-\infty}^{\infty} dt e^{ik_0 t} D^{>}(t)$$

$$D^{<}(k_0) = \int_{-\infty}^{\infty} dt e^{ik_0 t} D^{<}(t) = \int_{-\infty}^{\infty} dt e^{ik_0 t} D^{>}(t - i\beta) = e^{-\beta k_0} D^{>}(k_0)$$

$$\boxed{\sigma(k_0) = \frac{D^{>}(k_0) - D^{<}(k_0)}{2\pi} = \frac{1}{\pi} Im D_R(k_0)}$$

$$\Downarrow$$

$$D^{>}(k_0) = (1 + f(k_0))\sigma(k_0), \;\; f(k_0) = (e^{-\beta k_0} - 1)^{-1}$$

$$\Downarrow$$

$$\boxed{\Delta(\tau) = \int_0^{\infty} dk_0 \sigma(k_0) \frac{\cosh(k_0 \cdot (\tau - \beta/2))}{\sinh(\beta k_0/2)}}$$

$$\sigma(k_0) = \frac{1}{Z(\beta)} \sum_{n,m} e^{-\beta E_n} \left[\delta(k_0 + E_n - E_m) - \delta(k_0 + E_m - E_n)\right] |\langle n|\hat{q}|m\rangle|^2$$

$$\sigma(k_0) = -\sigma(-k_0), \quad sgn(k_0)\sigma(k_0) > 0$$

Thermodynamics of scalar field theory

Straightforward generalization to infinite number of degrees of freedom $q(t) \to \phi_x(t) \equiv \phi(t,x)$

$$L = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{g^2}{4!}\phi^4$$

$$\Downarrow$$

$$S_E(\beta) = \int_0^\beta d\tau \int d^3x \left(\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\partial_i \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g^2}{4!}\phi^4\right)$$

$$Z(\beta; j) = \int \mathcal{D}\phi \exp(-S_E(\beta) + \int_0^\beta d\tau \int d^3x j(\tau,x)\phi(\tau,x))$$

$$\phi(0,x) = \phi(\beta,x)$$

Free field limit ($g = 0$):

$$Z(\beta; j) = \int \mathcal{D}\phi \exp\left[-\int d^4x_E \frac{1}{2}\phi(-\partial_\tau^2 - \nabla^2 + m^2)\phi + \int_0^\beta d^4x_E j(x_E)\phi(x_E)\right]$$

$$x_E = (\tau, x)$$

Gaussian integration:

$$Z_0(\beta; j) = Z(\beta) \exp\left[\int_0^\beta d^4x_E dy_E \; j(x_E)\Delta_0(x_E - y_E)j(y_E)\right]$$

$$Z(\beta) = (det\Delta_0)^{1/2} = \mathrm{Tr}\ln\Delta_0$$

$$\left[-\partial_\tau^2 - \nabla^2 + m^2\right]\Delta_0(x_E - y_E) = \delta(\tau_x - \tau_y)\delta(x - y)$$

$$\Downarrow$$

$$(\omega_n^2 + k^2 + m^2)\Delta_0(i\omega_n, k) = (\omega_n^2 + \omega_k^2)\Delta_0(i\omega_n, k) = 1$$

$$\omega_n = 2\pi T n, \;\; \omega_k^2 = k^2 + m^2$$

$$\Downarrow$$

$$\Delta_0(i\omega_n, k) = \frac{1}{\omega_n^2 + \omega_k^2}$$ -Matsubara propagator

Mixed (Saclay) representation:

$$\Delta_0(\tau, k) = T\sum_n e^{-i\omega_n\tau}\Delta_0(i\omega_n, k)$$

$$[-\partial_\tau^2 + \omega_k^2]\Delta_0(\tau, k) = \delta(\tau_x - \tau_y), \;\; \Delta_0(\tau - \beta) = \Delta(\tau)$$

$$\rightarrow \Delta_0(\tau) = \frac{1}{2\omega_k}((1 + f(\omega_k))e^{-\omega_k\tau} + f(\omega_k)e^{\omega_k\tau}), f(\omega_k) = (e^{\beta\omega_k} - 1)^{-1}$$

$$\ln Z(\beta) = \frac{1}{2}\mathrm{Tr}\ln\Delta_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_k \ln\beta^2\Delta(i\omega_n,k) =$$

$$-\frac{1}{2}\sum_{n=-\infty}^{\infty} V\int\frac{d^3k}{(2\pi)^3}\ln\beta^2[\omega_n^2+\omega_k^2] =$$

$$\sum_n \frac{d}{d\omega_k^2}\ln[\omega_n^2+\omega_k^2] = \sum_n\frac{1}{\omega_n^2+\omega_k^2} = \beta\Delta_0(\tau=0,k) = \frac{\beta}{2\omega_k}(1+2f(\omega_k))$$

$$\Downarrow$$

$$\sum_n \ln\beta^2(\omega_n^2+\omega_k^2) = \beta\omega + 2\ln(1+e^{-\beta\omega_k}) + const$$

$$\ln Z(\beta) = -V\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\beta\omega_k + \ln(1-e^{-\beta\omega_k})\right]$$

$$F(T,V) = T\ln Z(\beta),\ p = -\partial F(T,V)/\partial V,\ S = -\frac{\partial F(T,V)}{\partial T}, U = F+TS$$

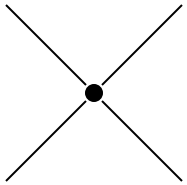
Massless case ($m=0\rightarrow\omega_k=k$):

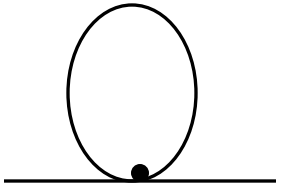
$$p = \int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}k + T\ln(1-e^{-\beta k}))\right] = \frac{\pi^2T^4}{90}$$

$$\epsilon(T) = U(T,V)/V = 3p,\ \ s(T) = S(T,V)/V = 4/3\epsilon(T)$$

Perturbative expansion:

$$e^{-S_E(\beta)} \simeq e^{-S_E^0(\beta)} \left(1 - \frac{g^2}{4!} \int d^4x_E \phi^4(x_E)\right)$$



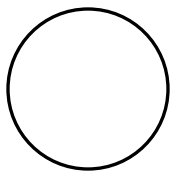
$\Pi =$ 

$$= \frac{g^2}{2} T \sum_n \int \frac{d^3k}{(2\pi)^3} \Delta_0(i\omega_n, k)$$

$$= \frac{g^2}{2} T \int \frac{d^3k}{(2\pi)^3} \Delta_0(\tau = 0, k)$$

$$= \frac{g^2}{2} T \int \frac{d^3k}{(2\pi)^3} (\frac{1}{2}\omega_k + n(\omega_k))$$

Massless case $(\omega_k = k)$: $\Pi = \dfrac{g^2 T^2}{24}$



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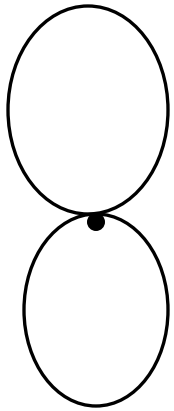
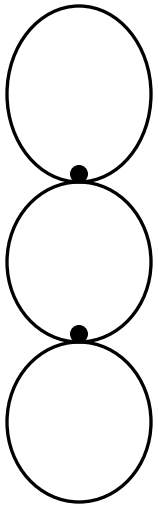


$$T\sum_n \int \frac{d^3k}{(2\pi)^3} \ln \Delta_0(i\omega_n, k) \quad + \quad \frac{g^2}{8}\left(T\sum_n \int \frac{d^3k}{(2\pi)^3} \Delta_0(i\omega_n, k)\right)^2$$

Massless case:

$$P = \frac{\pi^2 T^4}{90}\left(1 - \frac{5g^2}{64\pi^2}\right)$$

Infrared problems at finite temperature: the next-to-leading correction to the pressure is not of order λ^2 but $\lambda^{3/2}$ from $m = 0$!



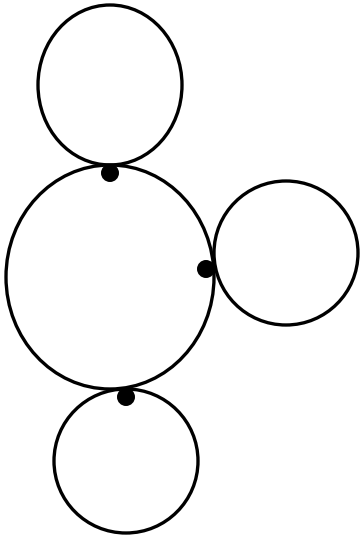
$$\frac{g^4}{16}\left(T\sum_l \int \frac{d^3p}{(2\pi)^3}\frac{1}{(p^2+\omega_l^2)^2}\right)\left(T\sum_n \int \frac{d^3k}{(2\pi)^3}\Delta_0(i\omega_n,k)\right)^2$$

the $l = 0$ term is IR divergent as $\int d^3p/p^4$

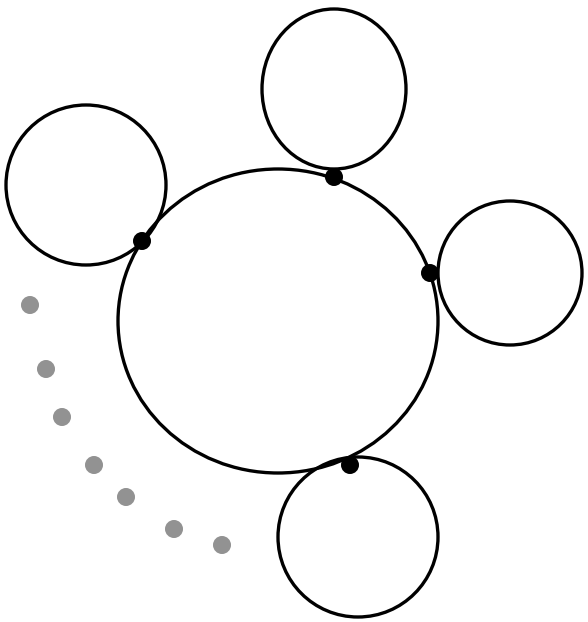
In the 4-loop diagram

$$\sim g^6\left(T\sum_l \int \frac{d^3p}{(2\pi)^3}\frac{1}{(p^2+\omega_l^2)^3}\right)\left(T\sum_n \int \frac{d^3k}{(2\pi)^3}\Delta_0(i\omega_n,k)\right)^3$$

the $l = 0$ term is IR divergent as $\int d^3p/p^6$



We need to resum all diagrams of the following type (ring diagrams)



$$= -\frac{1}{2} VT \sum_n \int \frac{d^3p}{(2\pi)^3} \sum_{N=2}^{\infty} \frac{1}{N} \left(\frac{(-\Pi)}{(\omega_n^2 + p^2)} \right)^N$$

$$= -\frac{1}{2} VT \sum_n \int \frac{d^3p}{(2\pi)^3} \left(\ln \left(1 + \frac{\Pi}{p^2 + \omega_n^2} \right) - \frac{\Pi}{p^2 + \omega_n^2} \right)$$

keeping only the contribution from (IR sensitive) $n = 0$ and using $\Pi = g^2 T^2 / 24$ we get

$$F_{ring} = \frac{VT^4}{12\pi} \left(\frac{g^2}{24} \right)^{3/2}$$

Collective effects in the medium have to be taken into account at all orders !

$$P = \frac{\pi^2 T^4}{90} \left(1 - \frac{15}{8} \left(\frac{g^2}{24\pi^2} \right) + \frac{15}{2} \left(\frac{g^2}{24\pi^2} \right)^{3/2} + \ldots \right)$$

Resummation and screened perturbation theory

$$\mathcal{P} = \mathcal{P}_{\rm ideal}\left[1 - \frac{5}{4}\alpha + \frac{5\sqrt{6}}{3}\alpha^{3/2} + \frac{15}{4}\left(\log\frac{\mu}{2\pi T} + 0.40\right)\alpha^2 - \frac{15\sqrt{6}}{2}\left(\log\frac{\mu}{2\pi T} - \frac{2}{3}\log\alpha - 0.72\right)\alpha^{5/2} + \mathcal{O}(\alpha^3\log\alpha)\right]$$

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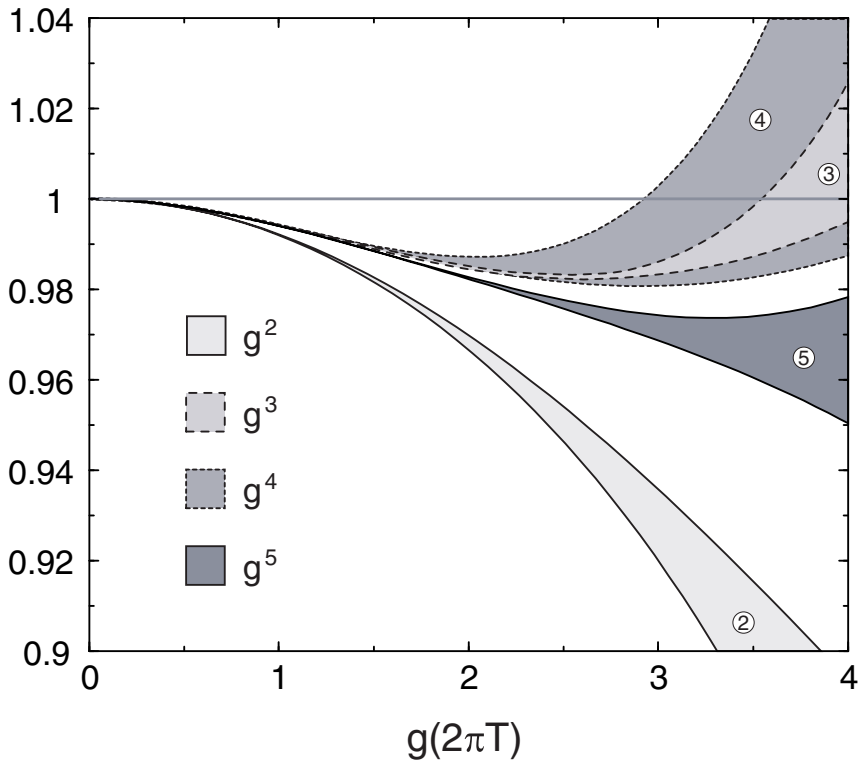
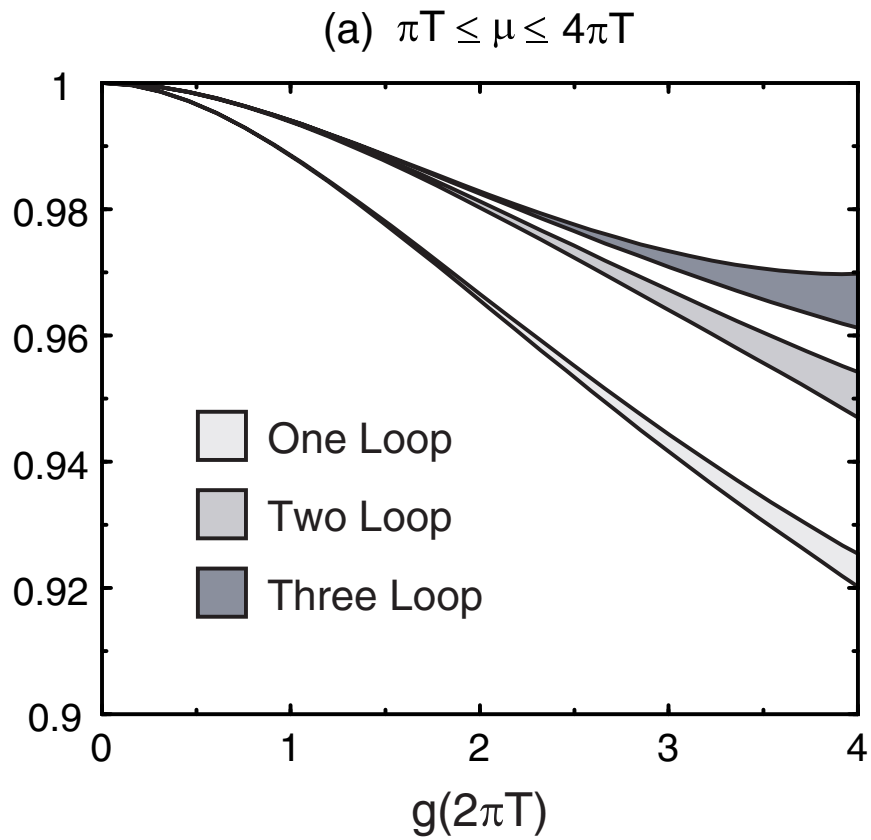
$\alpha = g^2/(16\pi^2)$





Karsch, Patkos, PP, PLB 401 (97)69 , Andersen, Braaten, Strickland PRD63 (01) 105008

Homework:

Prove the integral equation :

$$\Delta(\tau) = \int_0^\infty dk_0 \sigma(k_0) \frac{\cosh(k_0 \cdot (\tau - \beta/2))}{\sinh(\beta k_0/2)}$$

Show that:

$$\sigma(k_0) = \frac{1}{Z(\beta)} \sum_{n,m} e^{-\beta E_n} \left[\delta(k_0 + E_n - E_m) - \delta(k_0 + E_m - E_n)\right] |\langle n|\hat{q}|m\rangle|^2$$

use relation of $\sigma(k_0)$ and $D^{>,<}(k_0)$ and insert a complete set of energy eigenstates into $D^{>,<}(t)$)

Prove the sum rule

$$\int_{-\infty}^{\infty} k_0 \sigma(k_0) dk_0 = 1$$

For questions see me in my office, E6-124