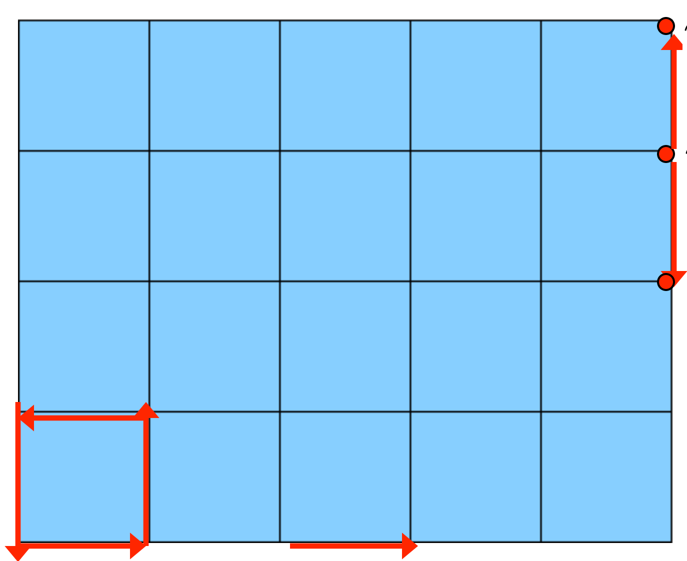
Lecture 3: basics of lattice QCD

- Scalar fields on the lattice
- Lattice regularization and gauge symmetry : Wilson gauge action, fermion doubling
- Different fermion formulations
- Meson correlation function and Wilson loops
- Scale setting, continuum limit and lines of constant physics (LCP)

Quarks and gluon fields on a lattice



$\bar{\psi}(x+\nu)$

$\psi(x)$ $\Rightarrow$ $\bar{\psi}\ \gamma_\nu D_\nu\ \psi$

$\bar{\psi}(x-\nu)$

$g = 0$:

$$\hat{\partial}_\nu \psi(x) = \tfrac{1}{2}[\psi(x+\nu) - \psi(x-\nu)]$$

$$\langle \psi(x)\bar{\psi}(x)\rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \frac{-i\sum_\nu \gamma_\nu \hat{p}_\nu + m}{\sum_\nu \hat{p}_\nu^2 + m^2}$$

$$U_\mu(x) = \exp(igaA_\mu(x))$$

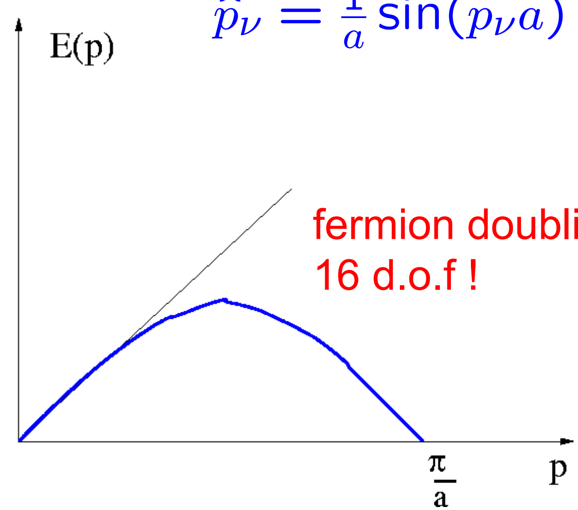
$$U_\mu(x) \simeq 1 + igaA_\mu(x)$$

$$U_P(x) = U_{\mu\nu}(x) = S^{1\times 1}_{\mu\nu}(x) =$$
$$U_\mu(x)U_\nu(x+\mu)U^\dagger_\mu(x+\mu+\nu)U^\dagger_\nu(x)$$

$$S_{Wilson} \ = \ \beta \sum_x (1 - \frac{1}{3} Re\ trU_P(x)),\ \beta = \frac{6}{g^2}$$

$$S_{Wilson}|_{a\to 0} = \int d^4x\ trF^2_{\mu\nu}$$

$$\hat{p}_\nu = \frac{1}{a}\sin(p_\nu a)$$



fermion doubling !
16 d.o.f !

Wilson fermions

$$S_f^W = \int_x \left[\bar{\psi}\ \gamma_\nu D_\nu\ \psi - a\frac{r}{2}\bar{\psi}\ \Box\ \psi - m\bar{\psi}\psi \right]$$ Wilson (1975)

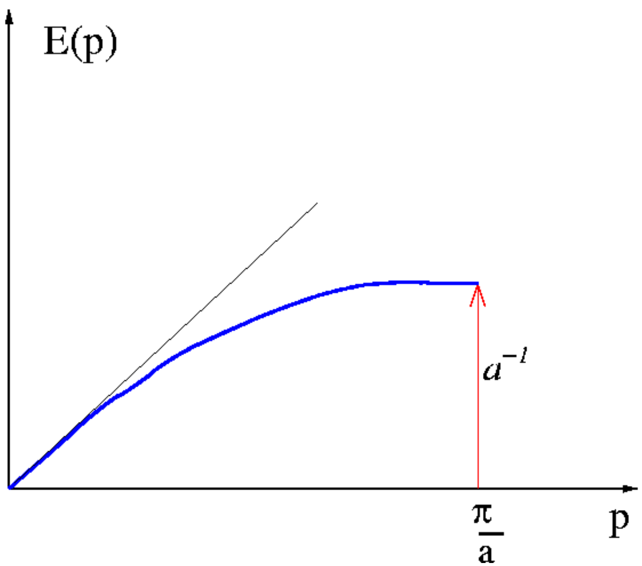
$$S_f^W = \int_x \bar{\psi}\ D^W\ \psi, \quad \int_x = \sum_x a^4$$

$$D^w(x,y) = \delta_{x,y}(4+m) + \sum_\mu (1+\gamma_\mu)\delta_{x+\mu,y} - (1-\gamma_\mu)\delta_{x-\mu,y}$$

$U_\mu(x)$ $\quad$ $U_\mu^\dagger(x)$

$$\langle \psi(x)\bar{\psi}(x) \rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \frac{-i\sum_\nu \gamma_\nu \hat{p}_\nu + m'(p)}{\sum_\nu \hat{p}_\nu^2 + m'^2(p)}$$

$$m'(p) = m + \frac{2r}{a}\sum_\mu \sin^2\left(\frac{p_\mu a}{2}\right)$$



$$SU(3)_V \times SU(3)_A \to SU(3)_V$$

chiral symmetry is broken even in the massless case !

⇓

additive mass renormalization

⇓

Wilson Dirac operator is not bounded from below

⇓

difficulties in numerical simulations

Discretization errors ~ $a\, g^2$, used for study of hadron properties, spectral functions

Staggered fermions

$$\psi(x) = T(x)\chi(x), \quad \bar{\psi}(x) = \bar{\chi}T^\dagger(x)$$

Kogut, Susskid (1975)

$$T(x)\gamma_\mu T^\dagger(x+\mu) = 1 \cdot \eta_\mu(x)$$

$$\eta_\mu(x) = (-1)^{x_1+\ldots+x_{\mu-1}}, \eta_1(x) = 1$$

$$T(x) = \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\gamma_4^{x_4}$$

$$S_f^{stagg} = \sum_x \sum_\alpha [\eta_\mu(x)\bar{\chi}_\alpha(x)\hat{\partial}_\mu \chi_\alpha(x) + m\bar{\chi}_\alpha(x)\chi_\alpha(x)]$$

$\Downarrow$ omit index α and the sum (16 $\rightarrow$ 4)

$$\sum_{x,y}[\bar{\chi}(x)D_{stagg}(x,y)\chi(y)], \quad D_{stag}(x,y) = \delta_{x,y}m + \sum_\mu \eta_\mu(x)(\delta_{x+\mu,y} - \delta_{x-\mu,y})$$

different flavors, spin componets sit in different corners of the Brillouin zone or in 2^4 hypercube

$\sum_{x,y}[\bar{\chi}(x)D(x,y)\chi(y)] \rightarrow$ 4-flavor theory

$$U(4)_V \times U(4)_A \rightarrow U_{e-o}(1) \subset SU_A(4)$$

$$\chi(x) \rightarrow U_o\chi(x), \quad \bar{\chi}(x) \rightarrow \bar{\chi}(x)U_e^\dagger, \quad \sum_{i=1}^{4} x_i \text{ even}$$

$$\chi(x) \rightarrow U_e\chi(x), \quad \bar{\chi}(x) \rightarrow \bar{\chi}(x)U_o^\dagger, \quad \sum_{i=1}^{4} x_i \text{ odd}$$

$$\bar{\chi}(x)(\chi(x+\mu) - \chi(x-\mu)) \rightarrow \bar{\chi}(x)U_e^\dagger U_e(\chi(x+\mu) - \chi(x-\mu))$$

$$m\bar{\chi}(x)(\chi(x) \rightarrow m\bar{\chi}(x)U_e^\dagger U_o\chi(x)$$

$||D_{stagg}|| > m$ useful in numerical simulations !

Chiral fermions on the lattice ?

$$S_F = a^4 \sum_{x,y} \bar{\psi}(x) D(x-y) \psi(y)$$

We would like the following properties for the lattice Dirac operator:

1. $D(x)$ should be local, i.e. $||D(x)|| \leq C \exp(-\gamma x)$

2. $D(p) = i \sum_\mu \gamma_\mu p_\mu + O((ap)^2)$ (cubic symmmetry)

3. no doubler exist, i.e. $D(p)$ is invertible for $p \neq 0$

4. $\gamma_5 D + D \gamma_5 = 0$ (chiral symmetry)

Nielsen-Ninomiya no-go theorem :
conditions one 1-4 cannot be satisfied simultaneously
Nielsen, Ninomiya (1981)

Wilson fermion formulation gives up 4)
Staggered fermion formulation gives up 3)

Ginsparg-Wilson fermions

$$\gamma_5 D + D\gamma_5 = 0$$

$$\Downarrow$$

$$\gamma_5 D^{-1} + D^{-1}\gamma_5 = a2R\gamma_5$$

$$\gamma_5 D + D\gamma_5 = aD2R\gamma_5 D$$

R local operator ($R = 1/2$ in most of applications)

Ginsparg, Wilson (1982)

- anti-commutation properties are recovered in the continuum limit (a->0)
- the r.h.s. of the Ginsparg-Wilson relation is zero for the solutions

$\downarrow$

mildest way to break the chiral symmetry on the lattice : physical consequences of the chiral symmetry are mantained (e.g. chiral perturbation theory)

Generalized chiral symmetry and topology

$$\psi(x) \to \psi(x) + \delta\psi(x), \bar{\psi}(x) \to \bar{\psi}(x) + \delta\bar{\psi}(x)$$

$$\delta\psi_i = iT^a_{ij}\phi^a\gamma_5(1 - a\tfrac{1}{2}D)\psi_j(x) \ \ flavor \ \ non - singlet$$

$$\delta\psi_i = \gamma_5(1 - a\tfrac{1}{2}D)\psi_i(x) \ \ flavor \ \ singlet$$

$$\delta\bar{\psi}_i = \bar{\psi}_j\gamma_5(1 - a\tfrac{1}{2}D)i\phi^a T^a_{ij} \ \ flavor \ \ non - singlet$$

$$\delta\bar{\psi}_i = \bar{\psi}_i\gamma_5(1 - a\tfrac{1}{2}D) \ \ flavor \ \ singlet$$

GW relation ⇒ $\delta(\bar{\psi}D\psi) = 0$ Luescher (1998)

flavor singlet transformation :

$$\delta[d\bar{\psi}d\psi] = \mathrm{Tr}(a\gamma_5 D)[d\bar{\psi}d\psi] = 2N_f(n_- - n_+)[d\bar{\psi}d\psi]$$

$$index(D) = n_- - n_+$$

$$q(x) = \tfrac{1}{2}\mathrm{tr}\gamma_5 D(x,x) \text{ -topological charge density}$$

Hasenfratz, Laliena, Niedermeyer (1998)

for flavor non-singlet transformation $\mathrm{Tr}(a\gamma_5 D T^a) = 0$ no anomaly !

Constructing chiral fermion action I

Overlap fermions :

$$A = 1 - aD_w$$
$$D = \frac{1}{a}\left(1 - \frac{A}{\sqrt{AA^\dagger}}\right)$$

Neuberger (1998)

using $\gamma_5 D_w \gamma_5 = D_w^\dagger$ it can be shown that

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

GW relation with *R=1/2*

no $\mathcal{O}(a^2)$ -discretization errors

large numerical cost due to evaluation of $(AA^\dagger)^{-1/2}$ (polynomial approximation), especially for gauge field configurations which change topology

costs $>$ 100$\times$ costs of Wilson formulations

Constructing chiral fermion action II

Domain wall fermions : introduce the fictitious 5th dimension of extent N_s :

$$S_{dwf} = -\sum_{x,y,s,s'} \bar{\psi}^5_{y,s'}(D_{x,y}\delta_{s,s'} + D_{s,s'}\delta_{x,y})\psi^5_{x,s}$$ Shamir (1993)

$$D_{x,y} = \frac{1}{2}\sum_\mu((1+\gamma_\mu)U_{x,\mu}\delta_{x+\mu,y} + (1-\gamma_\mu)U^\dagger_{x,\mu}\delta_{x-\mu,y} + (M-4)\delta_{x,y})$$

$$D_{s,s'} = (P_R\delta_{s+1,s'} + P_L\delta_{s-1,s'} - \delta_{s,s'}) + (P_R\delta_{1,s'} + P_L\delta_{N_s-2,s'}) - (mP_L\delta_{N_s-1,s'} + mP_R\delta_{0,s'} + \delta_{0,s'} + \delta_{N_s-1,s'})$$

$$P_{R,L} = (1 \pm \gamma_5)/2$$

$$\psi_x = \frac{1+\gamma_5}{2}\psi^5_{x,0} + \frac{1-\gamma_5}{2}\psi^5_{x,N_s-1}$$

$N_s \to \infty$ two chiral fermions bounded to the 4d walls

$$m_q = mM(M-2)$$

costs $= N_s \times$ costs of Wilson formulations $N_s = 16-32$

Extensively used in mumerical simulations : (see P. Boyle, 2007 for review)

QCD at finite baryon density

The naive continuum prescription of introducing chemical potential by adding a term $\mu \int d^4 x_E \bar{\psi}\gamma_0\psi$ does not work !

$$S = a^3 \sum_x \left[ma\bar{\psi}_x\psi_x + \mu a\bar{\psi}_x\gamma_0\psi_x + \frac{1}{2}\sum_\mu (\bar{\psi}_x\gamma_\mu\psi_{x+\mu} - \bar{\psi}_{x-\mu}\gamma_\mu\psi_x) \right]$$

$$\Downarrow$$

$\epsilon(\mu) \sim \mu^2/a^2$ instead of $\epsilon(\mu) \sim \mu^4$

The correct prescription is

$$U_0(x) \to e^{\mu a} U_0(x), \quad U_0^\dagger(x) \to e^{-\mu a} U_0^\dagger(x)$$

Hasenfratz, Karsch, PLB 125 (83) 308

$$S = (\overline{\psi}_x e^{\mu a} U_0(x)\psi_{x+0} - \overline{\psi}_x e^{-\mu a} U_0^+(x)\psi_{x-0}) + \sum_{x,i} \eta_i(x)\, (\overline{\psi}_x U_i(x)\psi_{x+i} - \overline{\psi}_x U_i^+(x)\psi_{x-i}) + am\sum_x \overline{\psi}_x\psi_x$$

$$\Downarrow$$

det M is complex => sign problem *det M exp(-S)* cannot be a probability

Meson correlators and Wilson loops

Meson states are created by quark bilenear operators:

$$J(x,y;\tau)\bar{\psi}(\tau,x)\Gamma\mathcal{U}(x,y)\psi(\tau,y)$$

Fixes the quantum number of of mesons, Γ is one
Of the Dirac matrices

Most often one considers point operators *x=y* and their correlation function:

$$G(\tau)=\langle J(\tau)J^\dagger(0)\rangle=\sum_{n=1}^{\infty} f_n^2 e^{-M_n\tau},\, f_n=|<0|J|n>|$$

decay constant

$$\tau\to\infty : G(\tau)\simeq f_1 e^{-M_1\tau}$$

Consider static quarks :

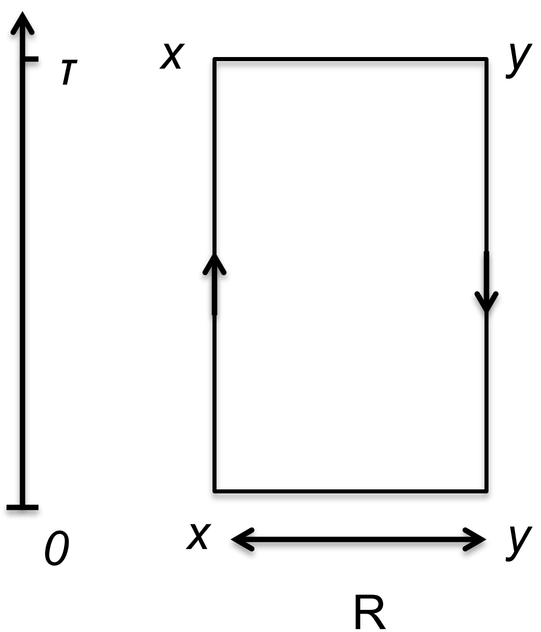
$$(-i\partial_\tau - gA_0(\tau,x))\,\psi(\tau,x)=0$$

$$\text{formal solution } \psi(\tau,x)=\mathcal{P}\exp\left(ig\int_0^\tau d\tau' A_0(\tau',x)\right)\psi(0,x)=L(\tau;x)\psi(0,x)$$

$$\text{lattice : } L(\tau;x)=\prod_{x_0=0}^{\tau} U_0(x,x_0)$$

Static meson correlation function functions after integrating out the static quark fields:

$$G(\tau) = \langle J(\tau)J^\dagger(0)\rangle = \mathrm{Tr}[L(\tau;x)\mathcal{U}(x,y)L^\dagger(\tau;y)\mathcal{U}^\dagger(x,y)] = W_C(\tau,R)$$



$$W_C(\tau,R) = \sum_{n=1}^{\infty} c_n e^{-E_n(R)\tau},$$

Static quark anti-quark potential

$\tau \to \infty$ ground state $E_1 = V(R)$ dominates

$$V(R) = -\alpha/R + \sigma R$$

confinement

String tension

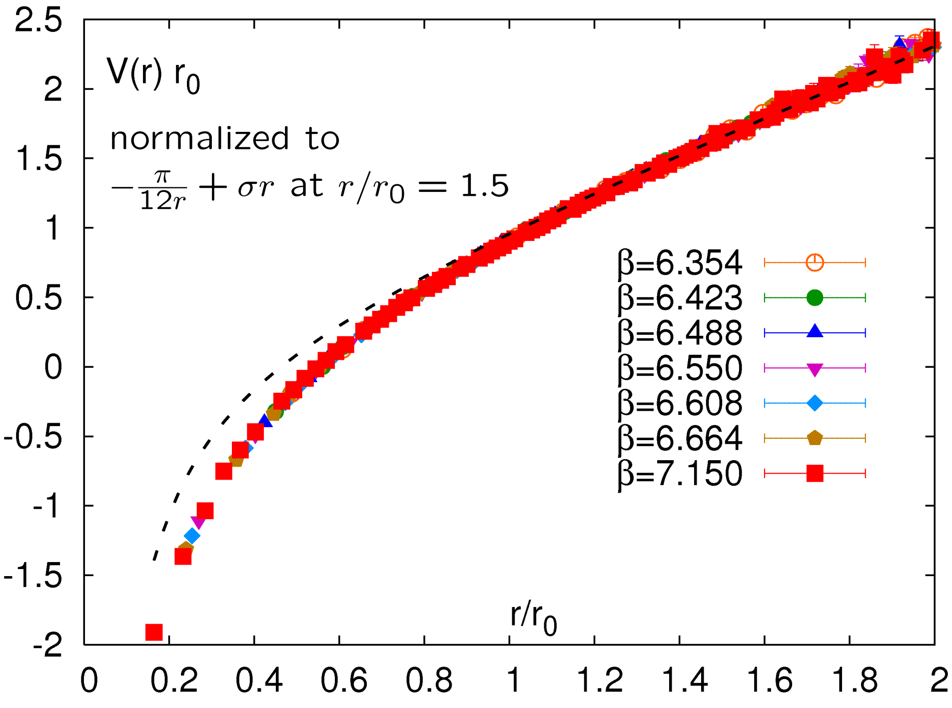
$$W_C(\tau,R) = \exp(-\sigma R\tau),$$

area law for large R and τ

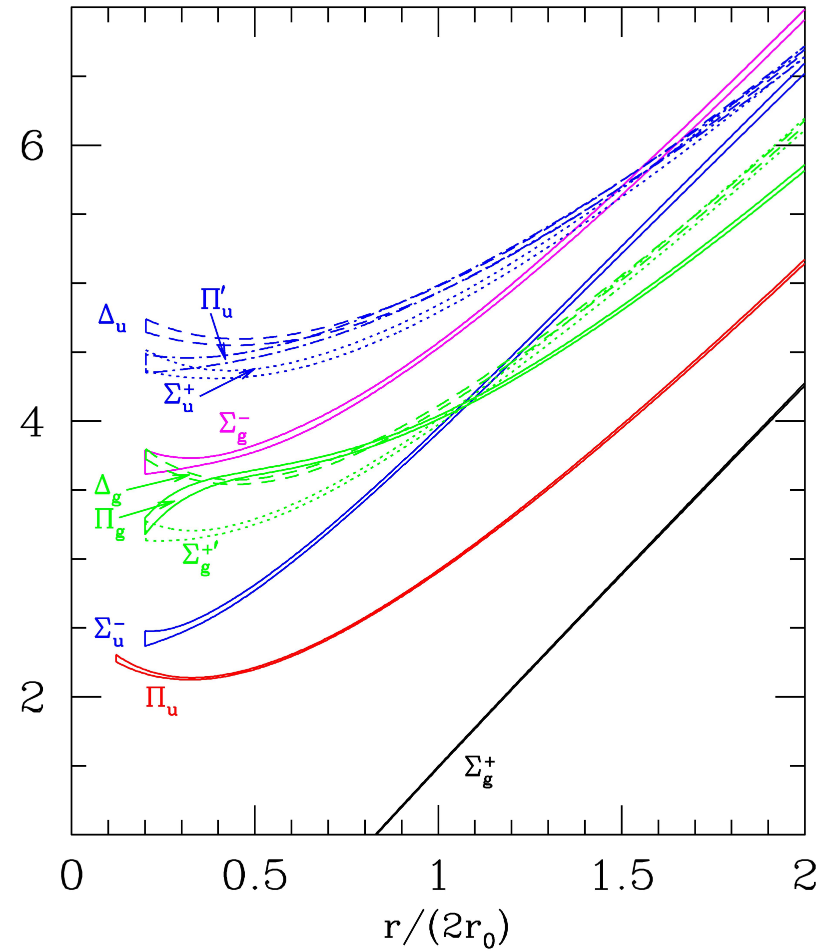
n=2 and larger : hybrid potentials

Numerial results on the potentials

Static quark anti-quark potential



Hybrid potentials



$$\sqrt{\sigma} \simeq 470 \text{ MeV}$$

$$\left(r^2 \frac{dV_{q\bar{q}}(r)}{dr}\right)_{r=r_0} = 1.65, \quad r_0 = 0.468(4)\, fm$$

Sommer scale

Scale setting in lattice QCD and continuum limit

Hadron masses in lattice QCD are dimensionless: $m = m_{phys}\, a$

Continuum limit: $a \to 0$

$m \to 0 \quad \Rightarrow \xi = 1/m \to \infty$ divergent correlation length $\Rightarrow$ second order phase transition (universality)

Physics does not depend on the details on the regularization, e.g dimensionless ratios :

$$r_0 m, \quad m/\sqrt{\sigma}$$

Should be independent of the lattice spacing

The gauge coupling constant depends on the lattice spacing:

$$a \to 0 \quad \Rightarrow g \to 0 \quad (\beta = 6/g^2 \to \infty) \quad a\Lambda = R(g) = (\beta_0 g)^{-\beta_1/(2\beta_0)} e^{-\frac{1}{2\beta_0 g^2}}$$

There are corrections to this asymptotic expression $\sim g^{2n} a^{2n}, \; n = 1, 2...$

$a\Lambda = c_0 R(g)(1 + c_2 g^2 R^2(g) + c_4 g^4 R^4(g) + ...)$

Allton Ansatz

We need to tune the bare quark mass m_{bare} on lattice to keep the physical ratio $m_{meson}/\sqrt{\sigma}$ fixed for each g:

$\Rightarrow m_{bare} = m_{bare}^{LCP}(g)$ defining the line of constant physics (LCP).

Homework

Show that the naive lattice fermion propagator has the form:

$$\frac{-i\sum_\nu \gamma_\nu \widehat{p}_\nu + m}{\sum_\nu \widehat{p}_\nu^2 + m^2} \qquad \widehat{p}_\nu = \frac{1}{a}\sin(p_\nu a)$$

Show that the Wilson action

$$S_{Wilson} \;=\; \beta \sum_x (1 - \frac{1}{3} Re\; tr U_P(x)),\;\; \beta = \frac{6}{g^2}$$

in the $a \to 0$ limit gives the Yang-Mills action up to corrections $\sim a^2$