Lecture 5-6: Chiral and deconfinement transition in QCD and the properties of matter at low and high temperatures

Effective field theory approach

Effective theories at low temperatures (chiral perturbation theory)

Chiral transition and axial symmetry restoration

Center symmetry, deconfinement transition, color screening

High temperature QCD: perturbation theory, magnetic screening, dimensional reduction, hard-thermal loop effective theory

Virial expansion and hadron resonance (HRG) model

Taylor expansion of the pressure and fluctuations of conserved charges

Effective field theories

Try to describe the physics of low momentum modes *(p < Λ)* in terms of the relevant dof only:

$$\phi = \phi_0(k < \Lambda) + \tilde{\phi}(k > \Lambda)$$

$$Z = \int \mathcal{D}\phi e^{-S[\phi;g_i]} = \int \mathcal{D}\phi_0 \underbrace{\int \mathcal{D}\tilde{\phi} e^{-S[\phi_0,\tilde{\phi};g_i]}}_{e^{-S_{eff}[\phi_0;g_j^{eff}]}} = \int \mathcal{D}\phi_0 e^{-S_{eff}[\phi_0;g_j^{eff}]}$$

How to calculate the effective action ?

1) Perform the function integral in perturbation theory (partial path integration) *difficult !*

2) Matching: write down the most general form of the effective Lagrangian as sum of different operators $g_j^{eff}(\Lambda; g_i) O_j / \Lambda^{d-4}$ consistent with the symmetries of the original theory

$$G^{(n)}(k = \Lambda; g_i) = G^{(n)}_{eff}(k = \Lambda; g_j^{eff})$$ *matching condition*

$$\Downarrow$$

$$g_j^{eff} = g_j^{eff}(\Lambda; g_i)$$

Linear sigma model and chiral perturbation theory

Goal to describe physics at scale < pion mass << Λ_{QCD} (for sufficiently small quark mass) taking into account spontaneous breaking of chiral symmetry

Linear sigma model

$$L = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{\lambda}{24}(\sigma^2 + \vec{\pi}^2 - v)^2 + h\sigma$$

Invariant under chiral rotations for $h \sim m_q = 0$: $\sigma \to \sigma + \vec{b} \cdot \vec{\pi}, \quad \vec{\pi} \to \pi - \vec{b}\sigma$

$$\sigma \leftrightarrow \bar{\psi}\psi, \quad \vec{\pi} \leftrightarrow \bar{\psi} i \vec{\tau} \gamma_5 \psi, \; \delta\psi = \vec{b} \cdot \vec{\tau} \gamma_5 \psi / 2$$

Useful for the discussion at $T>0$ and restoration of the chiral symmetry
Problem: $m_\sigma \approx 400$ MeV ~ Λ_{QCD} no well-defined power counting in a small parameter
Only two parameters λ and v, *cannot describe the experimental data => additional operators ??*

Chiral perturbation theory (non-linear signa model) :
Use only pion fields and chiral symmetry

$$U = \exp(i\vec{\tau} \cdot \vec{\pi}/F), \quad U \to G_R^\dagger U G_L, \quad G_{R,L} \subset SU(2)_{R,L}$$

Expansions in powers p^2/F^2, $(m_\pi/F)^2$, $F \sim \Lambda_{QCD}$

Gasser, Leutwyler, ‘84

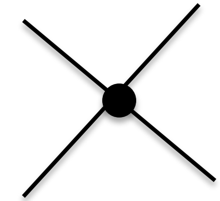
$$L_{eff} = L^{(2)} + L^{(4)} + L^{(6)} + \ldots.$$

$$L^{(2)} = \frac{1}{4} F^2 \mathrm{Tr}\left(\partial_\mu U \partial U_\mu^\dagger - M^2(U + U^\dagger)\right), \quad M^2 = (m_u + m_d)B \simeq m_\pi^2$$

Kinetic term is invariant under the chiral rotation and the mass term transforms like the quark mass term in the QCD Lagrangian

$$\mathrm{Tr}U + h.c. \to \mathrm{Tr}(G_L G_R^\dagger U) + h.c.$$

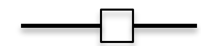
At leading order in $1/F^2$ expansion we recover the Lagrangian for massive pions, but interactions are proportional to p^2 (expand U in $1/F$) but operators with d>4 => the effective theory is non-renormalizable,
but no IR divergences

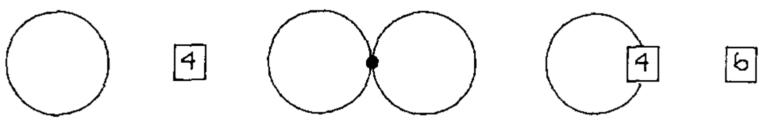


$\sim p^2/F^2$

Next-to-leading $\sim p^4/F^4$ order:

$$L^{(4)} = -l_1 \frac{1}{4} \left(\mathrm{Tr}\partial_\mu U^\dagger \partial_\mu U\right)^2 - \frac{1}{4} l_2 \mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U)\mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U) +$$
$$\frac{1}{8} l_4 M^2 \mathrm{Tr}\left(\partial_\mu U^\dagger \partial_\mu U\right) \mathrm{Tr}(U + U^\dagger) - \frac{1}{16}(l_3 + l_4) M^4 \left(\mathrm{Tr}(U + U^\dagger)\right)^2$$





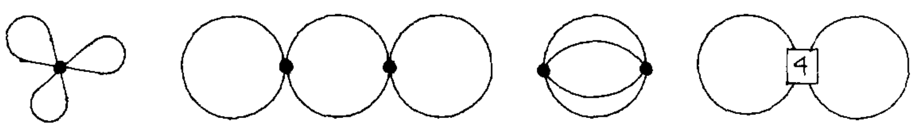
Gerber, Leutwyler, '89

6a 6b 6c

$$p(T) = \frac{\pi^2 T^4}{30}\left(1 + \frac{T^4}{36F^4}\ln\frac{\Lambda_p}{T}\right),$$

$$\Lambda_p \simeq 270 \text{ MeV}$$

for massless pions



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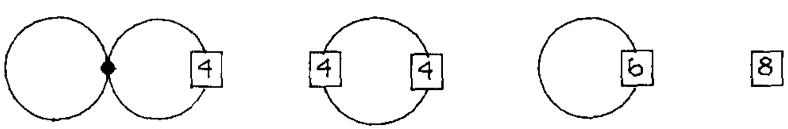


Hadronic interatcions are weak at low T, $T<F$ but increase with Increasing temperature

Is the expansion applicable in practice ?

$$F \simeq F_\pi \simeq 90 \text{ MeV}$$

Yes ! The true expansion parameter in the loop expansion is

$$1/(4\pi F) \sim 1/\Lambda_\chi \sim 1 GeV^{-1}$$

Not applicable close to the chiral transition

The chiral condensate and chiral susceptibility

$$\langle\bar\psi\psi\rangle = \frac{T}{V}\frac{\partial \ln Z}{\partial m_q}$$

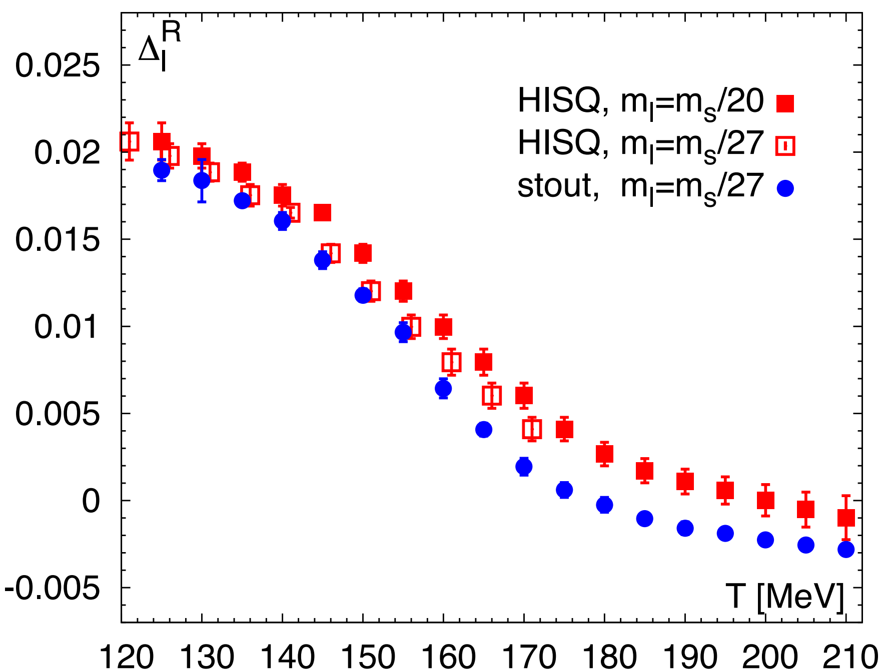
$$\chi_{m,q} = \frac{\partial}{\partial m_q}\langle\bar\psi\psi\rangle = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_q^2} = \chi_{disc} + \chi_{conn} =$$

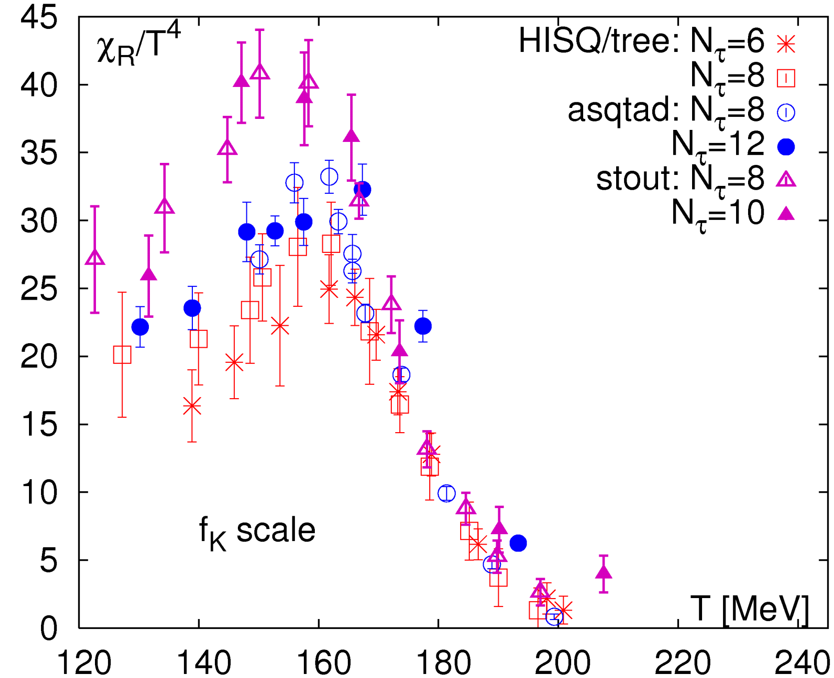
$$< (\bar\psi\psi)^2 > - < \bar\psi\psi >^2 \; - \mathrm{Tr} D_q^{-2}$$

Need renormalization

$$\langle\bar\psi\psi\rangle_q \Rightarrow \Delta_q^R(T) = m_s r_1^4 \left(\langle\bar\psi\psi\rangle_{q,T} - \langle\bar\psi\psi\rangle_{q,T=0}\right) + d, \quad \chi_R(T) = \frac{m_l^2}{T^4}\left(\chi_{m,l}^{(n_f=2)}(T) - \chi_{m,l}^{(n_f=2)}(T=0)\right)$$

$$d = \langle\bar\psi\psi\rangle_{m_q=0}^{T=0}$$





HotQCD : Phys. Rev. D85 (2012) 054503; Bazavov, PP, PRD 87(2013) 094505

How to define the chiral transition temperature ?

O(N) scaling and the chiral transition temperature

For sufficiently small m_l and in the vicinity of the transition temperature:

$$f(T,m_l) = -\frac{T}{V}\ln Z = f_{reg}(T,m_l) + f_s(t,h),\ t = \frac{1}{t_0}\left(\frac{T-T_c^0}{T_c^0} + \kappa\frac{\mu_q^2}{T^2}\right),\ H = \frac{m_l}{m_s}, h = \frac{H}{h_0}$$

governed by universal *O(4)* scaling $\quad M = -\frac{\partial f_s(t,h)}{\partial H} = h^{1/\delta} f_G(z),\ \ z = t/h^{1/\beta\delta}$

$SU_A(2) \sim O(4)$

T_c^0 is critical temperature in the mass-less limit, h_0 and t_0 are scale parameters

Pseudo-critical temperatures for non-zero quark mass are defined as peaks in the response functions (susceptibilities) :

$$\chi_{m,l} = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_l^2} \sim m_l^{1/\delta - 1} \qquad \chi_{t,l} = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_l \partial t} \sim m_l^{\frac{\beta-1}{\beta\delta}} \qquad \chi_{t,t} = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial t^2} \sim |t|^{-\alpha}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$T_{m,l} \quad = \quad T_{t,l} \quad = \quad T_{t,t} \quad = \quad \boldsymbol{T_c^0}$$

in the zero quark mass limit

$$\frac{\chi_{l,m}}{T^2} = \frac{T^2}{m_s^2}\left(\frac{1}{h_0} h^{1/\delta - 1} f_\chi(z) + reg.\right)$$

universal scaling function has a peak at $z=z_p$

$$\Downarrow$$

$$T_c(H) = T_{m,l} = T_c^0 + T_c^0 \frac{z_p}{z_0} H^{1/(\beta\delta)} + \ldots$$

Caveat : staggered fermions O(2)
$m_l \rightarrow 0$, $a > 0$,
proper limit $a \rightarrow 0$, before $m_l \rightarrow 0$

O(N) scaling and the transition temperature

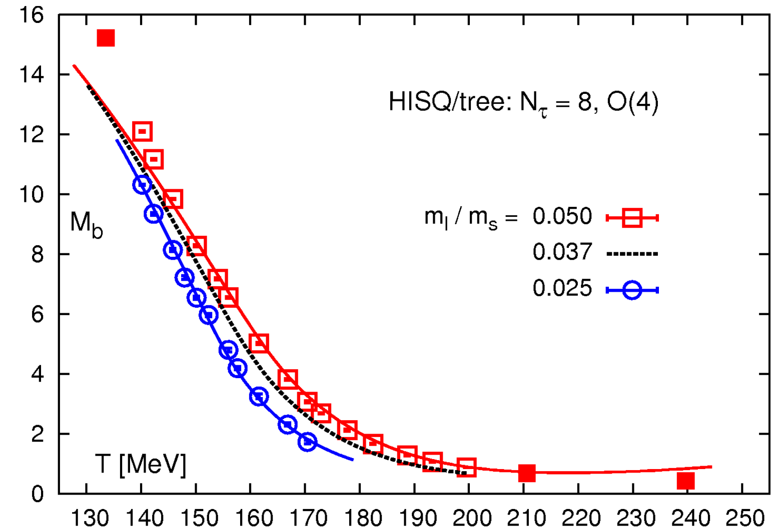
The notion of the transition temperature is only useful if it can be related to the critical temperature in the chiral limit : fit the lattice data on the chiral condensate with scaling form + simple Ansatz for the regular part

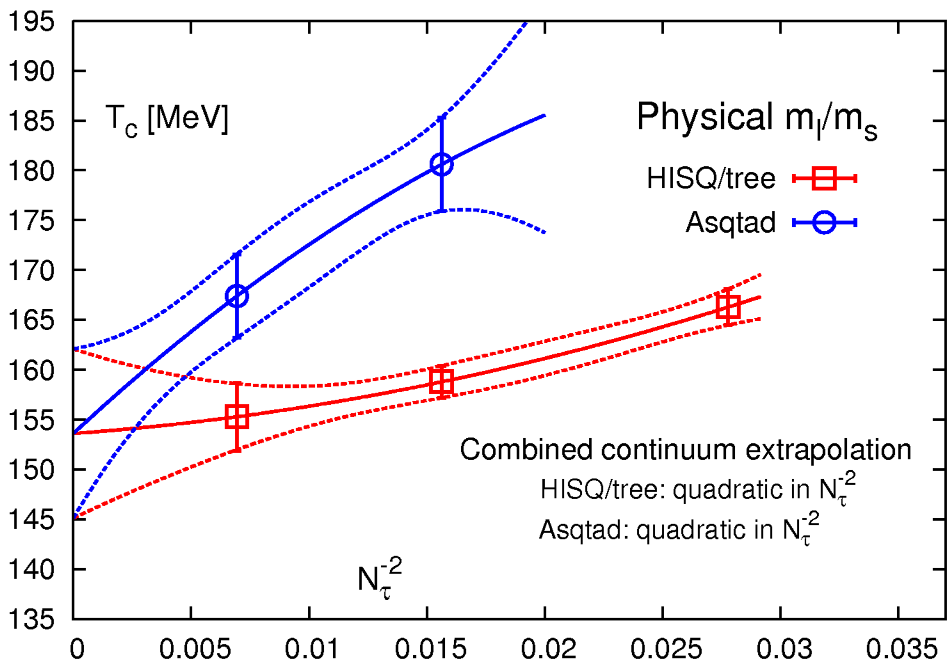
$$M_b = \frac{m_s \langle \bar{\psi}\psi \rangle_l}{T^4} = h^{1/\delta} f_G(t/h^{1/\beta\delta}) + f_{M,reg}(T,H)$$

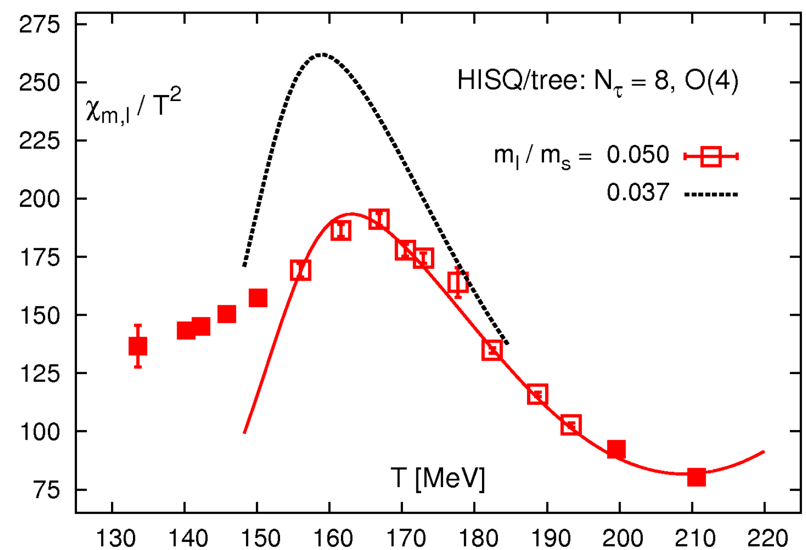
$$f_{reg}(T,H) = (a_1(T-T_c^0) + a_2(T-T_c^0)^2 + b_1)H$$

6 parameter fit : T_c^0, t_0, h_0, a_1, a_2, b_1

$$T_c = (154 \pm 8 \pm 1(scale))\text{MeV}$$



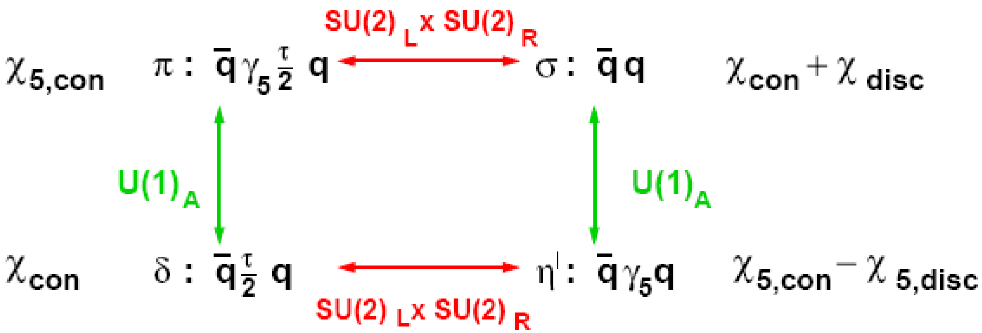




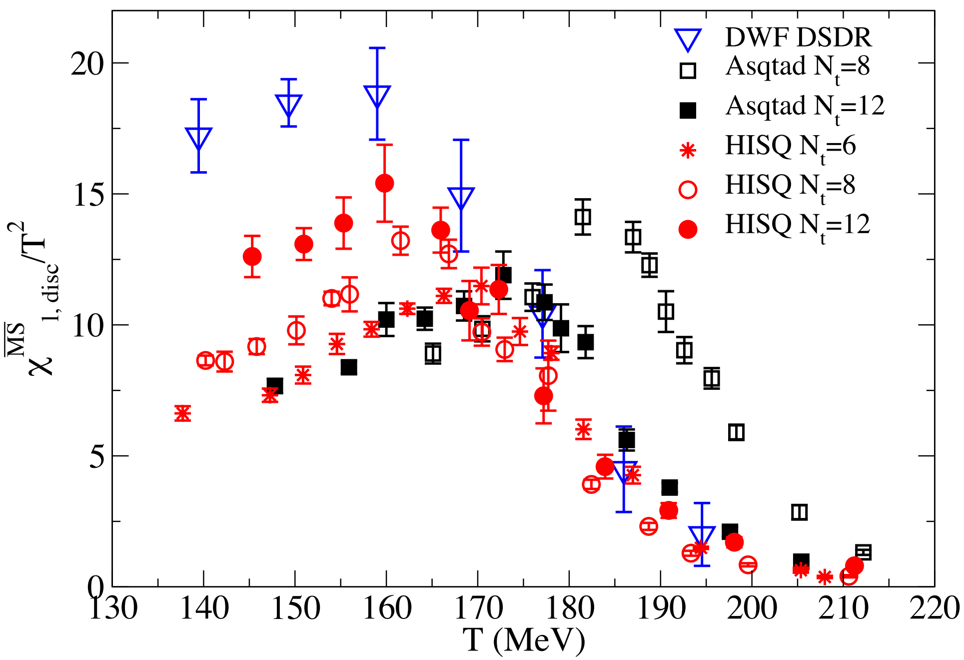
Domain wall Fermions and $U_A(1)$ symmetry restoration

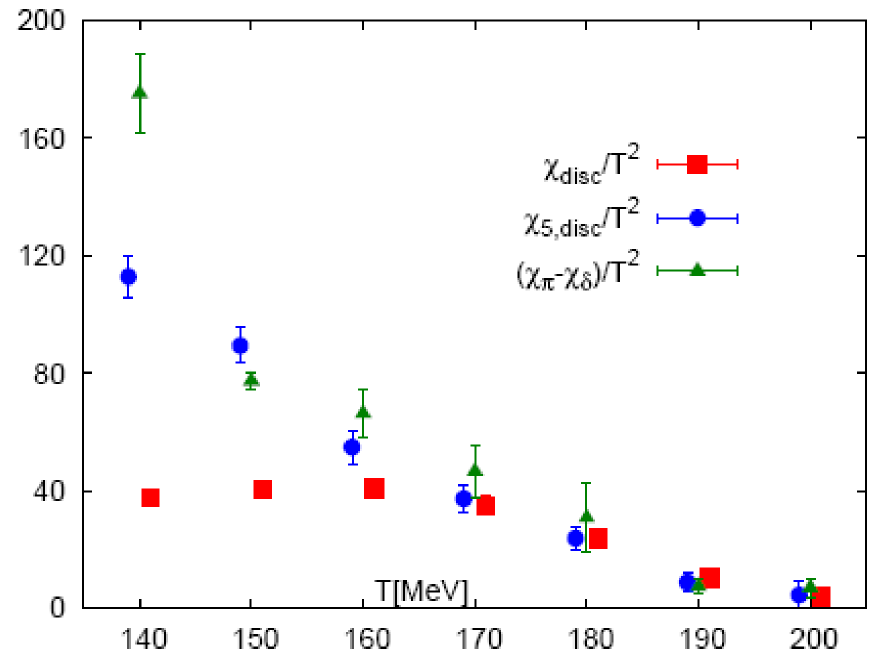
Domain Wall Fermions, Bazavov et al (HotQCD), PRD86 (2012) 094503

$$\chi_i = \int d^4x G_i(x)$$



chiral:	axial:
$\chi_\pi = \chi_\delta + \chi_{disc}$	$\chi_\pi = \chi_\delta$
$\chi_\delta = \chi_\pi - \chi_{5,disc}$	$\chi_\delta + \chi_{disc} = \chi_\pi - \chi_{5,disc}$
$\chi_{disc} = \chi_{5,disc}$	$\chi_{disc} = -\chi_{5,disc}$





Peak position roughly agrees with previous staggered results

axial symmetry is effectively restored *T*>200 MeV !

Center symmetry and deconfinement transition

Above the phase transition temperature Z(3) (center) symmetry of SU(3) gauge theory is broken
Quarks transform non-trivially under Z(3) symmetry group
=> static charges in fundamental representations can be screened by gluons !

Lattice set-up:

$U_\mu(\tau,x)=e^{igA_\mu(\tau,x)},\ N_\sigma^3\times N_\tau,\ T=1/(N_\tau a)$

Thermodynamic limit: $N_\sigma/N_\tau\to\infty$; Continuum limit : $N_\tau\to\infty$, T-fixed Temperature is set by $a\leftrightarrow\beta=2N_c/g^2$ allowable gauge transformations $U_\mu(x)\to\Omega(x+\mu)U_\mu(x)\Omega^\dagger(x)$

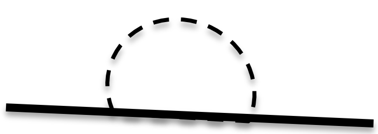
$$\Omega(0,\vec{x})=\Omega(\beta,\vec{x})C,\quad C=e^{2\pi i n/N_c}I\quad\to Z(N)-\text{symmetry}$$

$$L(\vec{x})=\frac{1}{N_c}\mathrm{tr}\prod_{\tau=1}^{N_\tau}U_0(\tau,\vec{x})$$

Polyakov loop is changed $L(\vec{x})\to e^{2\pi n i/N_c}L(\vec{x})$

$<L>\neq 0\ \to Z(N)$ spontaneously broken; $<L>=e^{-F_Q/T}$ -free energy of an isolated static quark is finite $\Rightarrow$ deconfinement

L is order parameter

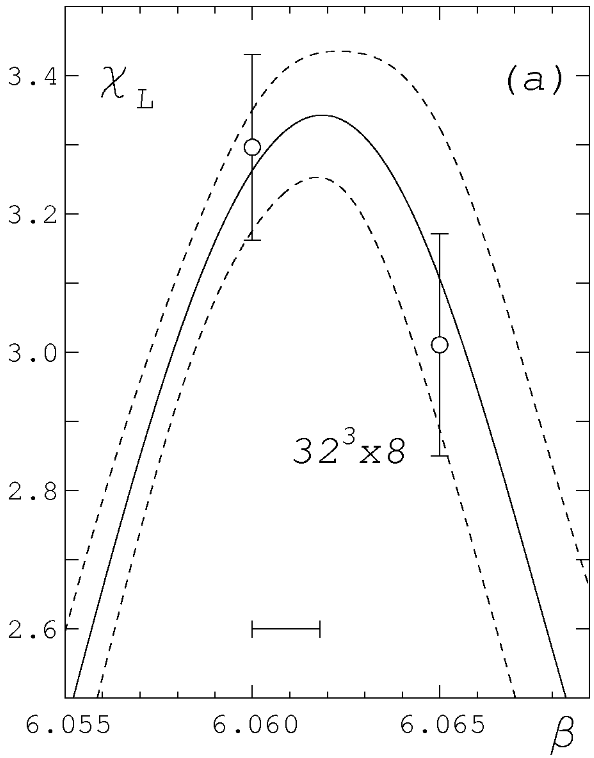


The free energy of static quark is infinite in the
Continuum limit due to linear *1/a* divergence => needs renormalization

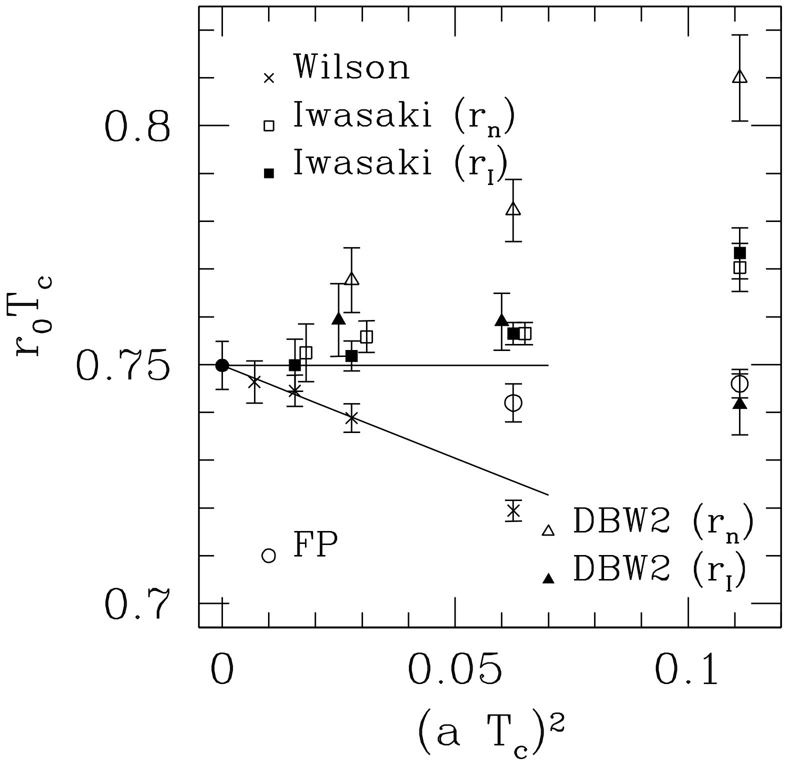
How to determine the deconfinement transition temperature ?

$$\frac{\chi_L}{T^2} = N_\sigma^3 \left(\langle L^2\rangle - \langle L\rangle^2\right) = \langle(\delta L)^2\rangle \text{ has a peak at } \beta_c$$

Boyd et al., Nucl. Phys. B496 (1996) 167



Necco, Nucl. Phys. B683 (2004) 167



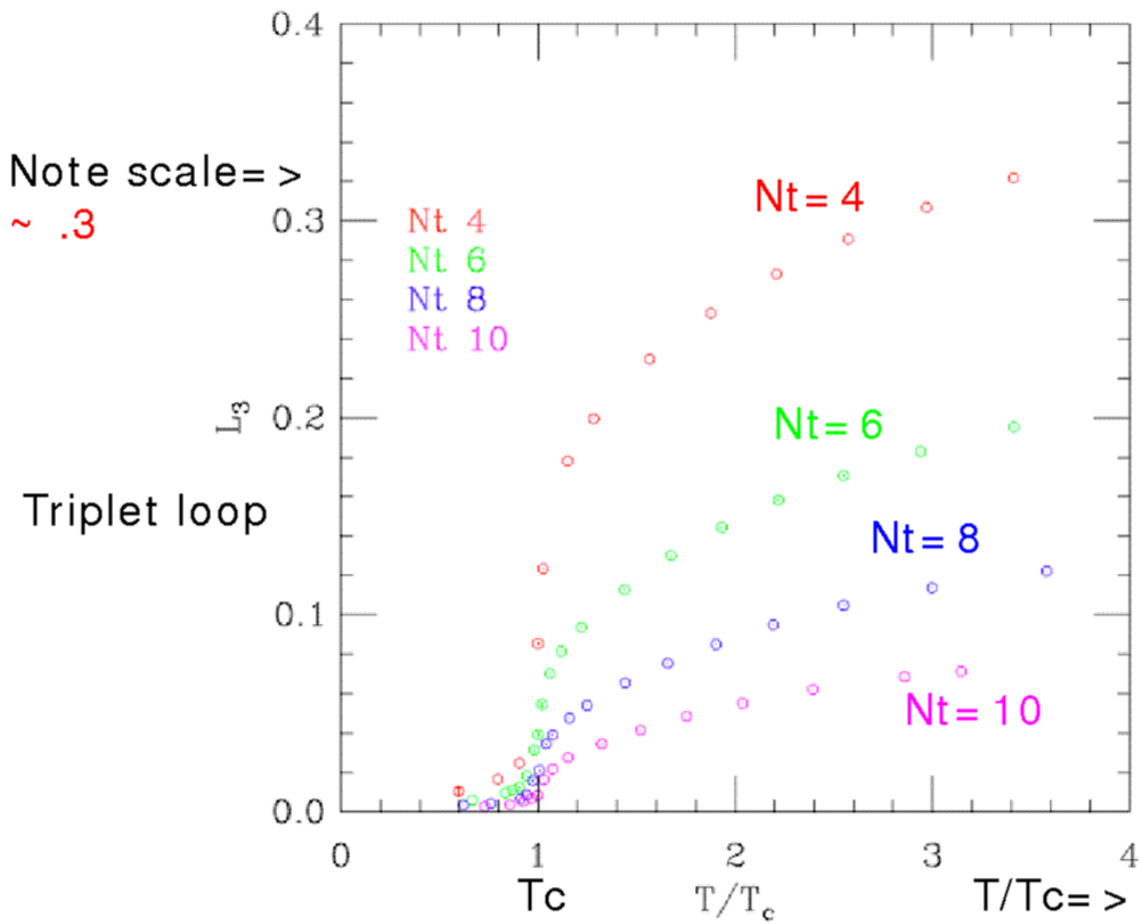
- Use different volumes and Ferrenberg-Swedsen re-weighting to combine information collected at different gauge couplings

Finite volume behavior can tell the order of the phase transition, e.g. for 1st order transition the peak height scales as spatial volume !

Continuum limit for *L* ?

Dumitru et al, hep-th/0311223

Bare triplet loop vs T, at different Nt



Nt = # time steps.

Bare loop vanishes as Nt -> ∞

needs renormalization !

Free energy of static quark anti-quark pair and other correlators

McLerran, Svetitsky, PRD 24 (81) 450

$\psi_a^\dagger(\tau,x),\ \psi_a(\tau,x)$-creation annihilation operators for static quarks at time τ and position x

$\psi_a^{\dagger c}(\tau,x),\ \psi_a^c(\tau,x)$-creation annihilation operators for static anti-quarks at time τ and position x

$$[\psi_a(\tau,x),\psi_b^\dagger(\tau,y)]_+ = \delta(x-y)\delta_{ab}$$

$$(-i\partial_\tau - gA_0(\tau,x))\,\psi(\tau,x) = 0$$

formal solution $\psi(\tau,x) = \mathcal{P}\exp\left(ig\int_0^\tau d\tau' A_0(\tau',x)\right)\psi(0,x) = W(x)\psi(0,x)$

lattice : $W(x) = \prod_{x_0=0}^{N_\tau-1} U_0(x,\tau)$

Free energy of static quark anti-quark pair

$$Z(\beta)e^{-\beta F(x,y)} = \sum_s < s|e^{-\beta H}|s >$$

$|s>$ denotes any state with a static quark at position x and static anti-quark at position y;

Let us denote by $|s'>$ states with no static quarks

$$e^{-\beta F(x,y)} = \sum_{s'} \frac{1}{N_c^2} \sum_{a=a', b=b'} < s'|\psi_a(0,x)\psi_b^c(0,y)e^{-\beta H}\psi^\dagger_{a'}(0,x)\psi^{\dagger c}_{b'}(0,y)|s'> \quad (1)$$

$$\boxed{e^{-\beta H} e^{\beta H}} \qquad e^{-\beta H} O(\tau) e^{\beta H} = O(\tau+\beta)$$

$$= \sum_{s'} \frac{1}{N_c^2} \sum_{a=a', b=b'} < s'|e^{-\beta H}\psi_a(\beta,x)\psi_b^c(\beta,y)\psi^\dagger_{a'}(0,x)\psi^{\dagger c}_{b'}(0,y)|s'>$$

$$= Z(\beta)\frac{1}{N_c^2} < \mathrm{Tr}W(x)\mathrm{Tr}W^\dagger(y) >= Z(\beta)G(r,T),\ \ r=|x-y|$$

$L(x) = \mathrm{Tr}W(x)$- Polyakov loop

Consider more general correlation function:

$$\mathcal{G}_{aa'bb'}(x,y;\beta,0) = \sum_{s'} < s'|e^{-\beta H}\psi_a(\beta,x)\psi_b^c(\beta,y)\psi^\dagger_{a'}(0,x)\psi^{\dagger c}_{b'}(0,y)|s'>$$

$$N_c=3,\ \ 3\otimes\bar{3} = 1\oplus 8: \qquad P_1 = \frac{1}{9} I\otimes I - \frac{2}{3} t^\alpha \bar{t}^\alpha,\ P_8 = \frac{8}{9} I\otimes I + \frac{2}{3} t^\alpha \bar{t}^\alpha$$

$$G_1(r,T) = \mathrm{Tr}(P_1\mathcal{G})/(\mathrm{Tr}P_1) = \frac{1}{3}\mathrm{Tr}\langle W(x)W^\dagger(y)\rangle$$

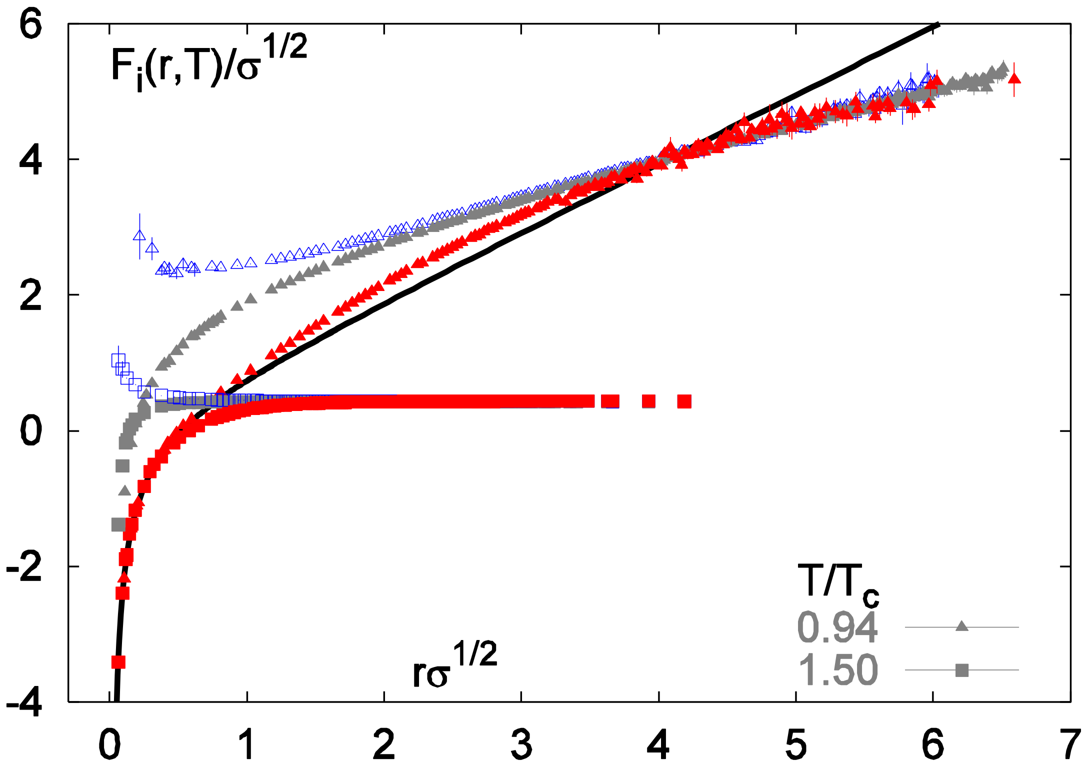
analog of the Wilson loop at *T=0*

$$G_8(r,T) = \mathrm{Tr}(P_8\mathcal{G})/(\mathrm{Tr}P_8) = \frac{1}{8}\langle \mathrm{Tr}W(x)\mathrm{Tr}W^\dagger(y)\rangle - \frac{1}{8}\frac{1}{3}\mathrm{Tr}\langle W(x)W^\dagger(y)\rangle$$

$$e^{-F(r,T)/T} = \frac{1}{9} G_1(r,T) + \frac{8}{9} G_8(r,T), \quad G_{1,8}(r,T) = e^{-F_{1,8}(r,T)/T}$$

$$G(r \to \infty, T) \equiv \langle L(r) L^\dagger(0) \rangle_{r\to\infty} = |\langle L \rangle|^2 = G_1(r \to \infty, T)$$

$$F(r \to \infty, T) = F_\infty(T) = 2F_Q$$



At short distance the quark anti-quark interactions should not depend on *T*:

$$F(r,T) + T \ln 9 =$$
$$= F_1(r,T) = V(r) + C,$$
$$rT \ll 1$$

$$V(r) = -\frac{\pi}{12r} + \sigma r$$

Kaczmarek, Karsch, P.P., Zantow, hep-lat/0309121

The spectral represenation of singlet and averaged correlators $(T < T_c)$:

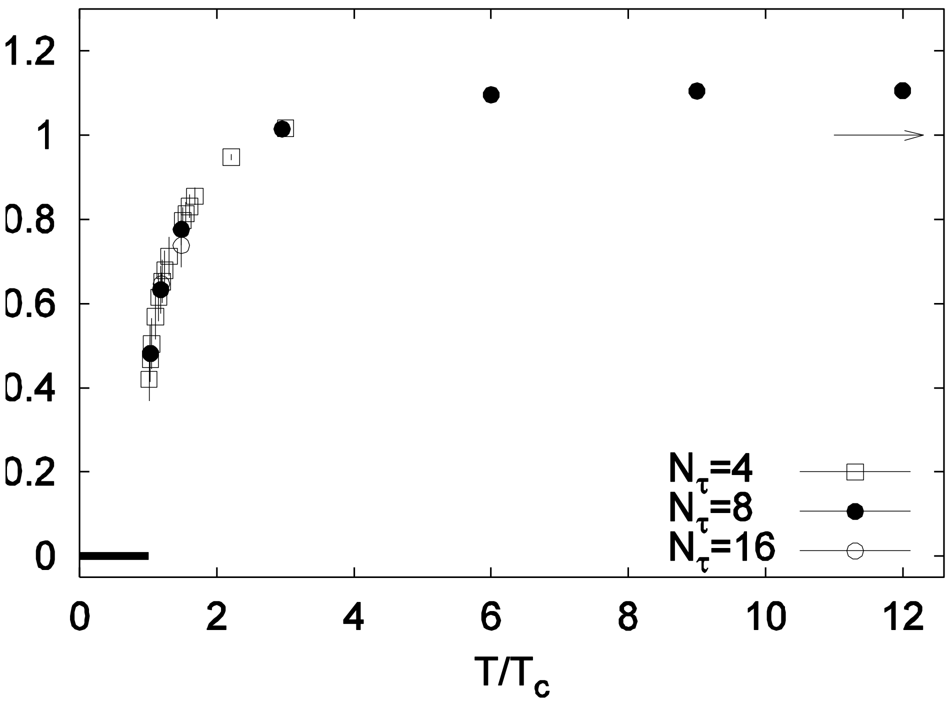
$$G_1(r,\beta) = \sum_{n=1}^{\infty} c_n(r) e^{-\beta E_n(r)}, \qquad G(r,\beta) = \frac{1}{9} \sum_{n=1}^{\infty} e^{-\beta E_n(r)}$$

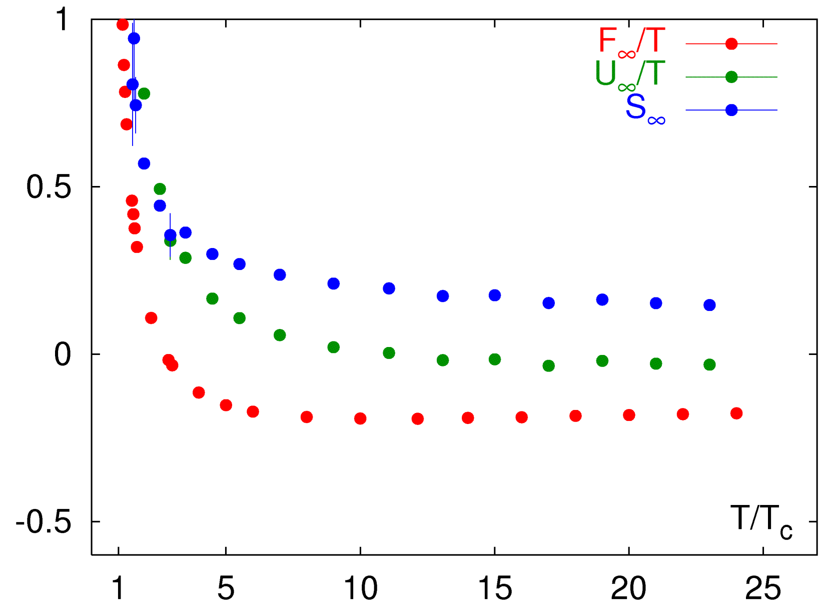
Jahn, Philipsen, PRD 70 (04) 0074504

The renormalized Polyakov loop in pure glue theory

$$L_{ren} = \exp(-F_\infty(T)/(2T))$$

Kaczmarek et al, PLB 543 (02) 41, PRD 70 (04) 074505, hep-lat/0309121





LO: $F_\infty \simeq -TS_\infty, \quad S_\infty = -\frac{4}{3}\alpha_s \frac{m_D}{T}$

Correlation length near the transition

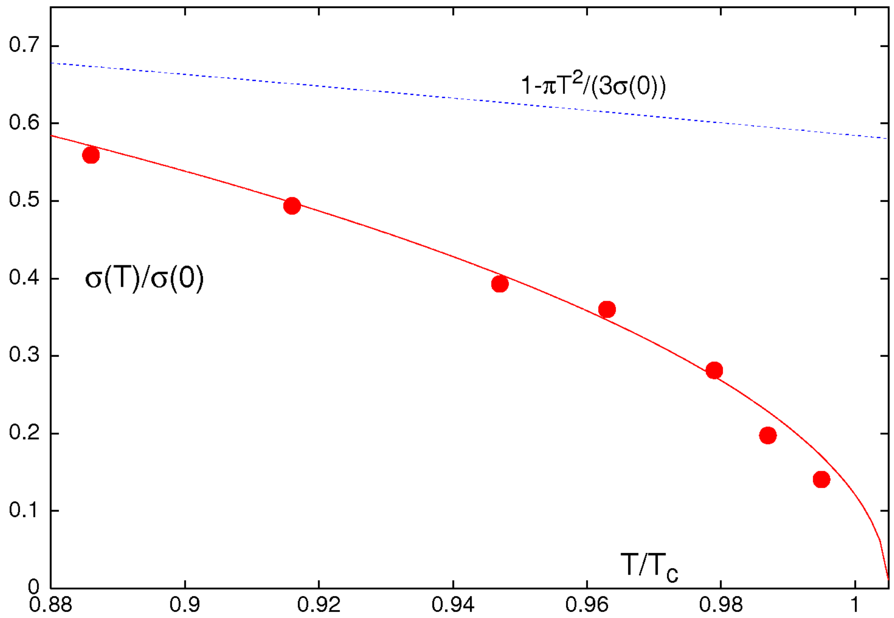
$T < T_c$:

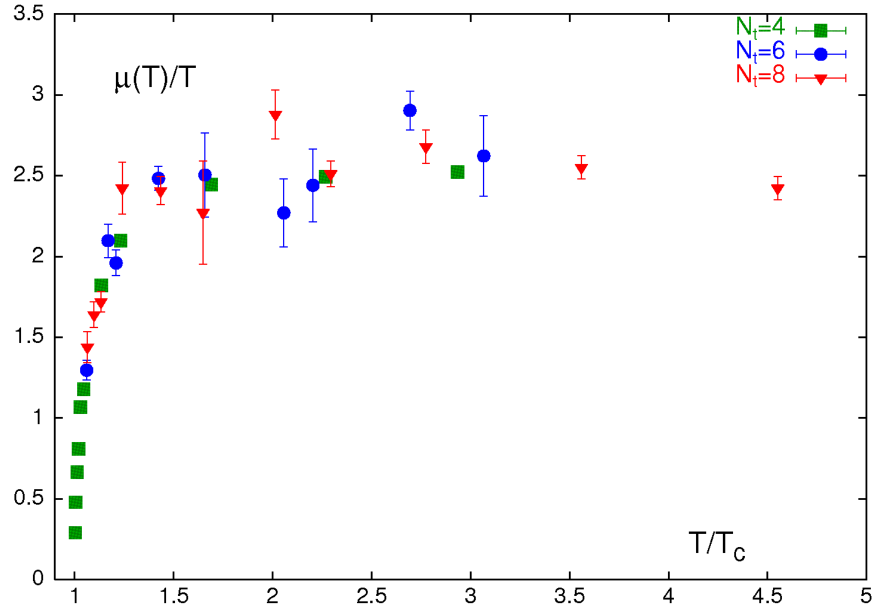
$$< L(r)L^\dagger(0) > \sim e^{-\sigma(T)r/T}$$

$T > T_c$:

$$\ln\left(\frac{< L(r)L^\dagger(0) >}{|< L >|^2}\right) \sim e^{-\mu(T)r}$$

Kaczmarek, Phys.Rev.D62 (00) 034021

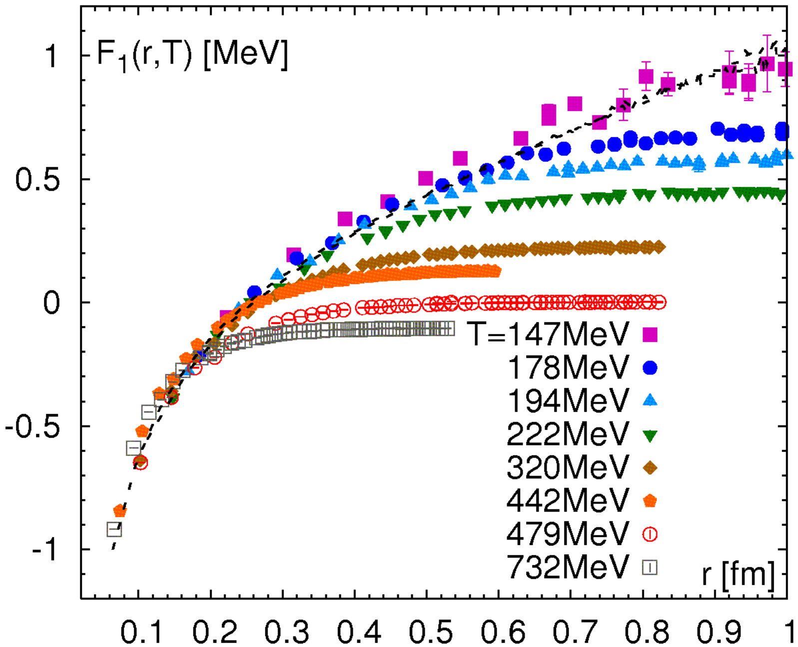


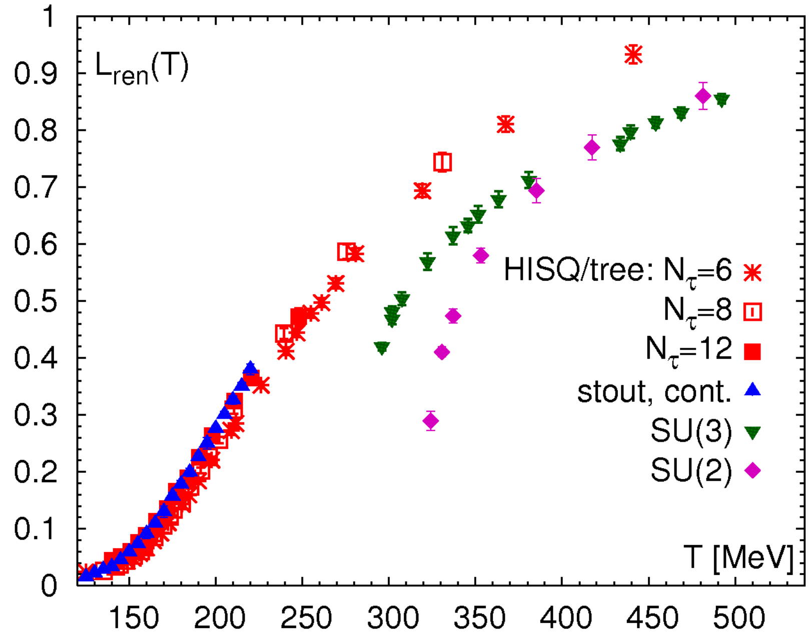


small inverse correlation length => weak 1st order phase transition

QCD is far from the large N-limit !

Singlet free energy and Polyakov loop in 2+1 flavor QCD





Pure glue ≠ QCD !

The free energy of static quark anti-quark pair is screened already at low temperature (even at *T=0*) => string breaking

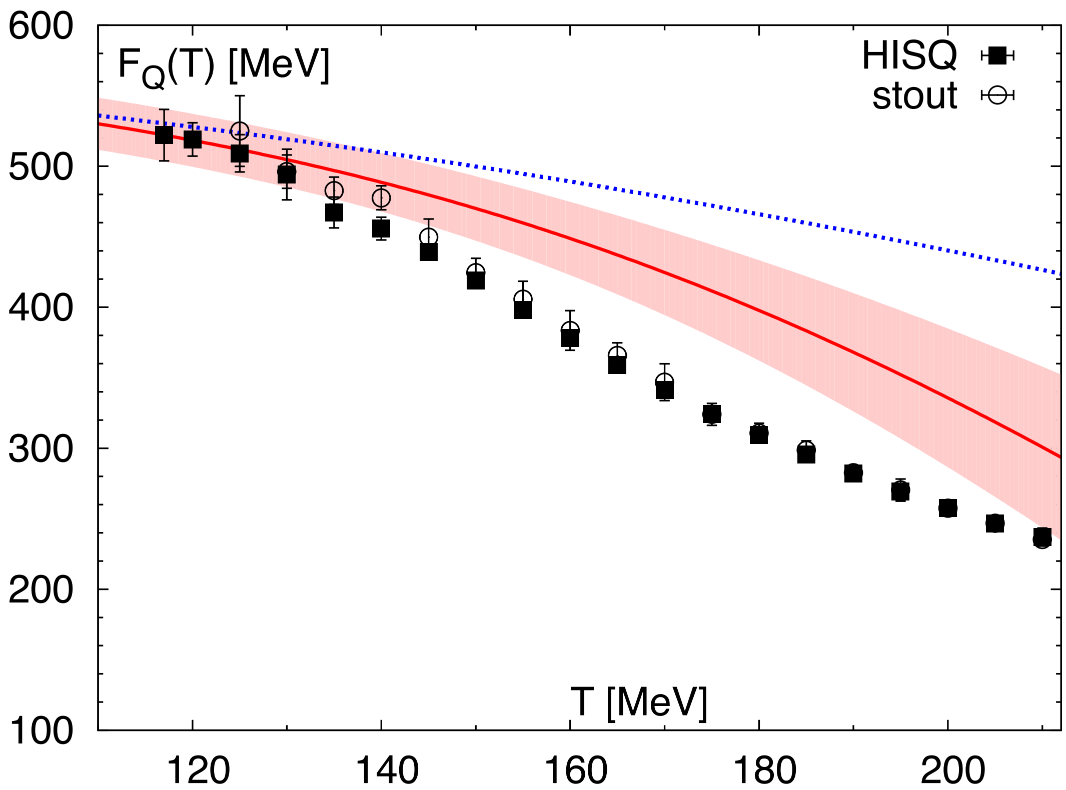
Deconfinement transition happens at lower temperature but the Polyakov loop behaves smoothly around T_c , *Z(3) symmetry plays apparent role*

Polyakov and gas of static-light hadrons

$$Z_{Q\bar{Q}}(T)/Z(T) = \sum_n \exp(-E_n^{Q\bar{Q}}(r \to \infty)/T)$$

Energies of static-light mesons: $E_n^{Q\bar{Q}}(r \to \infty) = M_n - m_Q$

Free energy of an isolated static quark: $F_Q(T) = -\frac{1}{2}(T \ln Z_{Q\bar{Q}}(T) - T \ln Z(T))$



Megias, Arriola, Salcedo, PRL 109 (12) 151601

Bazavov, PP, PRD 87 (2013) 094505

Ground state and first excited states are from lattice QCD

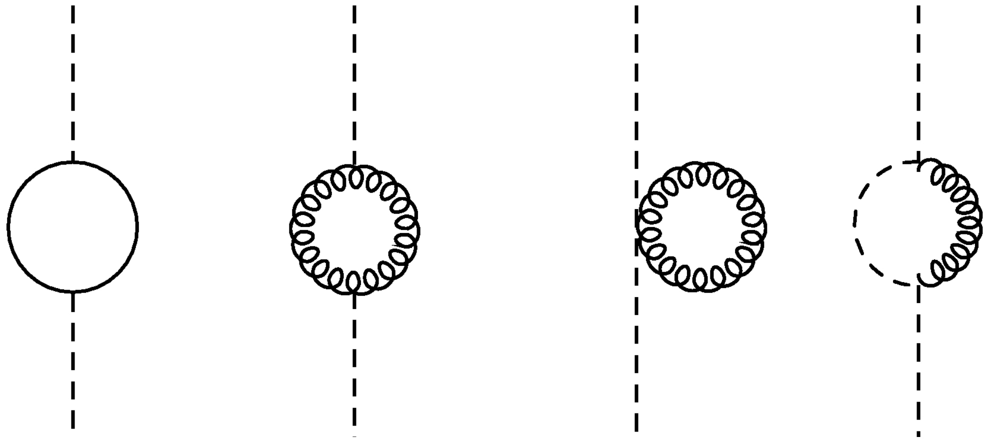
Michael, Shindler, Wagner, arXiv1004.4235

Wagner, Wiese, JHEP 1107 016,2011

Higher excited state energies are estimated from potential model

Gas of static-light mesons only works for *T* < *145* MeV

Gulon self energy and color screening in perturbation theory



Gluon self energy in the static limit:

$$\Pi_{00}(\omega_n = 0, k \to 0) = m_D^2 = (\frac{N_c}{3} + \frac{N_f}{6})g^2T^2$$

$$\Pi_{ii}(\omega_n = 0, k \to 0) = 0$$

$$V(r) \simeq -\frac{N_c^2 - 1}{2N_c} g^2 \int \frac{d^3x}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \frac{1}{k^2 + \Pi_{00}(k)} = -\frac{N_c^2 - 1}{2N_c} \alpha_s \frac{e^{-m_D r}}{r}$$

chromo-electric fields are screened but chromo-magnetic fields are not screened (at least in perturbation theory)

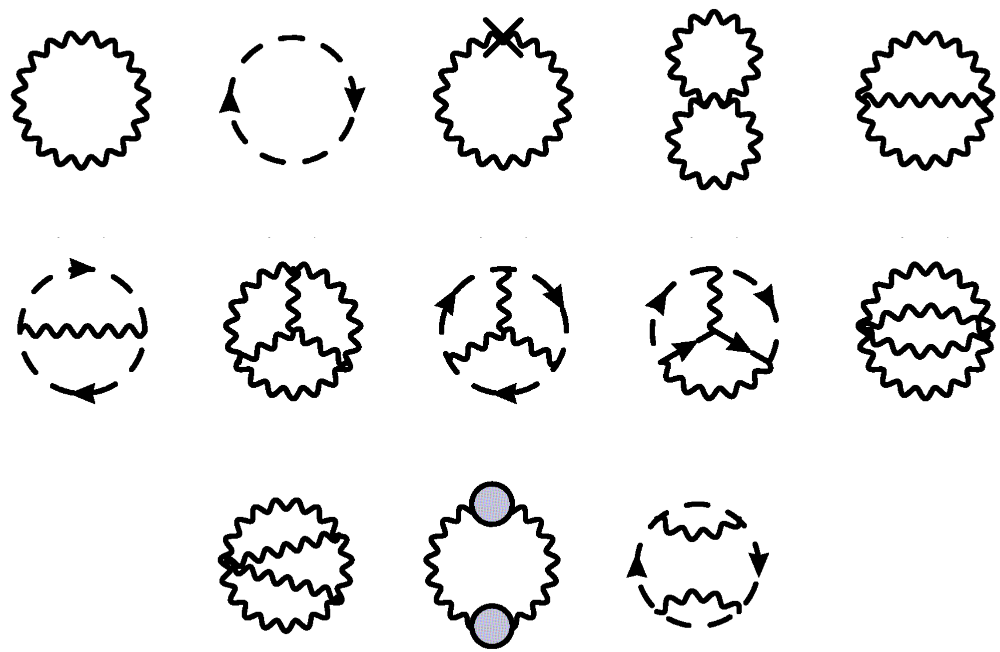
QCD at high temperatures

Because of asymptotic freedom thermodynamics quantities can be calculated in perturbation theory if $T \gg \Lambda_{QCD}$, at least in principle

Pressure has been calculated to 3-loop order

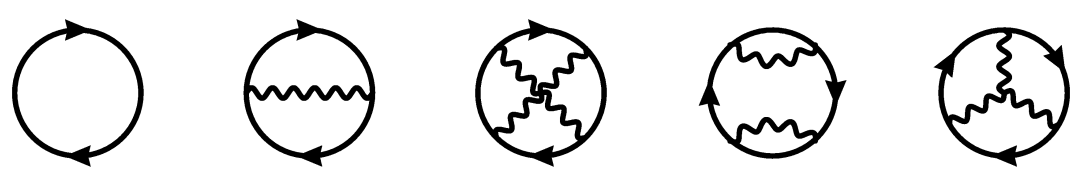
Arnold, Zhai, Phys.Rev. D51 (1995) 1906, Kastening, Zhai, Phys.Rev. D52 (1995) 7232

Bosonic contribution:



Static resummation:
$$\frac{1}{2} m_D^2 A_0^2 \delta_{\omega_n,0}$$

Fermionic contribution:



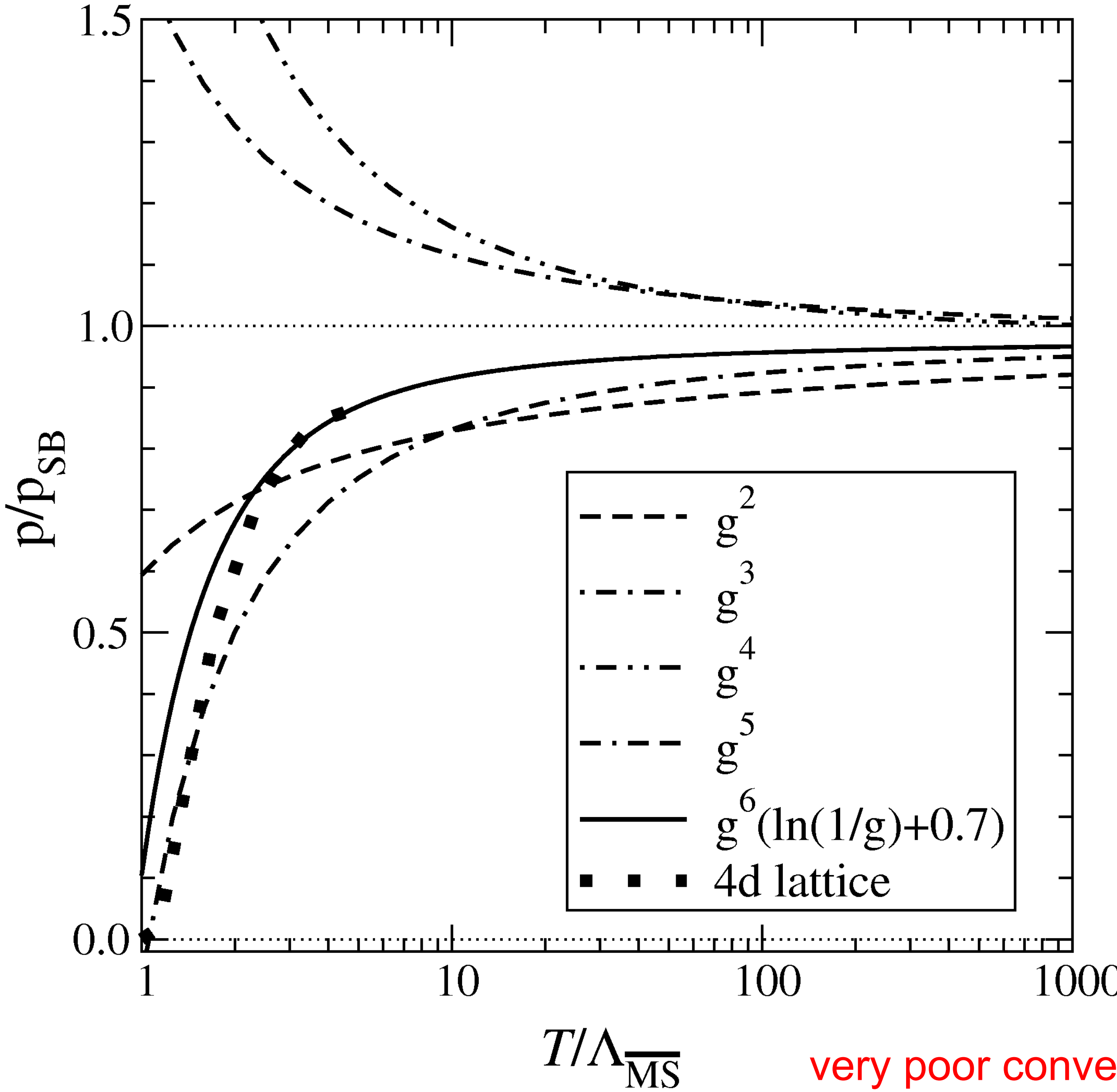
$$
\begin{aligned}
F \;=\; & d_{\mathrm{A}} T^4 \frac{\pi^2}{9}\Bigg\{ -\frac{1}{5}\left(1+\frac{7d_{\mathrm{F}}}{4d_{\mathrm{A}}}\right) + \left(\frac{g(\bar\mu)}{4\pi}\right)^2 \left(C_{\mathrm{A}} + \tfrac{5}{2}S_{\mathrm{F}}\right) \\
& -48\left(\frac{g(\bar\mu)}{4\pi}\right)^3 \left(\frac{C_{\mathrm{A}}+S_{\mathrm{F}}}{3}\right)^{3/2} - 48\left(\frac{g}{4\pi}\right)^4 C_{\mathrm{A}}(C_{\mathrm{A}}+S_{\mathrm{F}})\ln\left(\frac{g}{2\pi}\sqrt{\frac{C_{\mathrm{A}}+S_{\mathrm{F}}}{3}}\right) \\
& + \left(\frac{g}{4\pi}\right)^4 \Bigg[C_{\mathrm{A}}^2\left(\frac{22}{3}\ln\frac{\bar\mu}{4\pi T} + \frac{38}{3}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{148}{3}\frac{\zeta'(-1)}{\zeta(-1)} - 4\gamma_{\mathrm{E}} + \frac{64}{5}\right) \\
& \quad + C_{\mathrm{A}}S_{\mathrm{F}}\left(\frac{47}{3}\ln\frac{\bar\mu}{4\pi T} + \frac{1}{3}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{74}{3}\frac{\zeta'(-1)}{\zeta(-1)} - 8\gamma_{\mathrm{E}} + \frac{1759}{60} + \frac{37}{5}\ln 2\right) \\
& \quad + S_{\mathrm{F}}^2\left(-\frac{20}{3}\ln\frac{\bar\mu}{4\pi T} + \frac{8}{3}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{16}{3}\frac{\zeta'(-1)}{\zeta(-1)} - 4\gamma_{\mathrm{E}} - \frac{1}{3} + \frac{88}{5}\ln 2\right) \\
& \quad + S_{2\mathrm{F}}\left(-\frac{105}{4} + 24\ln 2\right)\Bigg] \\
& - \left(\frac{g}{4\pi}\right)^5 \left(\frac{C_{\mathrm{A}}+S_{\mathrm{F}}}{3}\right)^{1/2} \Bigg[C_{\mathrm{A}}^2\left(176\ln\frac{\bar\mu}{4\pi T} + 176\gamma_{\mathrm{E}} - 24\pi^2 - 494 + 264\ln 2\right) \\
& \quad + C_{\mathrm{A}}S_{\mathrm{F}}\left(112\ln\frac{\bar\mu}{4\pi T} + 112\gamma_{\mathrm{E}} + 72 - 128\ln 2\right) \\
& \quad + S_{\mathrm{F}}^2\left(-64\ln\frac{\bar\mu}{4\pi T} - 64\gamma_{\mathrm{E}} + 32 - 128\ln 2\right) \\
& \quad - 144 S_{2\mathrm{F}}\Bigg] + O(g^6)\Bigg\},
\end{aligned}
$$

Running coupling:
$g^2(\mu) = g^2(1 - \beta_0 \ln(\mu/\mu_0))$
$\beta_0 = \frac{1}{3(4\pi)^2}(11C_{\mathrm{A}} - 2N_f)$

$$
d_{\mathrm{A}} = N_c^2 - 1\,, \qquad C_{\mathrm{A}} = N_c\,, \qquad d_{\mathrm{F}} = N_c N_{\mathrm{f}}\,, \qquad S_{\mathrm{F}} = \frac{1}{2}N_{\mathrm{f}}\,, \qquad S_{2\mathrm{F}} = \frac{N_c^2-1}{4N_c}N_{\mathrm{f}}\,.
$$

Arnold, Zhai, Phys.Rev. D51 (1995) 1906, Kastening, Zhai, Phys.Rev. D52 (1995) 7232

Kajantie et al, Phys. Rev. D 67 (2004) 105008



very poor convergence !

Pressure at order g^6 and magnetic mass

Infrated sensitive contribution to the partition function at $l+1$-loop order:

$$g^{2l}\left(T\int d^3p\right)^{l+1} p^{2l}(p^2+m_{mag}^2)^{-3l}$$

$$g^{2l}T^4, \quad l=1,2$$

$$g^6T^4\ln(T/m_{mag}), \quad l=3$$

$$g^6T^4(g^2T/m_{mag})^{l-3}, \quad l>3$$

$m_{mag}\sim g^2T \Rightarrow$ infinitely many diagramls contribute at g^6 order !

Dimensional reduction at high temperatures

Decomposition in Matsubara modes

$$\phi(\tau,x)=\sum_n e^{i\omega_n\tau}\phi_n(x)$$

$$S_E=\int_0^\beta d\tau\int d^3x[(\partial_\mu\phi)^2+V(\phi)]\to\int d^3x(\sum_n(\partial_i\phi_n(x))^2+(2\pi Tn)^2\phi_n(x))+V(\phi_n))$$

$$F_{\mu\nu}=D_\mu A_\mu-D_\nu A_\mu$$

$$A_\mu\to\beta^{1/2}A_\mu$$

integrate out all $n\neq 0$ modes

Effective hight T theory for QCD $2\pi T\gg gT\gg g^2T$:

$$S_{eff}=\int d^3x\left(\frac{1}{2}TrF_{ij}^2+Tr(D_iA_0)^2+m_D^2TrA_0^2+\lambda_3(TrA_0^2)^2\right)$$

$$F_{ij}=\partial_iA_j-\partial_jA_i+ig_3[A_i,A_j],\ \ D_iA_0=\partial_iA_0+ig_3[A_i,A_0]$$

EQCD

the parameters $g_3^2\sim g^2T$, $m_D\sim gT$ and $\lambda_3\sim g^4T$ can be computed perturbatively to any order.

The effective theory is confining and non-perturbative at momentum scales $<g_3^2$ but can be solved on the lattice to calculate the weak coupling expansion of the pressure and other quantities

Integrate out A_0

⇩

3d YM theory MQCD

$F^{3d}\sim g_3^{\,6}$

Braaten, Nieto, PRD 51 (95) 6990, PRD 53 (96) 3421
Kajantie et al, NPB 503 (97) 357, PRD 67 (03) 105008

$F = F(\text{non-static}) + T\,F^{3d}$

Spatial string tension at *T>0* and dimensional reduction

$W(r(x,y),z) \sim \exp(-\sigma_s(T)\cdot r \cdot z)$
$\sigma_s(T) \simeq \sigma(T=0),\ T<T_c$

Cheng et al., Phys.Rev.D78 (08) 034506

EQCD :

$\sigma_s(T) = c_M \cdot g_3^4(T) \sim T^2,\ T \gg T_c$

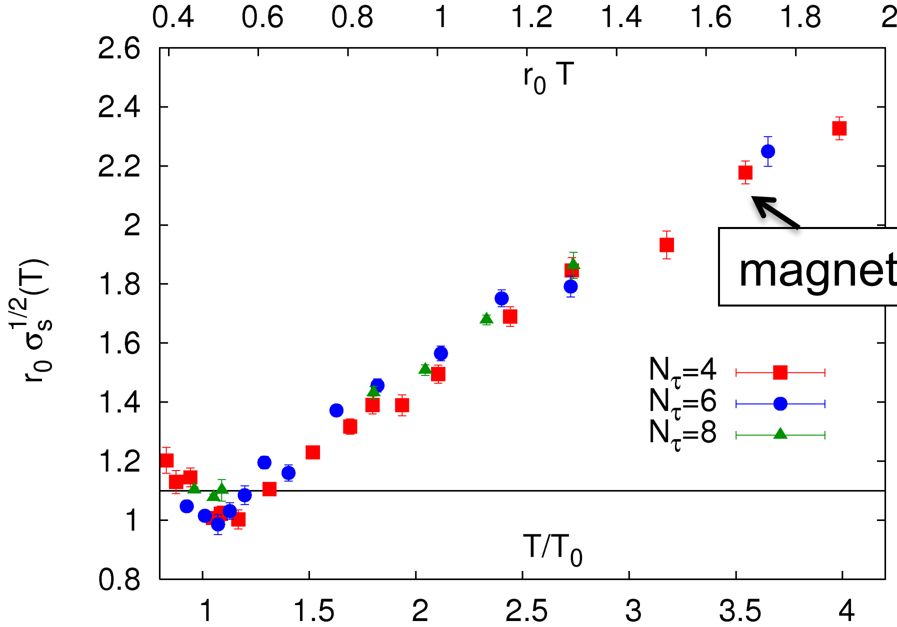
non-perturbative

$c_M = 0.55(1)$

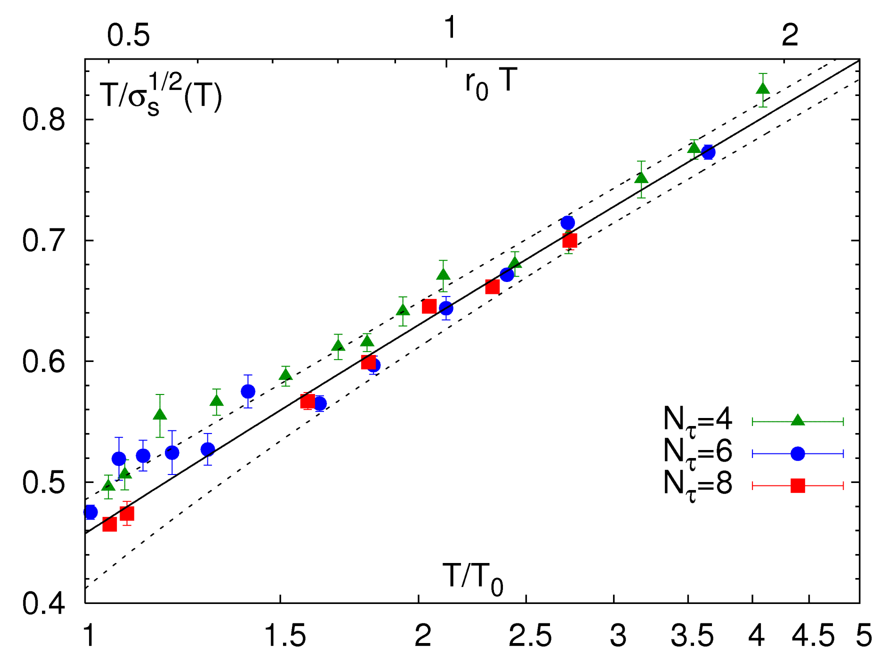
$g_3^2(T) = g^2(T)T(1 + c_1(N_f,T)g^2(T) + ...)$

Laine, Schröder, JHEP0503(05) 067

Calculated perturbatively



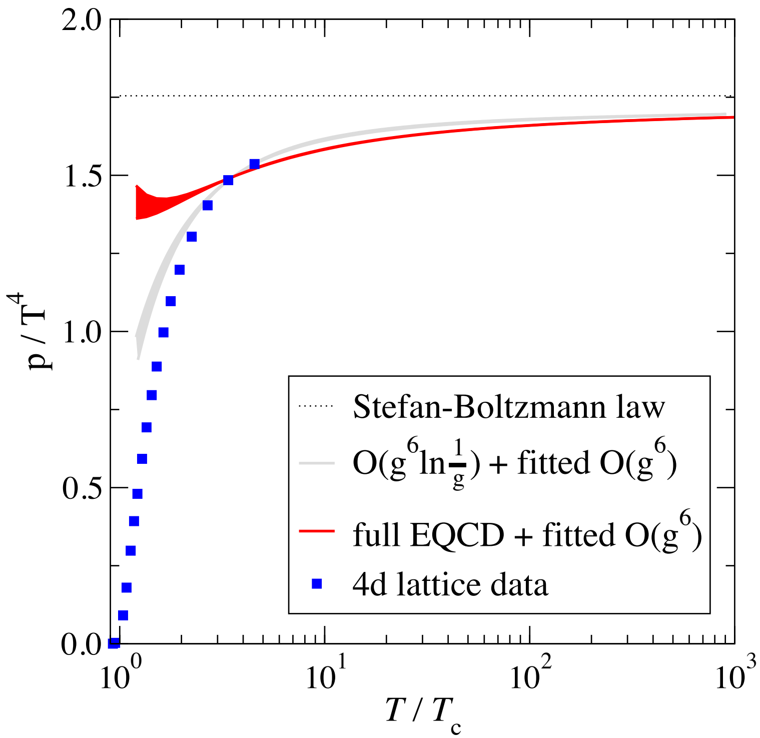
magnetic screening



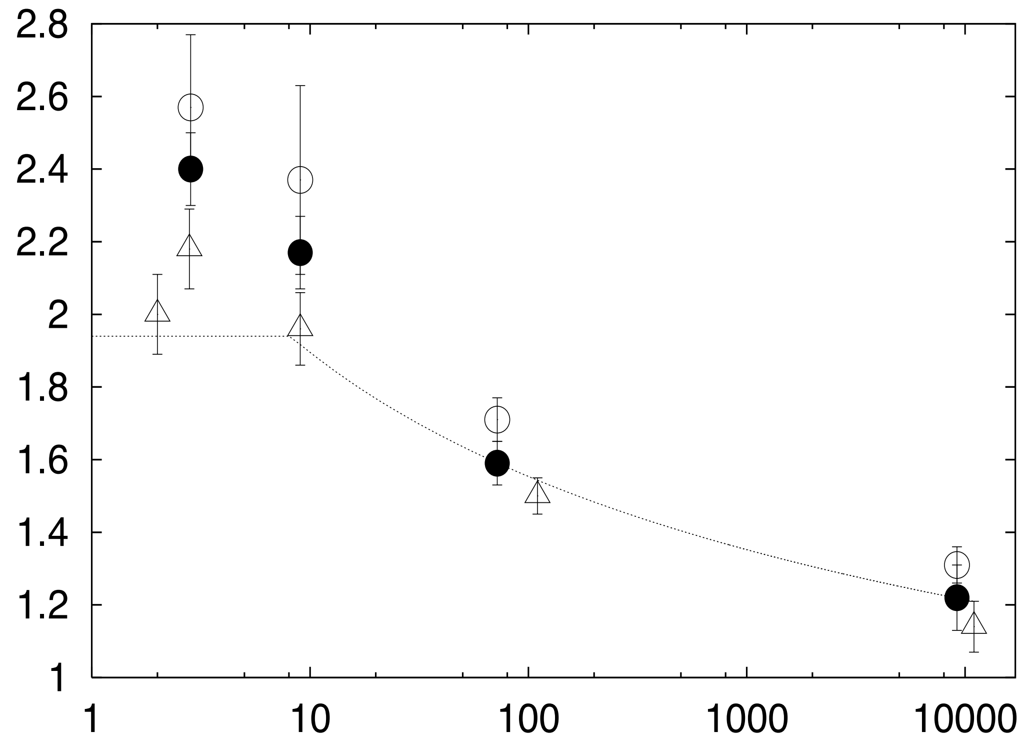
The T-dependence of spatial string tension is perturbative for $T > 1.5T_c$!

Pressure and screening mass in the 3d effective theory

Pressure in 3d theory
Kajantie et al, 2009



Electric screening mass
Cuchieri et al, 2001

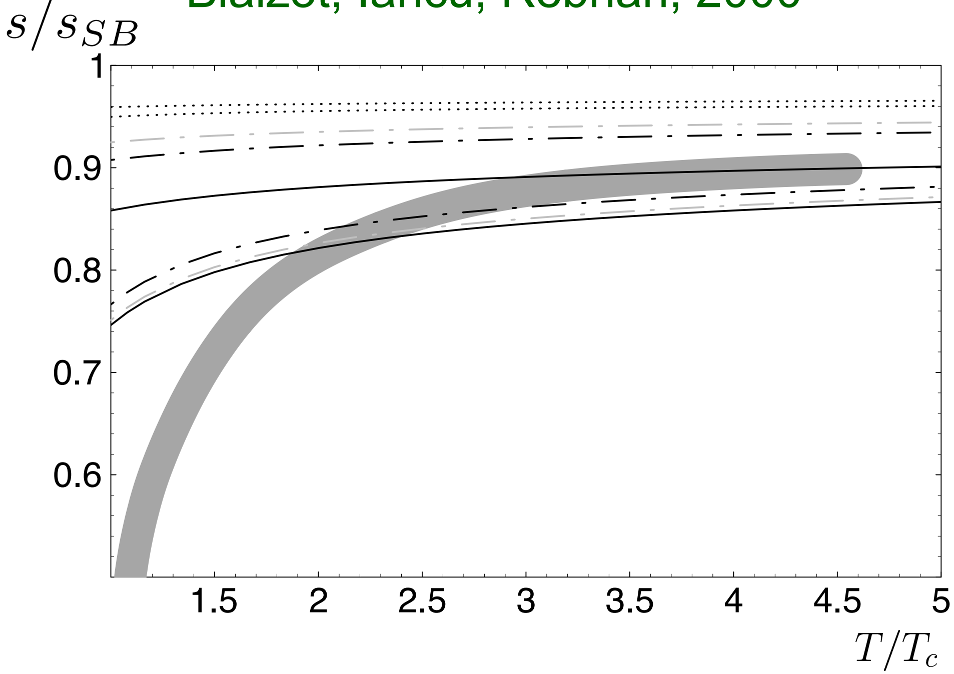


Line is the fit to the 4d lattice results. Different symbols correspond to different choices of the 3d mass parameter

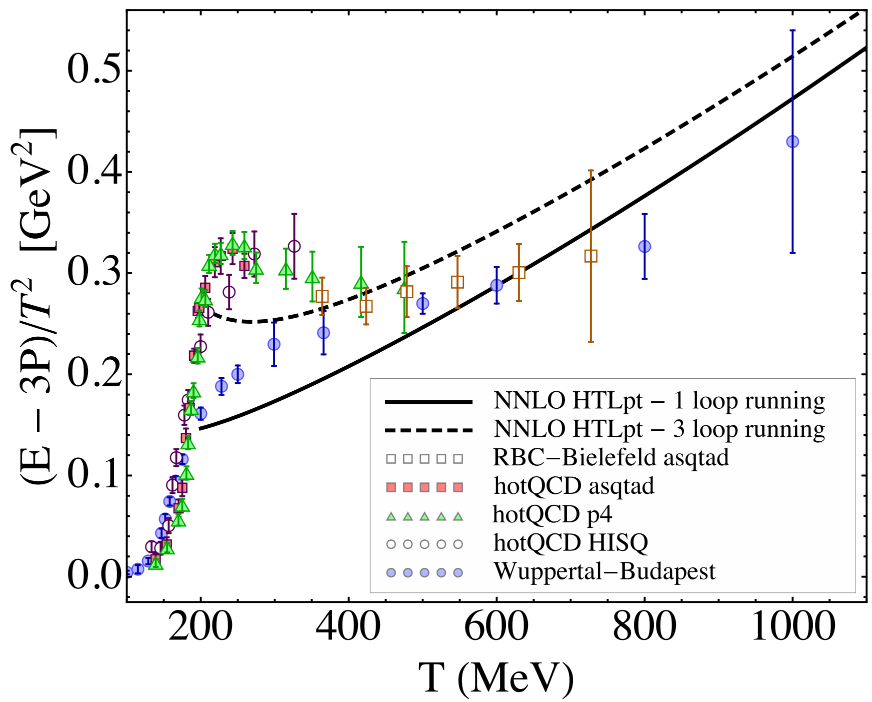
Pressure and trace anomaly in HTL perturbation theory

Perform resummation with δL_{HTL}

Blaizot, Iancu, Rebhan, 2000



Andersen et al, 2011



Free energy of a static quark anti-quark pair at high T

$$G(r,T) = \frac{1}{9}G_1(r,T) + \frac{8}{9}G_8(r,T) \equiv \frac{1}{9}e^{-F_1(r,T)/T} + \frac{8}{9}e^{-F_8(r,T)/T}$$

Perturbation theory ($T \gg T_c$):

$$F_1(r,T) = -\frac{4}{3}\frac{\alpha_s}{r}e^{-m_D r} - \frac{4}{3}\alpha_s m_D,$$

$$F_8(r,T) = +\frac{1}{6}\frac{\alpha_s}{r}e^{-m_D r} - \frac{4}{3}\alpha_s m_D,$$

The work to separate $Q\bar{Q}$ from distance r_1 to r_2

$$A = F(r_2) - F(r_1)$$

In leading order perturbation theory:

$$F(r,T) = -\frac{1}{9}\frac{\alpha_s^2}{r^2 T}\exp(-2m_D r)$$

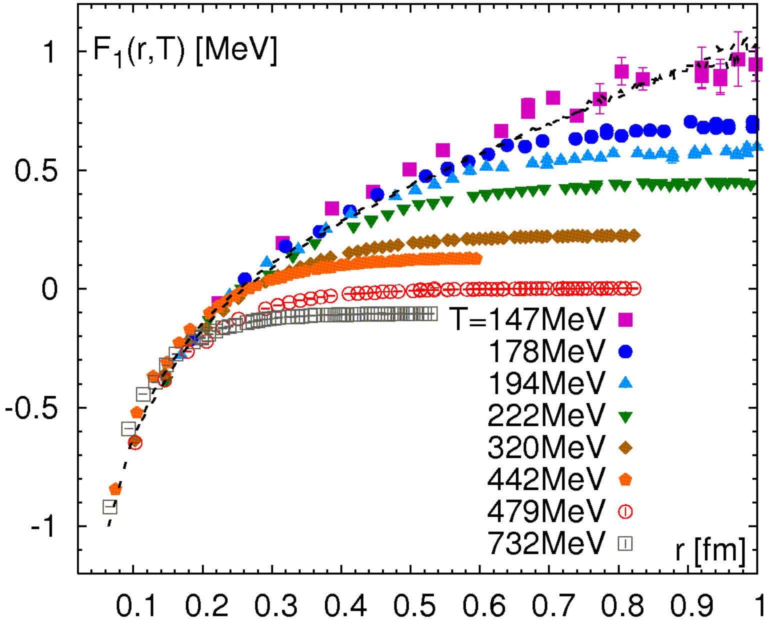
In QED

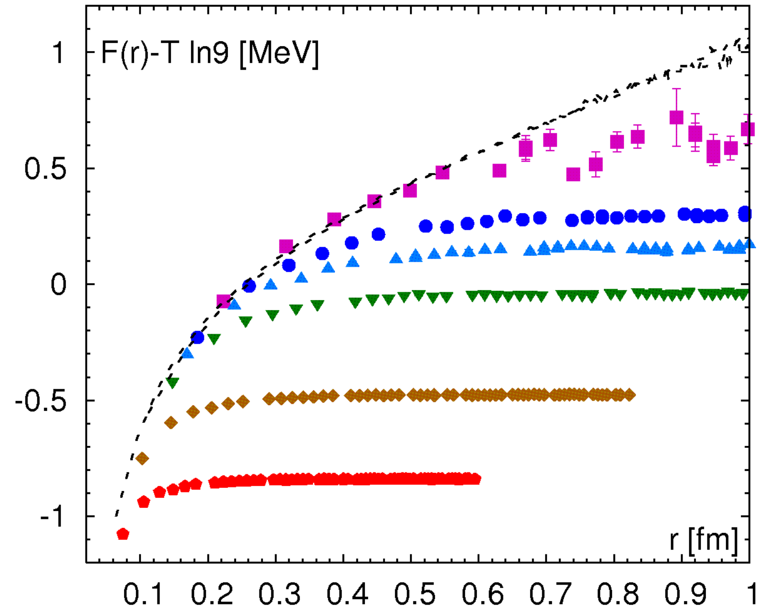
$$F(r,T) = -\frac{\alpha}{r}\exp(-m_D r)$$

In QCD the work is reduced due to cancelation between color singlet and octet contribution

Static quark anti-quark free energy in 2+1f QCD

HISQ action, $24^3 \times 6$, $16^3 \times 4$ (high T) lattices, $m_\pi \simeq 160$ MeV

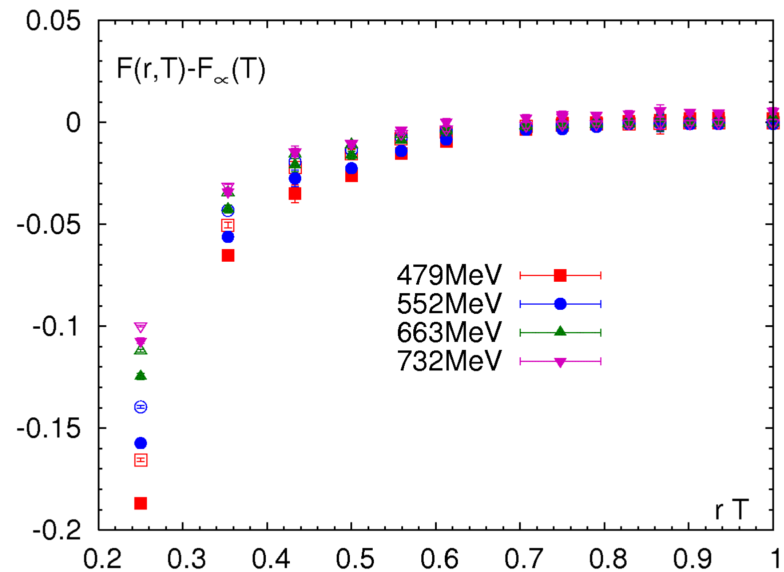




- The strong *T*-dependence for *T<200* MeV is not necessarily related to color screening

- The free energy has much stronger *T*-dependence than the singlet free energy due to the octet contribution

- At high T the temperature dependence of the free energy can be entirely understood in terms of F_1 and Casimir scaling $F_1=-8\ F_8$



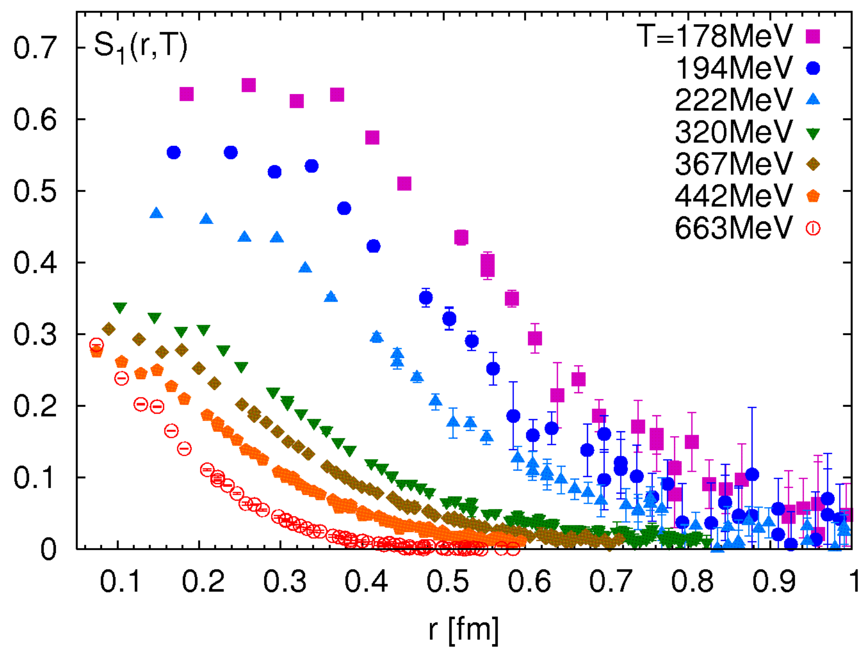
Static quark anti-quark free energy in 2+1f QCD (cont'd)

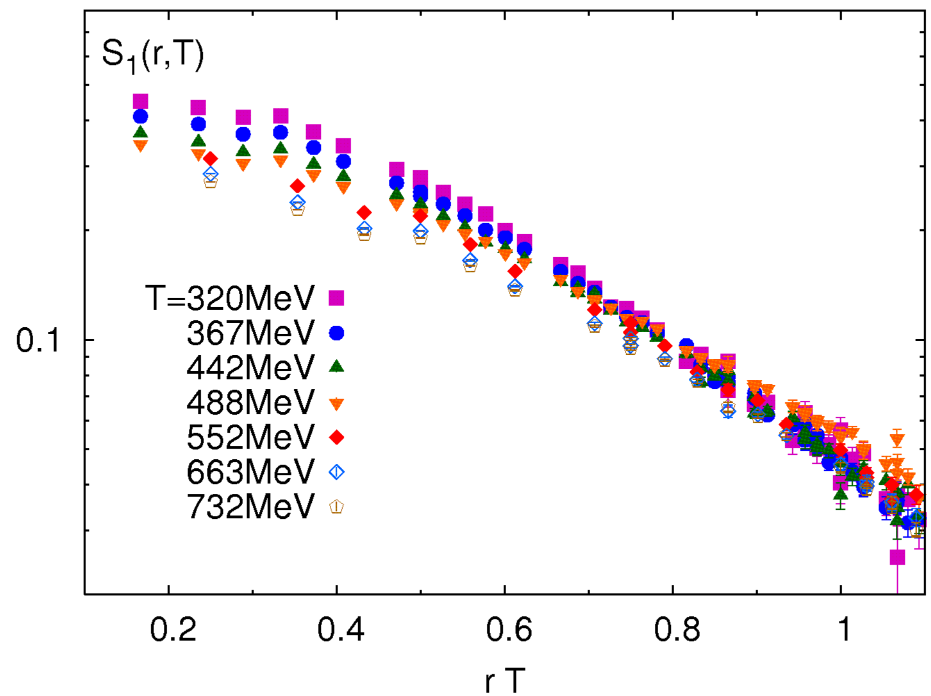
Leading order weak coupling :

$$F_1(r,T) = C_F \frac{\alpha_s}{r} \exp(-m_D r) + F_\infty(T)$$

Screening function: $$S_1(r,T) = (F_\infty(T) - F_1(r,T) \cdot r$$

$$rT \ll 1, \quad S_1(r,T) \sim \alpha_s \qquad\qquad rT \gg 1, \quad S_1(r,T) \sim \exp(-m_D r)$$





Virial expansion and Hadron Resonance Gas

Chiral perturbation theory is limited to pion gas. Other hadrons, resonances ?
Relativistic virial expansion : compute thermodynamic quantities in terms
as a gas of non-interacting particles and S – matrix
Dashen, Ma, Bernstein, '69

Free gas   interactions

$$\ln Z = \ln Z_0 + \sum_{i_1, i_2} e^{\mu_{i_1}/T} e^{\mu_{i_2}/T} b(i_1, i_2)$$

$$b(i_1, i_2) = \frac{V}{4\pi i} \int \frac{d^3p}{(2\pi)^3} \int dE e^{-(p^2+E^2)^{1/2}/T} \sum_{final} \left[AS\left(S^{-1}\frac{\partial S}{\partial E} - \frac{\partial S^{-1}}{\partial E} S\right) \right]$$

Elastic scattering only (final state = initial state)

$$S(E) = \sum_{l,I}{}' (2l+1)(2I+1) \exp(2i\delta_l^I(E))$$

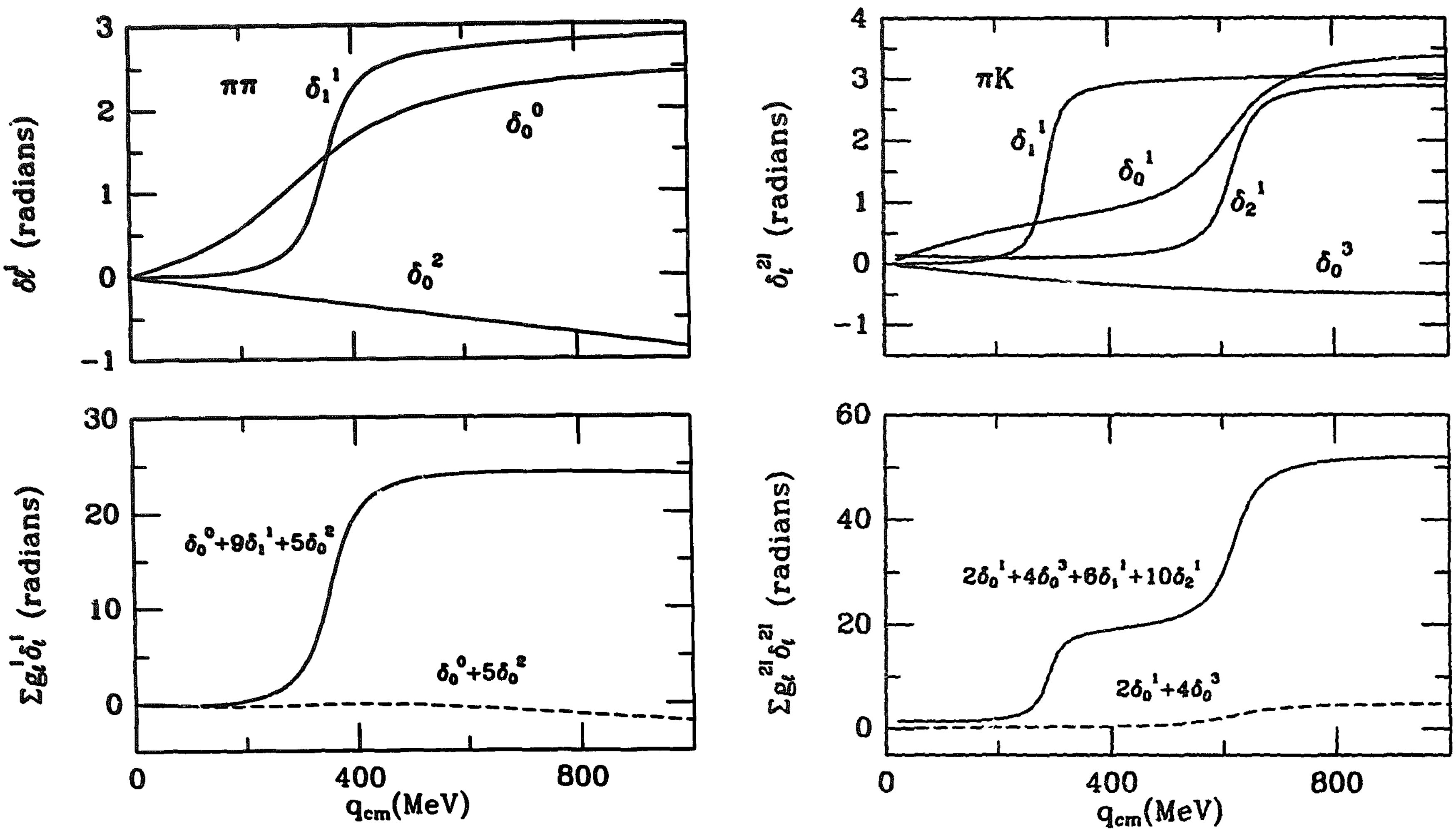
Partial wave decomposition

perform the integral over the 3-momentum

$$b_2 = \frac{T}{2\pi^3} \int_M^\infty dE E^2 K_2(E/T) \sum\nolimits_{l,I}' (2l+1)(2I+1) \frac{\partial \delta_l^I(E)}{\partial E}$$

invarian mass of the pair at threshold

Use experimental phase shifts to determine b_2 , Venugopalan, Prakash '92

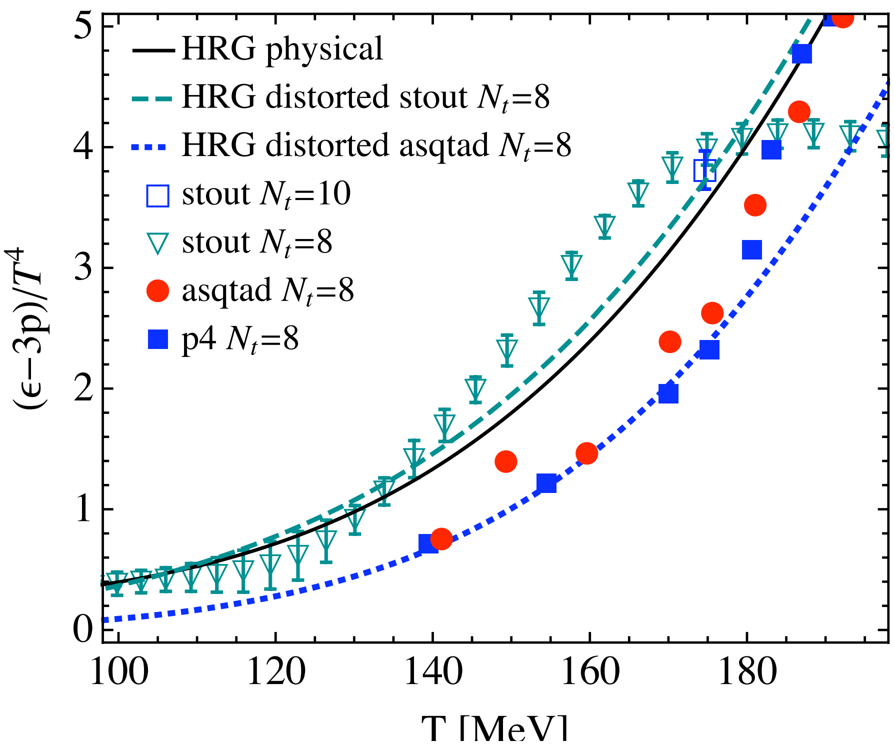


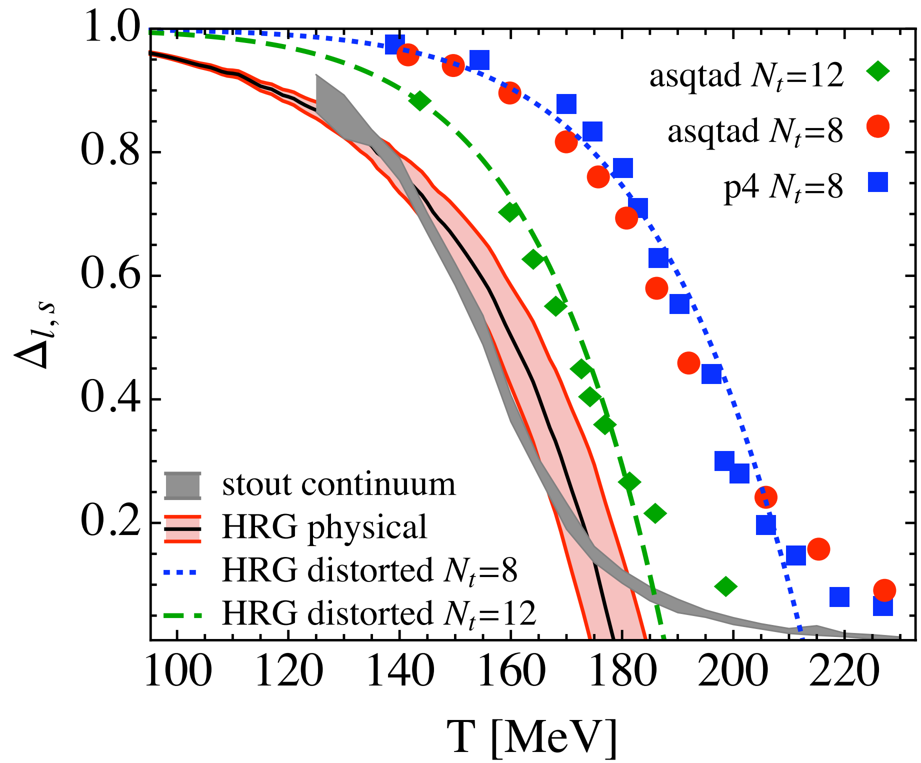
After summing all the channels only resonance contributions survives in ,

$$\sum_{l,I}(2l+1)(2I+1)\frac{\partial \delta_l^I(E)}{\partial E}$$

Interacting hadron gas = non-nteracting gas of hadrons and resonances

Hadron resonance gas versus lattice QCD calculations





Wuppertal-Budapest Collaboration, JHEP 1009 (2010) 073

QCD thermodynamics at non-zero chemical potential

Taylor expansion :

$$\frac{p(T,\mu_B,\mu_Q,\mu_S)}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!l!} \chi_{ijk}^{BQS} \cdot \left(\frac{\mu_B}{T}\right)^i \cdot \left(\frac{\mu_S}{T}\right)^j \cdot \left(\frac{\mu_Q}{T}\right)^k$$ hadronic

$$\frac{p(T,\mu_u,\mu_d,\mu_s)}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}^{uds} \cdot \left(\frac{\mu_u}{T}\right)^i \cdot \left(\frac{\mu_d}{T}\right)^j \cdot \left(\frac{\mu_s}{T}\right)^k$$ quark

$$\chi_{ijk}^{abc} = T^{i+j+k} \frac{\partial^i}{\partial \mu_a^i} \frac{\partial^j}{\partial \mu_b^j} \frac{\partial^k}{\partial \mu_c^k} \frac{1}{VT^3} \ln Z(T,V,\mu_a,\mu_b,\mu_c)|_{\mu_a=\mu_b=\mu_c=0}$$

Taylor expansion coefficients give the fluctuations and correlations of conserved charges, e.g.

$$\chi_2^X = \chi_X = \frac{1}{VT^3}(\langle X^2\rangle - \langle X\rangle^2) \qquad \chi_{11}^{XY} = \frac{1}{VT^3}(\langle XY\rangle - \langle X\rangle\langle Y\rangle)$$

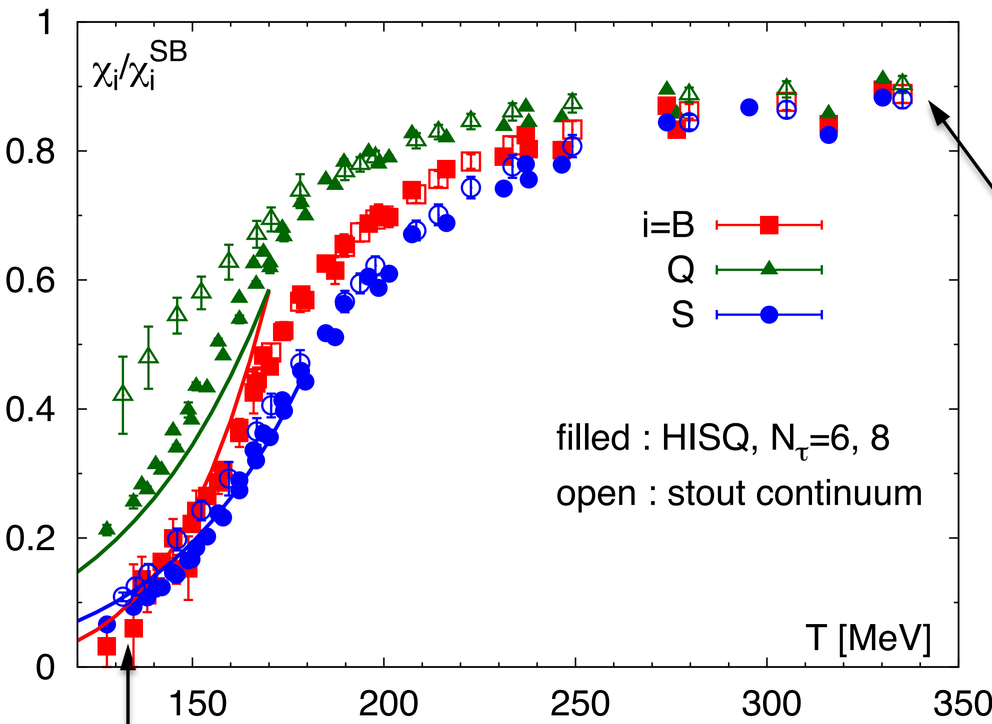
Computation of Taylor expansion coefficients reduces to calculating the product of inverse fermion matrix with different source vectors **=> can be done effectively on GPUs**

Deconfinement : fluctuations of conserved charges

$$\chi_{\rm B}^{\rm SB} = \frac{1}{VT^3}(\langle B^2\rangle - \langle B\rangle^2)$$ baryon number

$$\chi_{\rm Q}^{\rm SB} = \frac{1}{VT^3}(\langle Q^2\rangle - \langle Q\rangle^2)$$ electric charge

$$\chi_{\rm S}^{\rm SB} = \frac{1}{VT^3}(\langle S^2\rangle - \langle S\rangle^2)$$ strangeness



conserved charges are carried by massive hadrons

Ideal gas of massless quarks :

$$\chi_B^{\rm SB} = \frac{1}{3} \qquad \chi_Q^{\rm SB} = \frac{2}{3}$$

$$\chi_B^{\rm SB} = 1$$

conserved charges carried by light quarks

HotQCD: PRD86 (2012) 034509

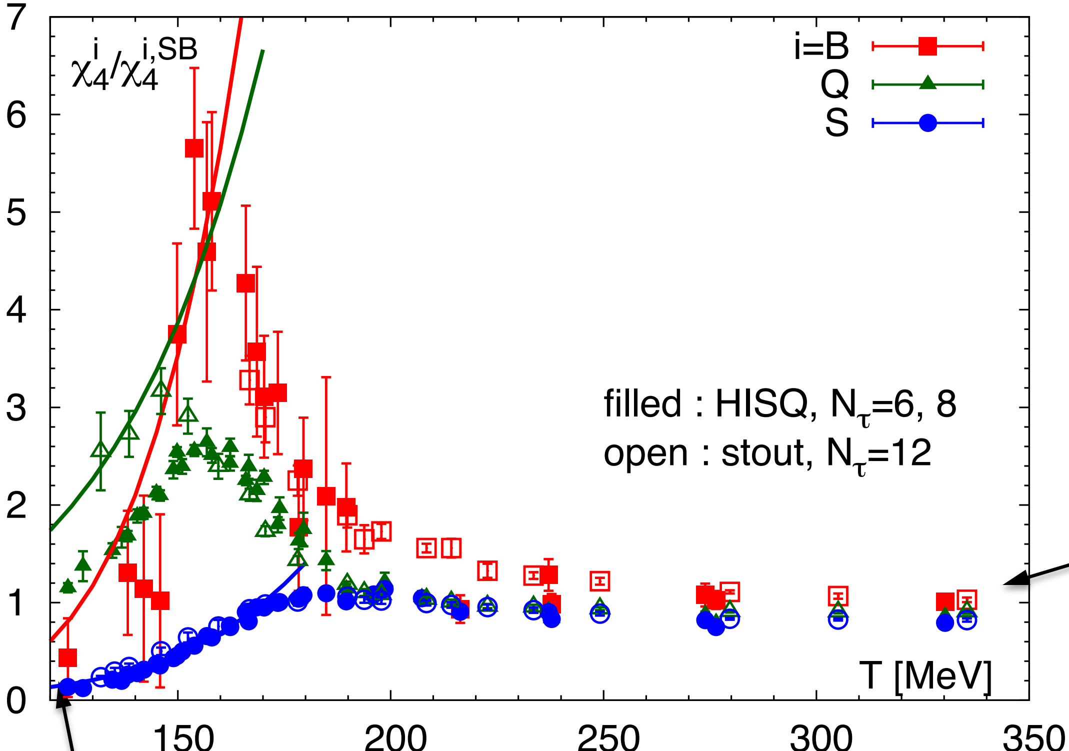
BW: JHEP 1201 (2012) 138,

Deconfinement : fluctuations of conserved charges

$$\chi_4^B = \frac{1}{VT^3}(\langle B^4\rangle - 3\langle B^2\rangle^2)$$ baryon number

$$\chi_4^Q = \frac{1}{VT^3}(\langle Q^4\rangle - 3\langle Q^2\rangle^2)$$ electric charge

$$\chi_4^S = \frac{1}{VT^3}(\langle S^4\rangle - 3\langle S^2\rangle^2)$$ strangeness



Ideal gas of massless quarks :

$$\chi_4^B{}_{\text{SB}} = \frac{2}{9\pi^2} \qquad \chi_4^Q{}_{\text{SB}} = \frac{4}{3\pi^2}$$

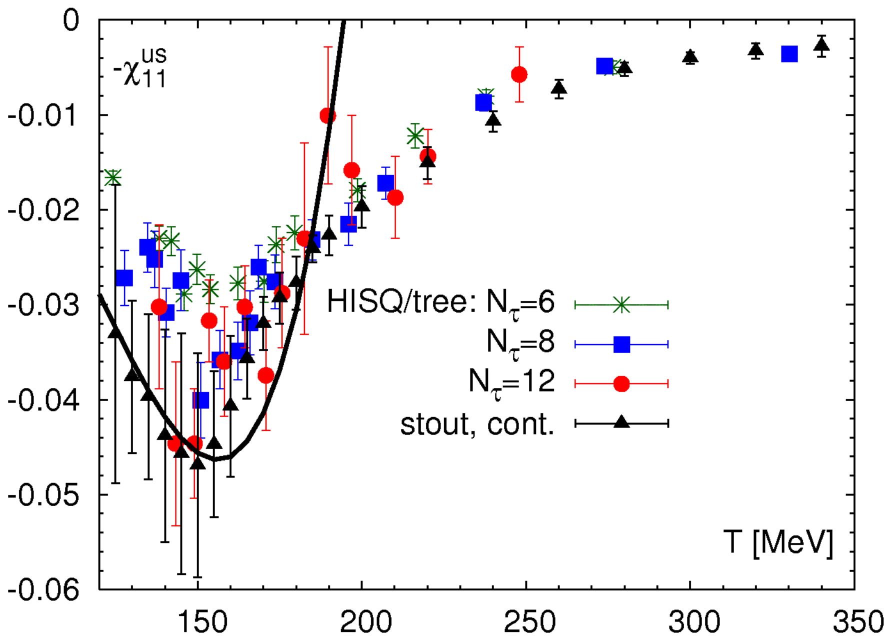
$$\chi_4^S{}_{\text{SB}} = \frac{6}{\pi^2}$$

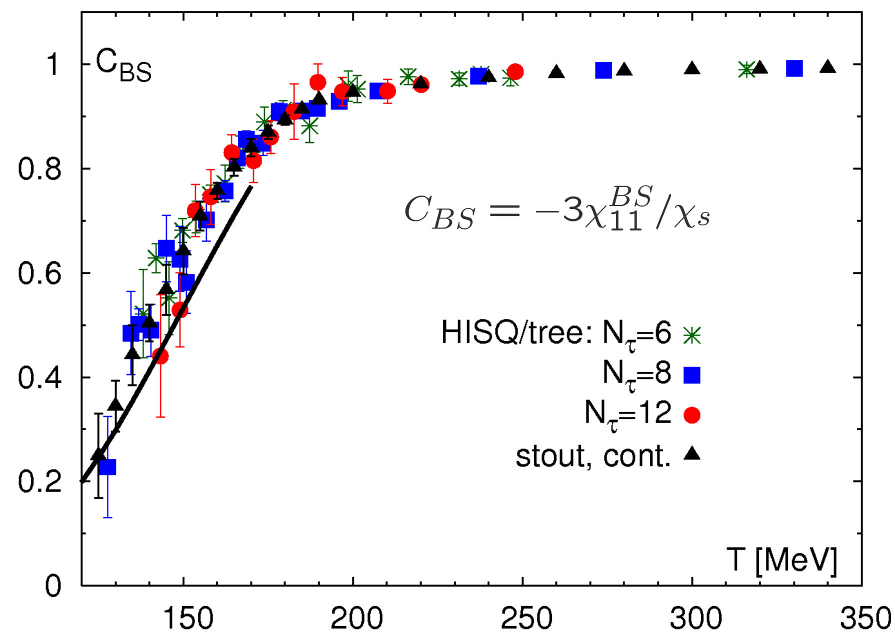
conserved charges carried by light quarks

conserved charges are carried by massive hadrons

BNL-Bielefeld : talk by C. Schmidt
BW: talk by Borsanyi
@ Confinement X conference

Correlations of conserved charges





P.P. J.Phys. G39 (2012) 093002

- Correlations between strange and light quarks at low *T* are due to the fact that strange hadrons contain both strange and light quarks but very small at high *T (>250* MeV*)* => weakly interacting quark gas

- For baryon-strangeness correlations HISQ results are close to the physical HRG result, at *T>250* MeV these correlations are very close to the ideal gas value

- The transition region where degrees of freedom change from hadronic to quark-like is broad ~ (100-150) MeV

Deconfinement of strangeness

Partial pressure of strange hadrons in uncorrelated hadron gas:

$$P_S = \frac{p(T) - p_{S=0}(T)}{T^4} = M(T)\cosh\left(\frac{\mu_S}{T}\right) +$$

$$B_{S=1}(T)\cosh\left(\frac{\mu_B - \mu_S}{T}\right) + B_{S=2}(T)\cosh\left(\frac{\mu_B - 2\mu_S}{T}\right) + B_{S=3}(T)\cosh\left(\frac{\mu_B - 3\mu_S}{T}\right)$$

$$v_1 = \chi_{31}^{BS} - \chi_{11}^{BS}$$

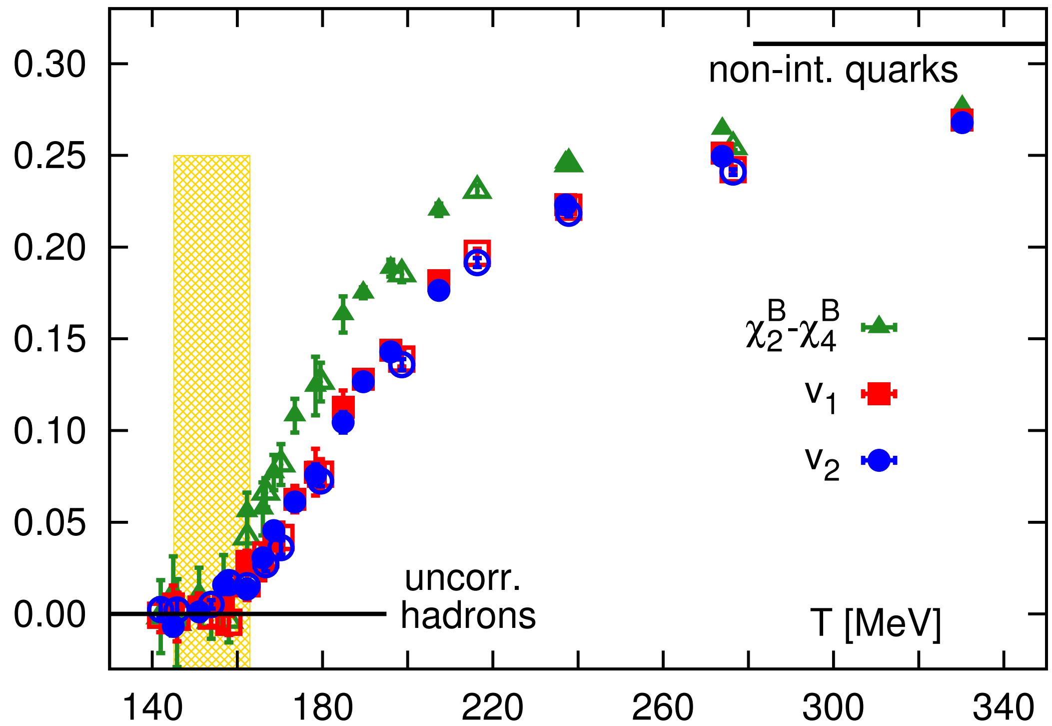
$$v_2 = \frac{1}{3}\left(\chi_4^S - \chi_2^S\right) - 2\chi_{13}^{BS} - 4\chi_{22}^{BS} - 2\chi_{31}^{BS}$$

should vanish !

- v_1 and v_2 do vanish within errors at low T

- v_1 and v_2 rapidly increase above the transition region, eventually reaching non-interacting quark gas values

BNL-Bielefeld, arXiv:1304.7220



Deconfinement of strangeness (cont'd)

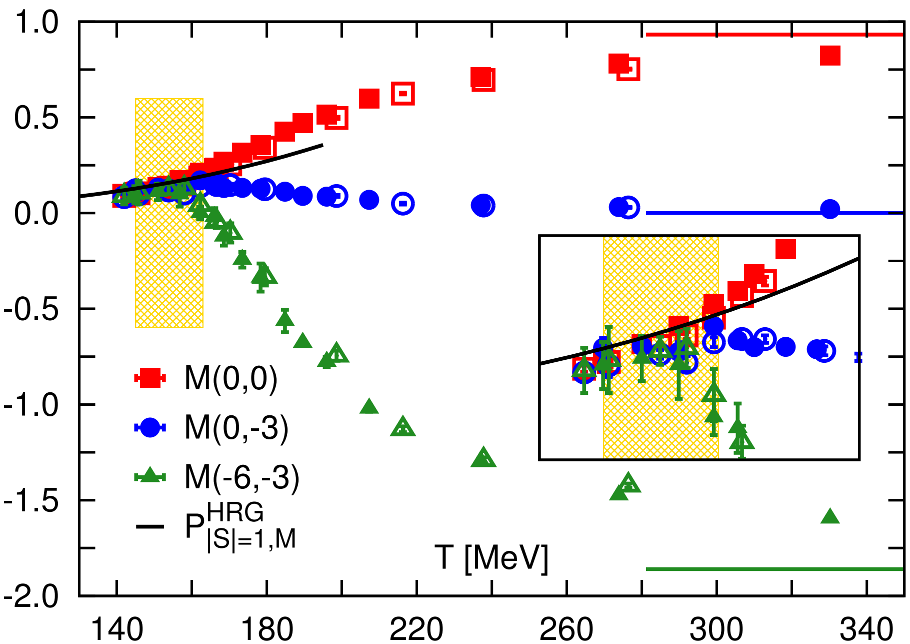
Using the six Taylor expansion coefficients related to strangeness

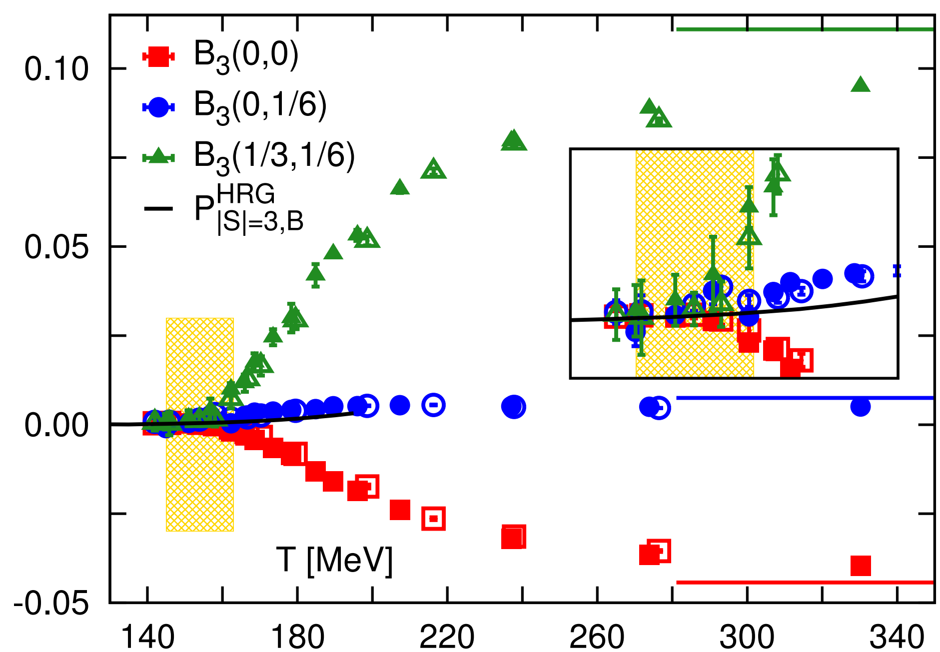
$$\chi_2^S, \ \chi_4^S, \ \chi_{13}^{BS}, \ \chi_{22}^{BS}, \ \chi_{31}^{BS}$$

it is possible to construct combinations that give

$$M(T), \ B_{S=1}(T), \ B_{S=2}(T), \ B_{S=3}(T)$$

up to terms $c_1 v_1 + c_2 v_2$





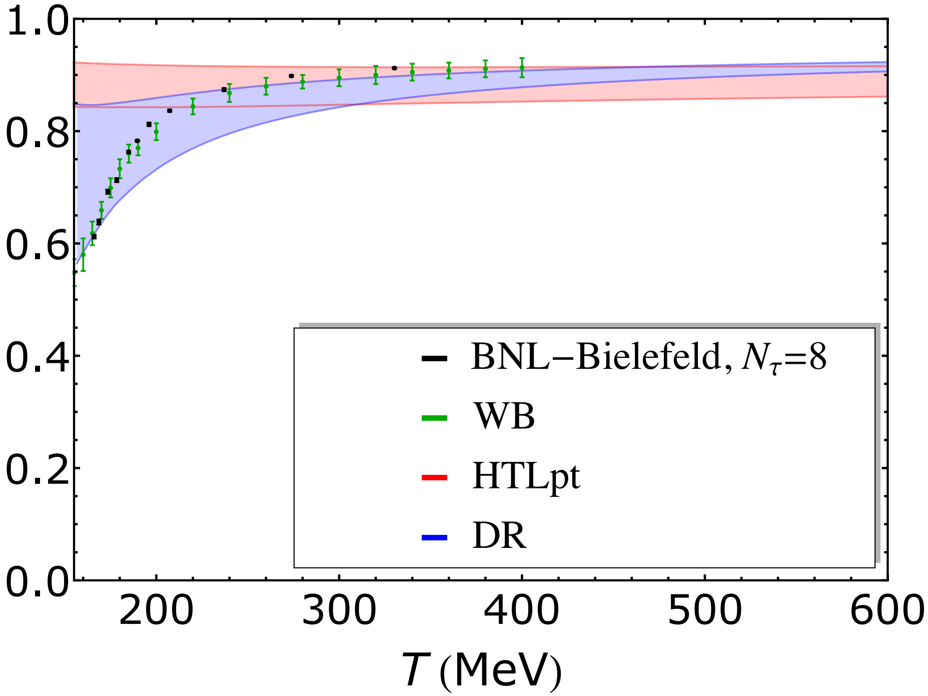
Hadron resonance gas descriptions breaks down for all strangeness sectors above T_c

BNL-Bielefeld, arXiv:1304.7220

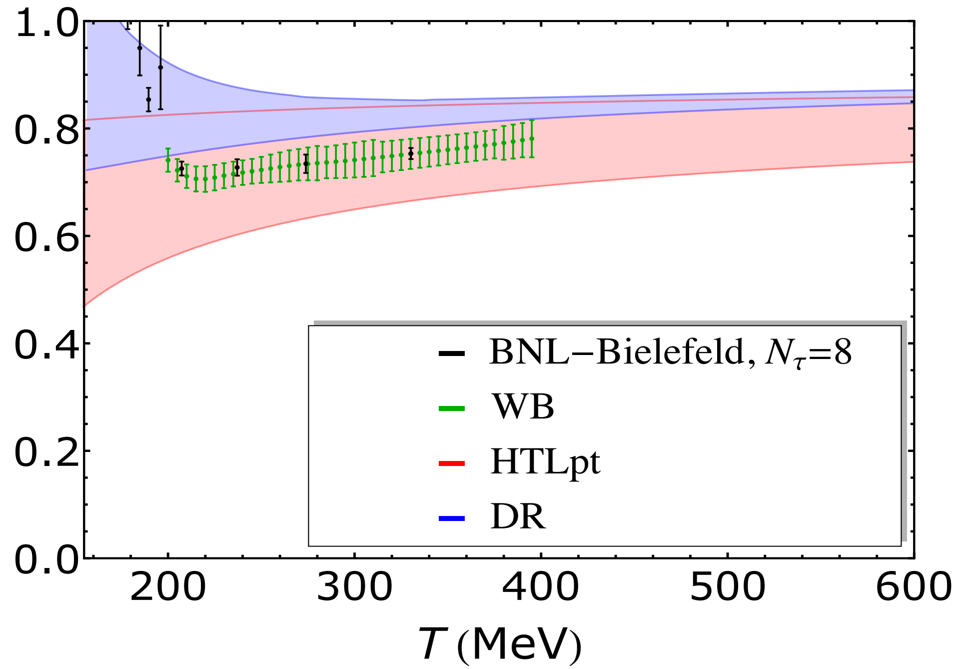
Quark number fluctuations at high T

At high temperatures quark number fluctuations can be described by weak coupling approach due to asymptotic freedom of QCD

2nd order quark number fluctuations



4th order quark number fluctuations



Andersen, Mogliacci, Su, Vuorinen, PRD87 (2013) 074003

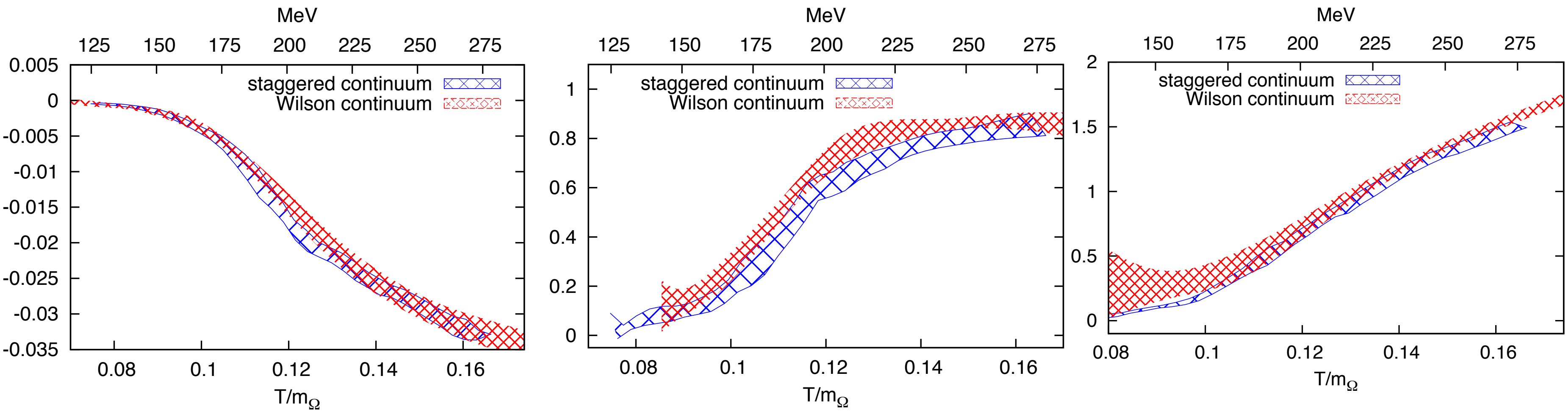
- Lattice results converge as the continuum limit is approached
- Good agreement between lattice and the weak coupling approach for 2nd order quark number fluctuations
- For 4th order the weak coupling results are in reasonable agreement with lattice

Staggered versus Wilson and Overlap Fermioms

Comparison with Wilson Fermion calculations, $m_\pi \approx 500$ MeV,

Borsányi et al, arXiv:1205.0440

Chiral condensate | Strangeness susceptibility | Polyakov loop



Comparison with overlap Fermion calculations, $m_\pi \approx 350$ MeV

Borsányi et al, PLB713 (12) 342