Heavy Quark Transport and Spectral Functions

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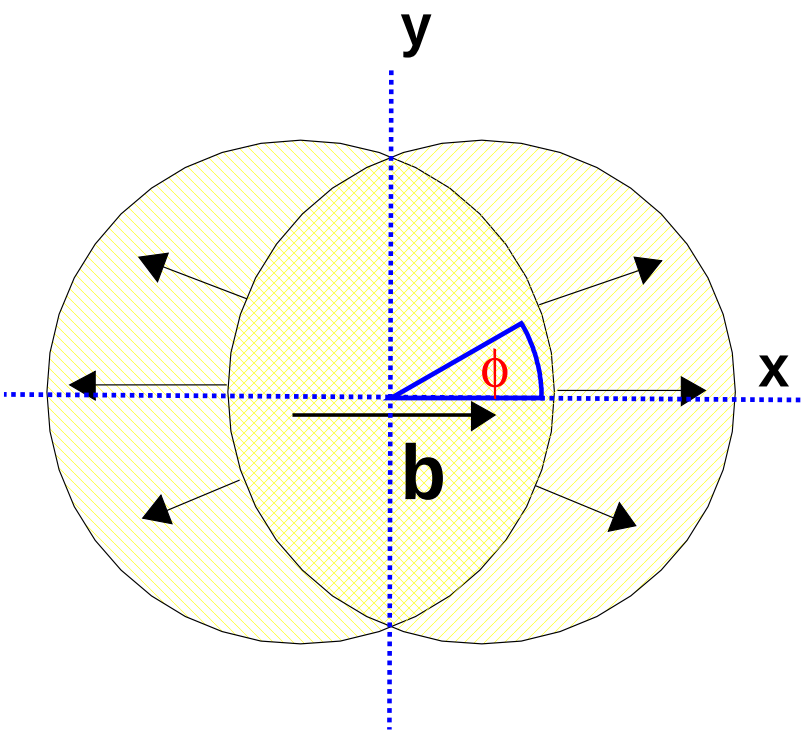
- This work was done in close collaboration with Jorge Casalderrey-Solana and Peter Petreczky
  - Jorge Casalderrey-Solana, DT; In progress
  - Jorge Casalderrey-Solana, DT; hep-th/0701123
  - Jorge Casalderrey-Solana, DT; hep-ph/0605199

Observation:



There is a large momentum anisotropy:

$$v_2 \equiv \frac{\left\langle p_x^2 - p_y^2 \right\rangle}{\left\langle p_x^2 + p_y^2 \right\rangle} \approx 20\%$$

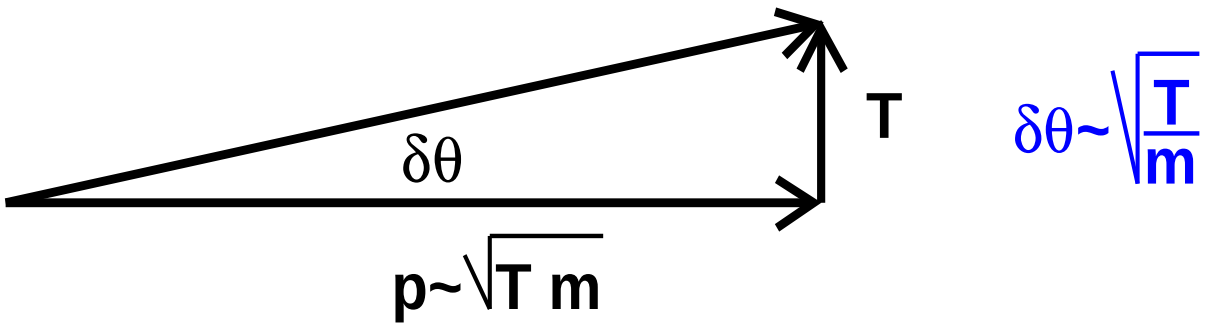
Interpretation

- The medium responds as a fluid to differences in X and Y pressure gradients

Hydro models “work”

Estimate of transport times with Heavy Quarks

- Put a heavy quark in this medium



- The charm quark undergoes a random walk suffering many collisions

- The relaxation time of the heavy quark is:

$$\tau_R^{\text{charm}} \sim \frac{M}{T}\tau_R^{\text{light}}$$

If you think you know the relaxation time you should be able to compute the charm spectrum.

Langevin description of heavy quark thermalization:

- Write down an equation of motion for the heavy quarks.

$$\frac{dx}{dt} = -\frac{p}{M}$$
$$\frac{dp}{dt} = -\underbrace{\eta_D p}_{\text{Drag}} + \underbrace{\xi(t)}_{\text{Random Force}}$$

- The drag and the random force are related

$$\langle \xi_i(t)\xi_j(t')\rangle = \frac{\kappa}{3}\delta_{ij}\,\delta(t-t') \qquad \eta_D = \frac{\kappa}{2MT}$$

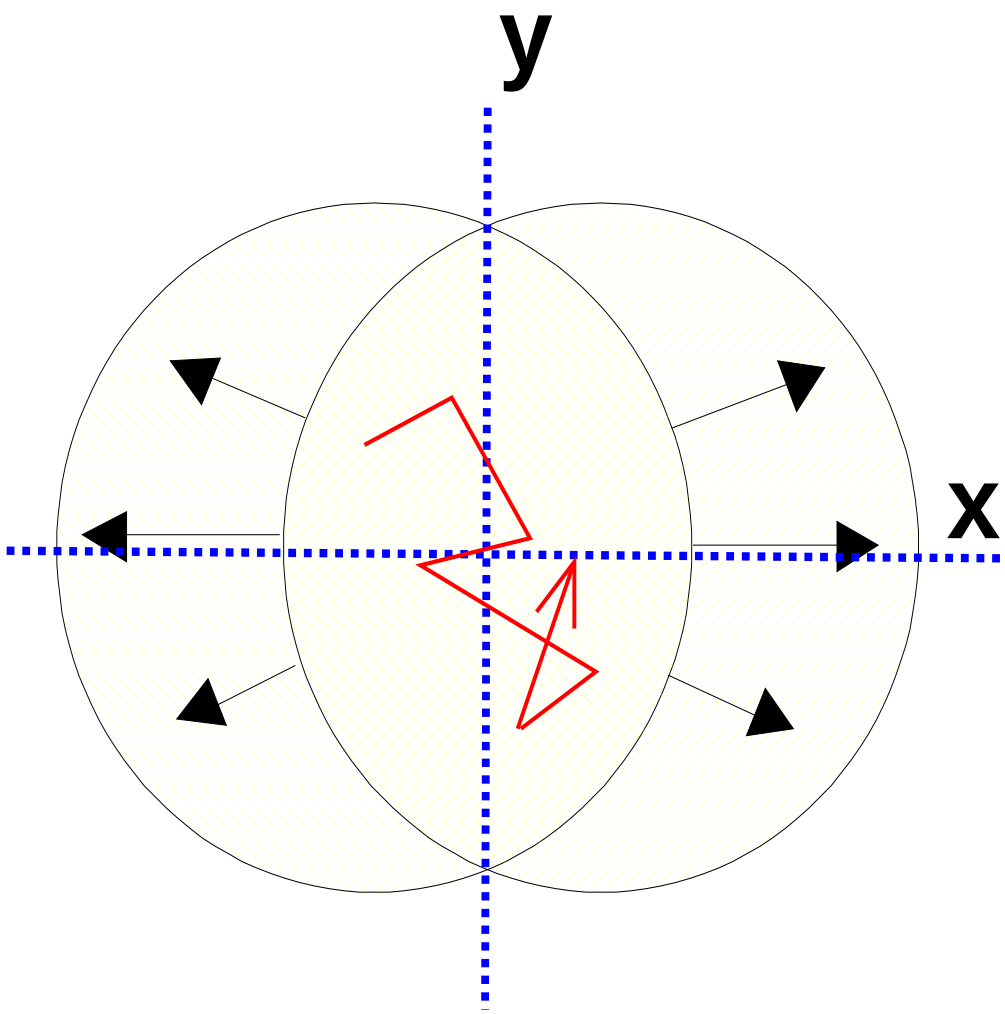
$\kappa =$ Mean Squared Momentum Transfer per Time

- Einstein related the diffusion coefficient to the mean squared momentum transfer

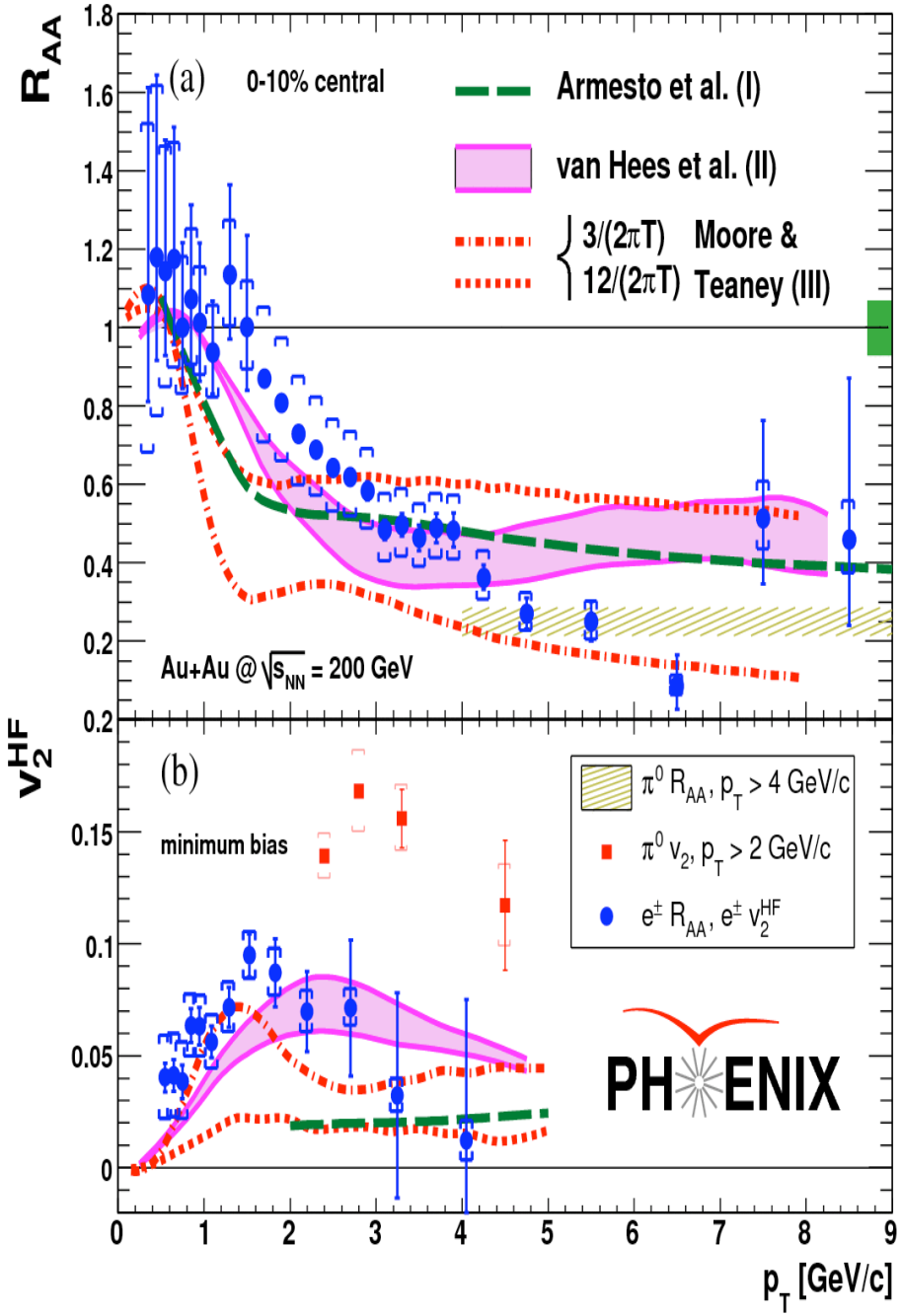
$$D = 2T^2/\kappa$$

All parameters are related to the heavy quark diffusion coefficient or κ

Hydro + Brownian Heavy Quarks



The heavy quarks will either relax to the thermal spectrum and show the same v_2 as all thermal particles or not depending on the Drag/Diffusion coefficients and p_T.



Summary

1. Suppression and Elliptic Flow are intimately related.
2. From the suppression pattern, we estimate that

$$D \lesssim \frac{12}{2\pi T}$$

With this diffusion coefficient, I can't produce enough elliptic flow.

I want to know the heavy quark diffusion coefficient

- Lattice $\longrightarrow$ Spectral Functions
- Compute at strong coupling $\longrightarrow$ AdS/CFT

Matching Langevin to a Microscopic Theory

- Heavy Quarks are Quasi Classical

$$\lambda_{\text{de Broglie}} \sim \frac{\hbar}{\sqrt{MT}} \ll \frac{\hbar}{T}$$

- Compare the Langevin process to the microscopic theory

Langevin

$$\frac{dp}{dt} = -\eta_D p + \xi(t)$$

Microscopic Theory

$$\frac{dp}{dt} = \mathcal{F}(t, \mathbf{x}) = qE(t, \mathbf{x})$$

- Match the Langevin to the Microscopic Theory

Langevin

$$\kappa = \int dt \, \langle \xi(t) \, \xi(0) \rangle$$

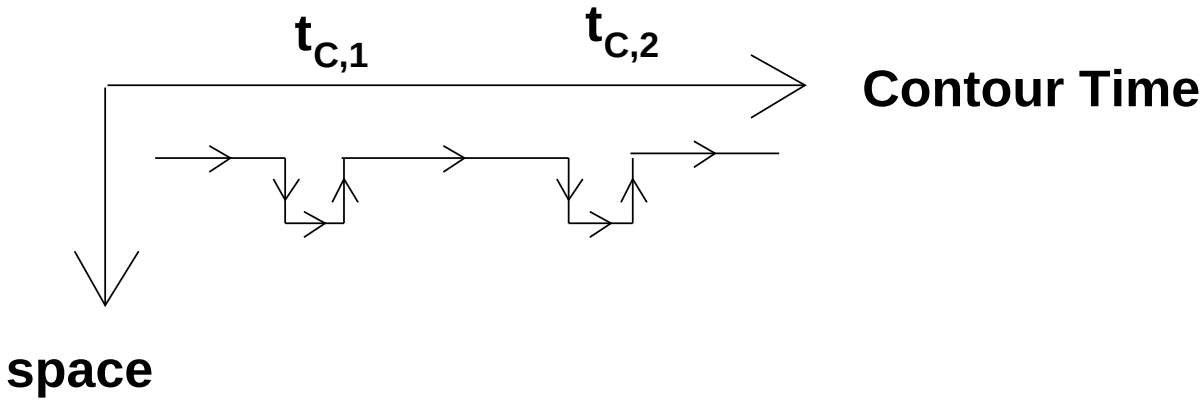
Microscopic Theory

$$\kappa = \int dt \, \langle \mathcal{F}(t, \mathbf{x}) \mathcal{F}(0, \mathbf{x}) \rangle_{HQ}$$

Diffusion Coefficient $\leftrightarrow$ Electric Field Correlator

Wilson Lines and the Diffusion Coefficient:

- Variations of Wilson Lines give electric field correlators

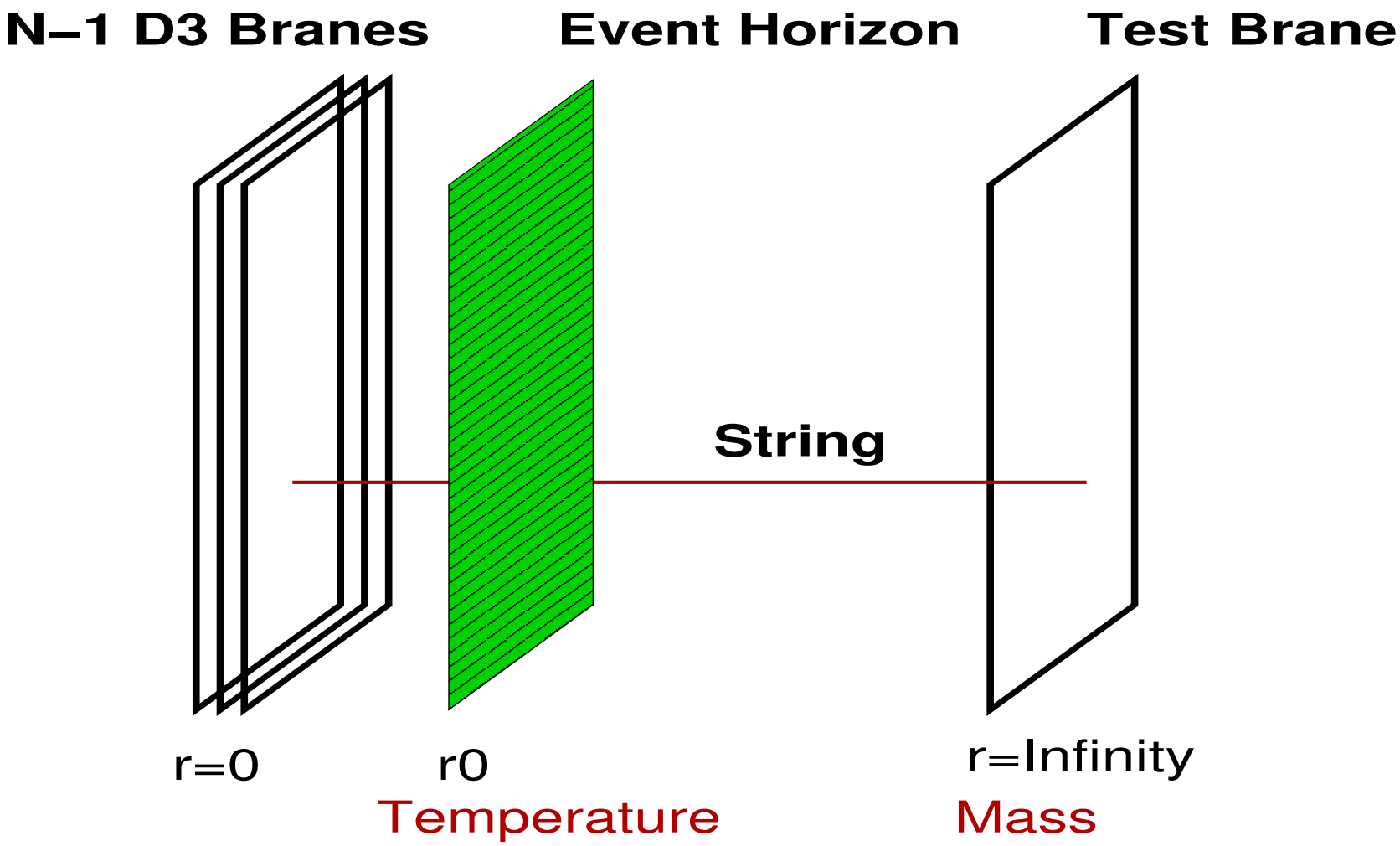


$$\left\langle \frac{\delta^2 W[\delta y]}{\delta y(t_1)\,\delta y(t_2)} \right\rangle_{YM} =, \langle\, T_C\,[\,E(t_2)E(t_1)\,]\,\rangle$$

- The varied Wilson line $=$ Force generating functional.

$$\left\langle e^{i\int_C dt\,\delta y(t)\,\mathcal{F}(t)} \right\rangle_{HQ} = \frac{1}{\langle W_C[0]\rangle}\,\langle W_C[\delta y]\rangle$$

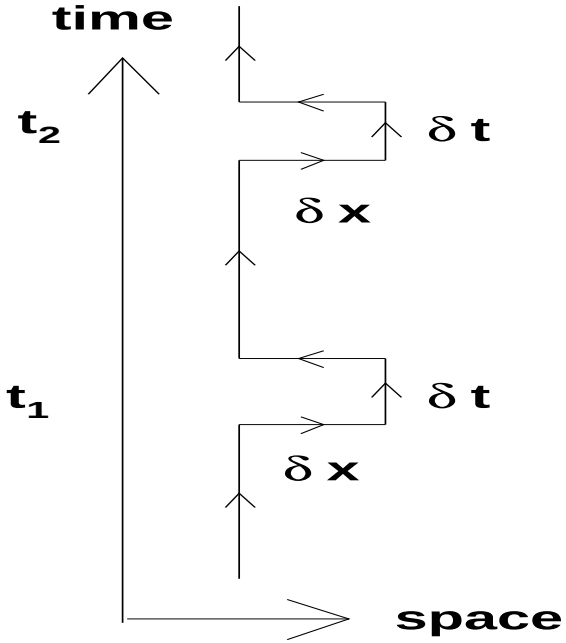
N=4 at Finite Temperature



Temperature of gauge theory is the hawking temperature of the Black hole

Testing the string dynamics:

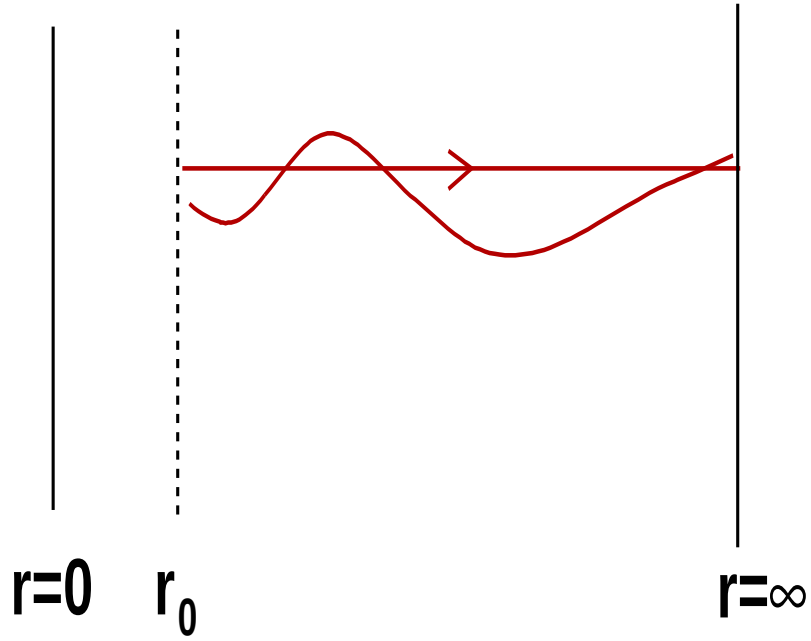
Boundary



$W[\delta x]$

$\Longrightarrow$

Bulk



$S_{cl}[\delta x]$

$$\frac{1}{\langle W_C[0]\rangle}\,\langle W_C[\delta y]\rangle = \frac{1}{e^{iS_{cl}[0]}}\,e^{iS_{cl}[\delta y]}$$

Summary of Technical Steps

- Fluctuate the end point of a string at $r = \infty$ at very small frequencies: $u = 1/r^2$

$$S_{NG} = \frac{R^2}{2\pi\ell_s^2} \int \underbrace{dt\, dr}_{\text{Area}} \left[1 - \underbrace{\frac{1}{2}\left(\frac{\dot{y}_{\parallel}^2}{f} - 4fu\left(y'_{\parallel}\right)^2 \right)}_{\text{Quadratic Fluctuations}} \right]$$

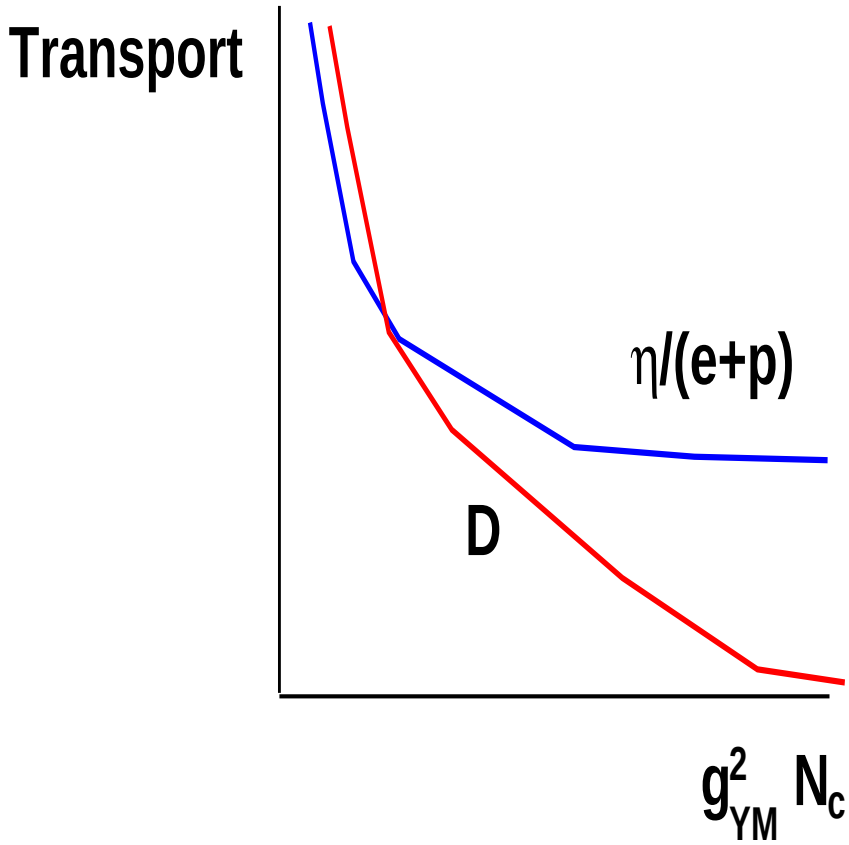
- Solve for the waves along the string in the $AdS_5 \times S_5$ metric
  - Impose incoming boundary conditions for retarded propagator .
- Differentiating twice with respect to δy yields the retarded force force correlator

$$\langle \mathcal{F}(t)\mathcal{F}(0)\rangle \sim \frac{\delta^2}{\delta y(t)\delta y(0)} \langle W[\delta y]\rangle \sim \frac{\delta^2}{\delta y(t)\delta y(0)} e^{iS_{NG}[\delta y]}$$

κ is the imaginary part of the retarded force-force correlator

Result

$$D = \frac{2}{\sqrt{\lambda}\pi T} \qquad \frac{\eta}{e+p} = \frac{1}{4\pi T}$$



QCD Guesses: Strong Coupling

- Strong Coupling: $\mathcal{N} = 4$ SUSY. $\lambda \approx 5 \leftrightarrow 20$

$$D = \frac{2}{\sqrt{\lambda}\pi T} \quad \longrightarrow \quad D \simeq \frac{1.0 \leftrightarrow 2.0}{2\pi T}$$

- Weak coupling (Aleski Vuorinnen)

  2 gluons + 6 scalars + 8 fermions $\neq$ 2 gluons

$$\frac{D_{QCD}}{D_{SYM}} = \frac{6}{1 + \frac{N_f}{2N_c}} \approx 4$$

- Best guess for QCD from strong coupling

$$D \approx \frac{4.0 \leftrightarrow 8.0}{2\pi T}$$

Compare to weak coupling best guess $D \approx 6/(2\pi T)$

Constraint On The Heavy Quark Mass

- To treat the heavy quark as a quasi-classical quasi-particle we need

$$\tau_R \gg \frac{\hbar}{T}$$

- Then we have

$$\tau_R \sim \frac{M}{T} D \qquad\qquad D = \frac{2}{\sqrt{\lambda}\pi T}$$

- This leads to a constraint on Mass/String Length

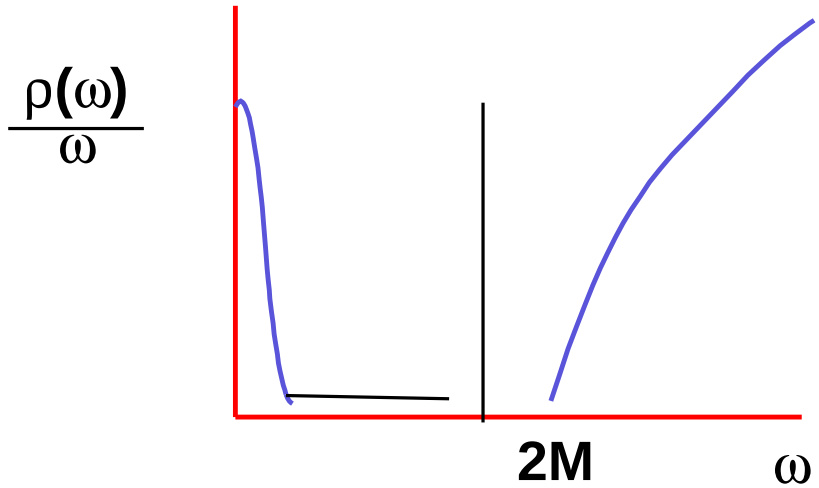
$$M \gg \frac{\pi T}{2}\sqrt{\lambda} \qquad\qquad L \gg r_o$$

- Substituting numbers we have

$$M \gg 1.7\text{GeV}\left(\frac{T}{0.250\,\text{GeV}}\right)\left(\frac{\alpha_{SYM}N}{1.5}\right)^{1/2}$$

Spectral Functions

$$\rho(\omega) = \int e^{+i\omega t} \langle [J(t), J(0)] \rangle$$



- Equations of motion predict the spectral weight

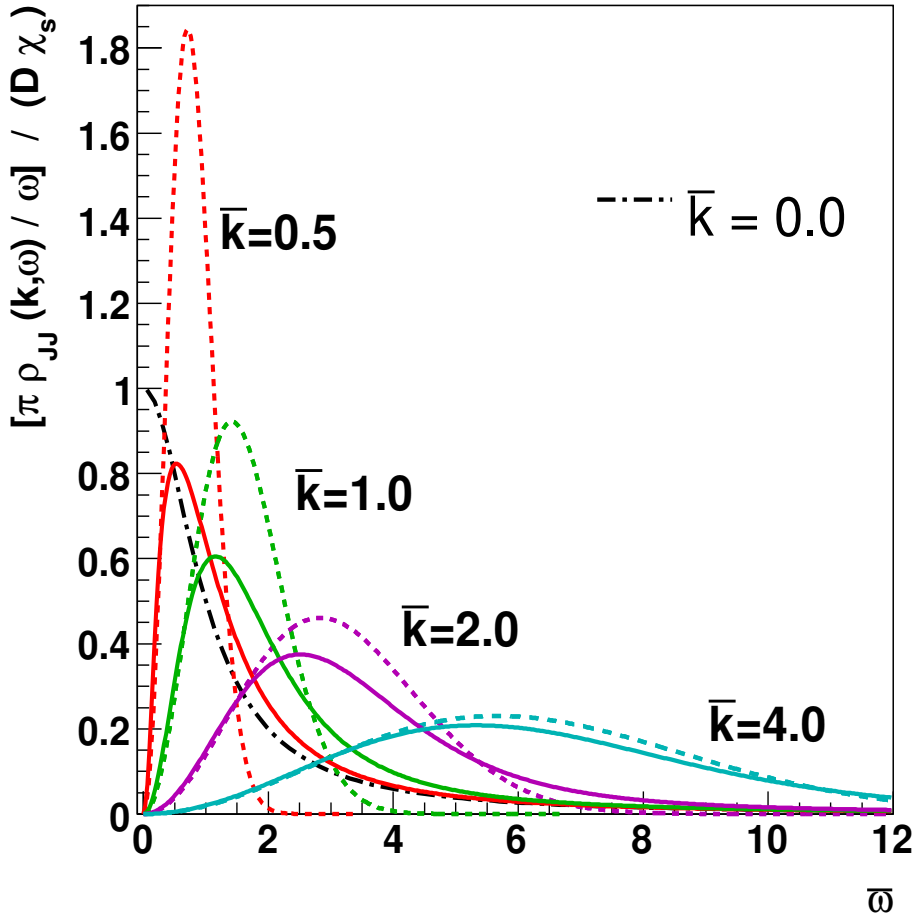
$$\frac{dv}{dt} = -\eta_D\, v + \xi \qquad\qquad \langle v(t) \rangle = v_0 e^{-\eta_D t}$$

- This makes the predictions

$$\frac{\rho(\omega)}{\omega} = \chi_s \frac{T}{M} \frac{\eta}{\omega^2 + \eta^2} \qquad \eta = \frac{T}{MD}$$

The Spectral Density

$$\bar{\omega} = \frac{\omega}{\eta_D} \qquad \bar{k} = \frac{k}{\sqrt{M/T}\eta_D}$$



How to get the full spectral function?

- We had the equation of motion

$$M_Q \frac{d^2x}{dt} + \underbrace{\frac{\kappa}{2T}}_{-\frac{\mathrm{Im}G_R}{\omega}} \times \underbrace{\frac{dx}{dt}}_{-i\omega} = \xi(t) \qquad \langle \xi(t)\xi(t')\rangle = \underbrace{\kappa}_{-\frac{2T}{\omega}\mathrm{Im}G_R} \delta(t-t')$$

- There is a finite frequency generalization of this

$$M_Q \frac{d^2x_r}{dt} + \int^t G_R(t-t')x_r(t') = \xi \qquad \langle \xi(\omega)\xi(-\omega)\rangle = (1+2n_B)\mathrm{Im}G_R(\omega)$$

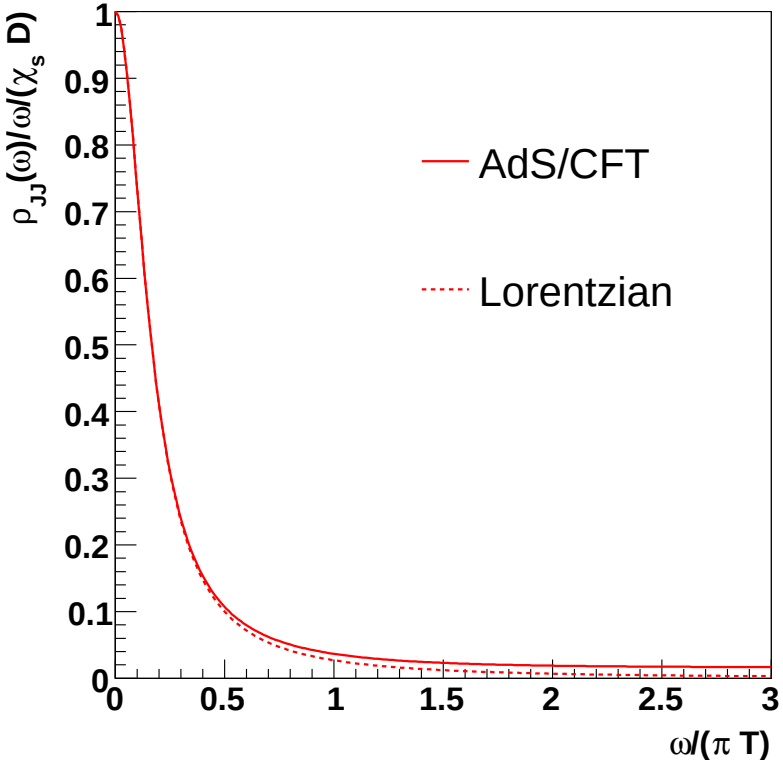
- The solution to this equation is

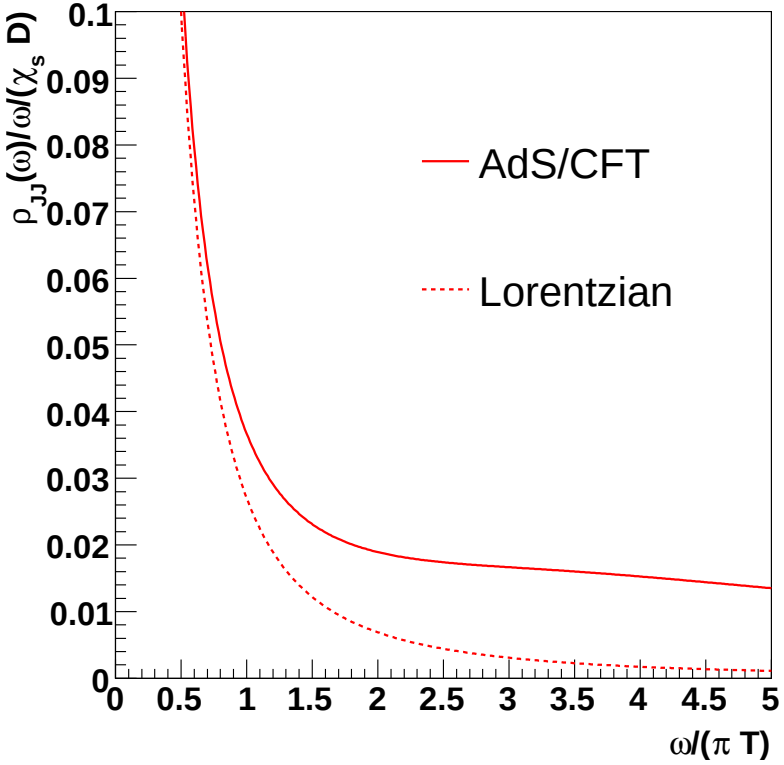
$$x_r(\omega) = \frac{\xi(\omega)}{(-M_Q\omega^2 + \mathrm{Re}G_R(\omega)) + i\mathrm{Im}G_R(\omega)}$$

Can relate $\langle x(\omega)x(-\omega)\rangle$ to $\rho_{JJ}(\omega)$

The answer:

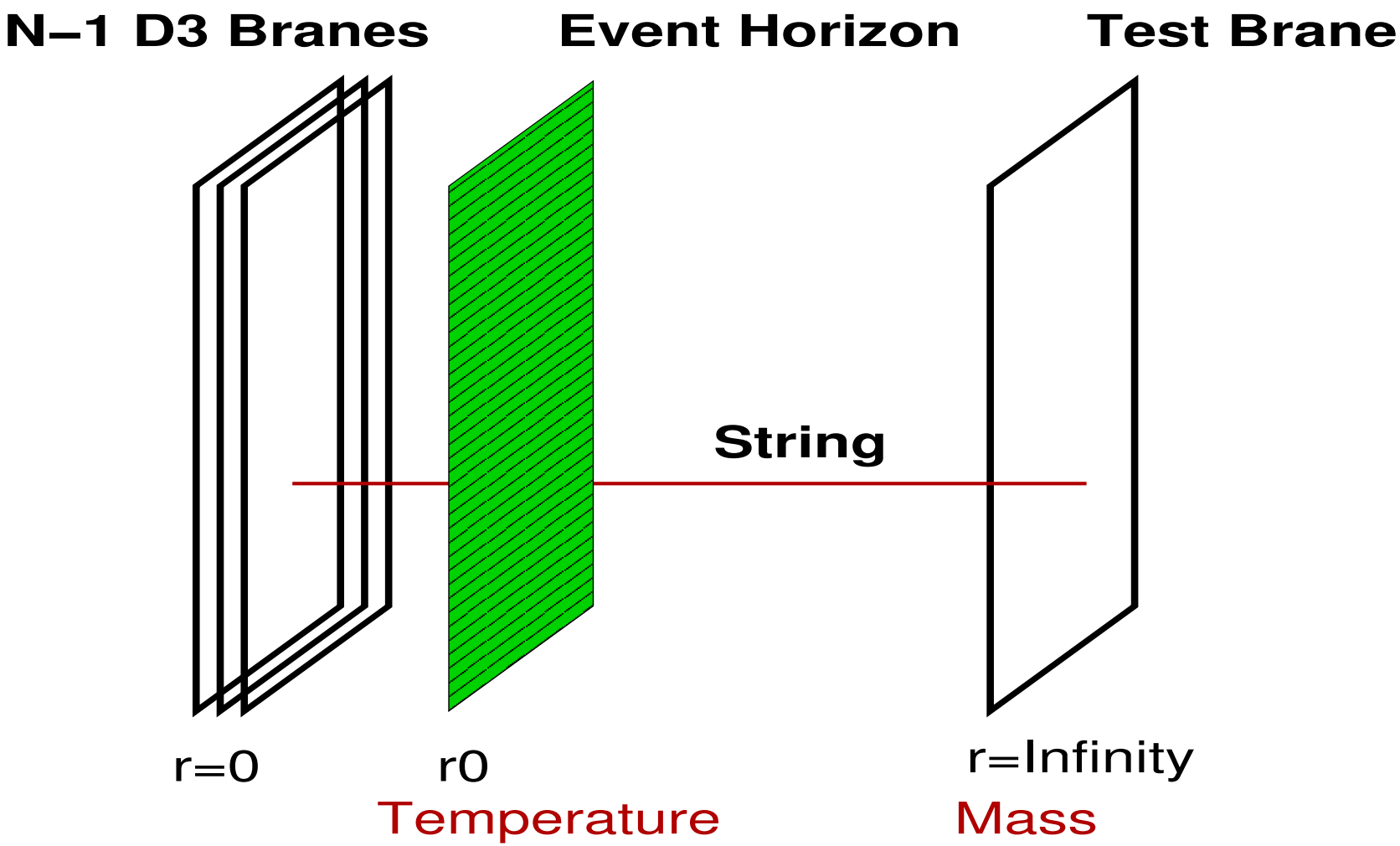
$$\rho_{JJ}(\omega) = \chi_s \frac{\omega^2 \mathrm{Im} G_R(\omega)}{(-M_Q\omega^2 + \mathrm{Re}G_R(\omega))^2 + (\mathrm{Im}G_R(\omega))^2}$$





Grass Contribution is larger than you think: $\frac{\sqrt{\lambda}T^2}{M^2}$

The problem:



The quark doesn't move. There is no noise in "standard" AdS/CFT

How to derive this? (Feynman, Vernon, Caldeira, Legget, C. Greiner)

$$M_Q \frac{d^2 x_r}{dt} + \int^t G_R(t-t') x_r(t') = \xi$$

- Consider a heavy particle coupled to bath a force on the contour

$$Z_Q = \left\langle \int Dx_1 Dx_2 \, e^{i\int \frac{1}{2}Mv_1^2 - i\int \frac{1}{2}Mv_2^2} \; e^{i\int dt_1 F_1 x_1} \; e^{-i\int dt_2 F_2 x_2} \right\rangle_{\text{Bath}}$$

- The force term is small compared to the inertia

$$\left\langle e^{i\int dt_1 F_1 x_1} \; e^{-i\int dt_2 F_2 x_2} \right\rangle_{\text{bath}} \simeq e^{-\frac{1}{2}\int dt\, dt' \, x_a(t) i G_{ab}(t-t') x_b(t')}$$

- Now switch to the “ra” basis

$$x_r = (x_1` + x_2)/2 \qquad x_a = x_1 - x_2$$

- Find

$$Z_Q = \int Dx_r Dx_a \, e^{-i\int x_a M\ddot{x}_r - i\int dtdt' x_a(t) G_R(t-t') x_r(t')}$$
$$\times \, e^{-\frac{1}{2}\int dt \int dt' x_a(t) G_{\rm sym}(t-t') x_a(t')}$$

- Replace Gaussian with fourier transform

$$e^{-\frac{1}{2}\int dt \int dt' x_a(t) G_{\rm sym}(t-t') x_a(t')} = \int D\xi e^{i\int \xi x_a} e^{-\frac{1}{2}\int \xi(t) G^{-1}_{\rm sym} \xi(t')}$$

- Finally do the integrals over x_a

$$Z_Q = \int Dx_r D\xi \, \delta\left(M\ddot{x}_r + \int iG_R(t-t')x_r(t') + \xi\right) e^{-\frac{1}{2}\xi(t) G^{-1}_{\rm sym} \xi(t')}$$

Important Points:

- Need the contour
- Need to integrate something out

The Real Time Thermal Fields and the Correspondence – Son& Herzog (2002)

- The AdS Black Hole Metric in canonical coordinates, $f \equiv 1 - (r_0/r)^4$

$$ds^2 = \frac{r^2}{R^2}\left[-f(r)dt^2 + d\mathbf{x}_{\parallel}^2\right] + \frac{R^2}{f(r)r^2}\,dr^2 + R^2 d\Omega_5^2$$

- The canonical observer sees only one patch of space time

- Change to Kruskal Coordinates

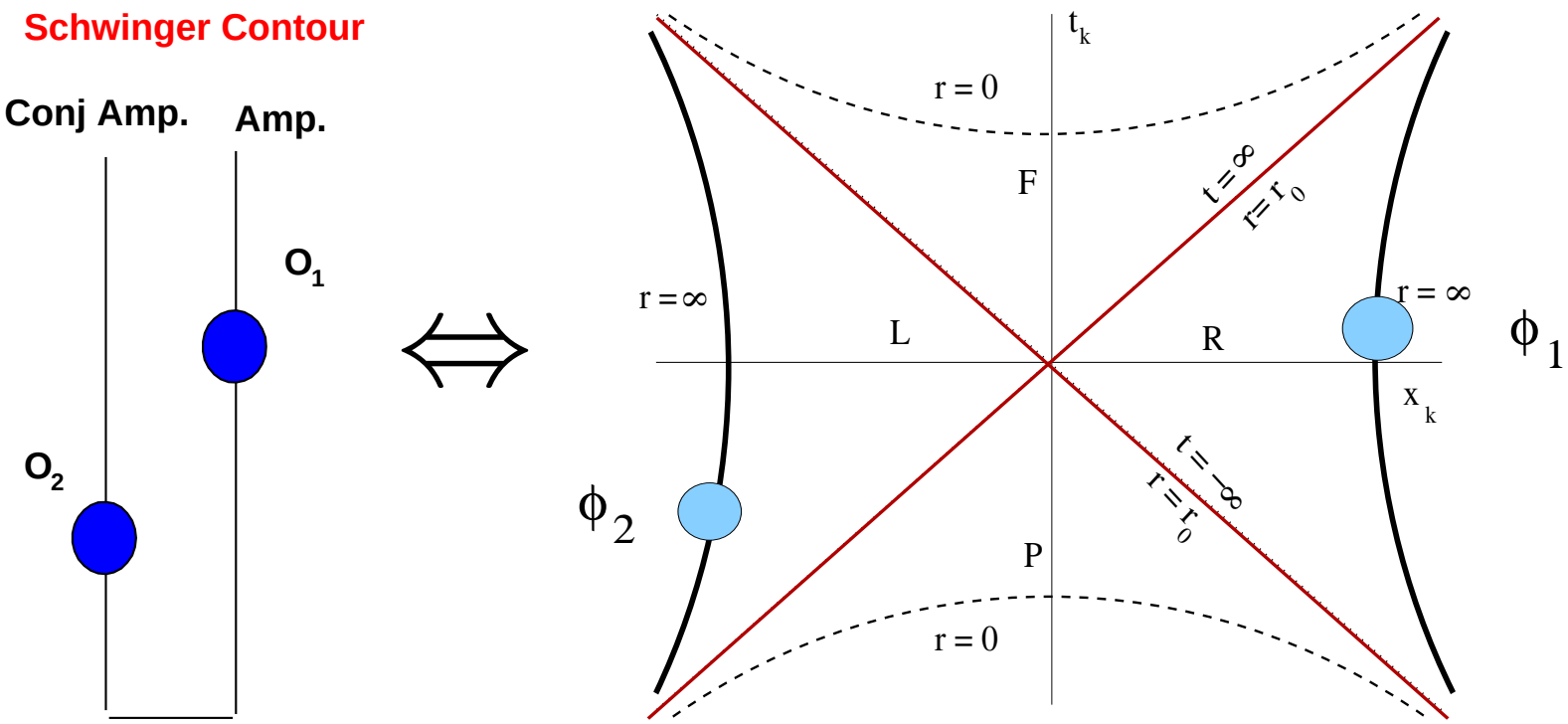
- Consider the scalar wave equation

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu \phi = 0$$

Kruskal Coordinates and the Correspondence (Son-Herzog/Unruh/Israel):

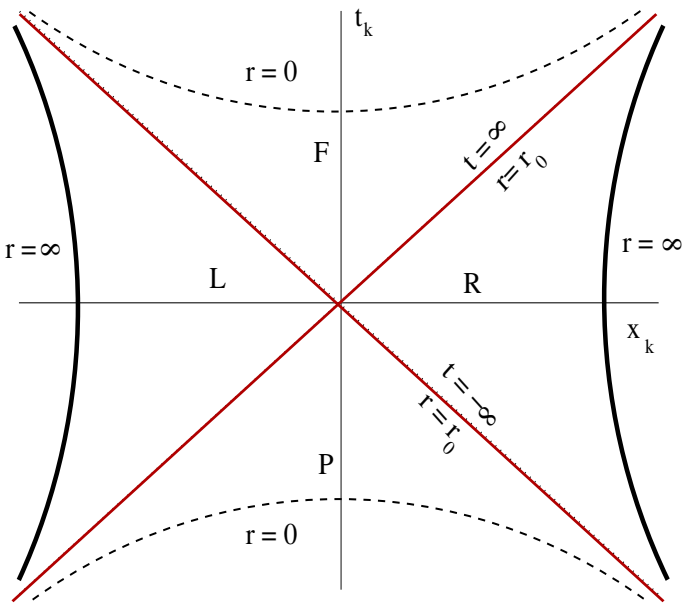
- Source fields for ”1” and ”2” operators live on the right and left quadrants

$$\mathcal{O}_1 \ , \ \mathcal{O}_2 \Leftrightarrow \phi_1 \ , \ \phi_2$$



$$\left\langle e^{i\int dt_1\ \phi_1\mathcal{O}_1} e^{-i\int dt_2\ \phi_2\mathcal{O}_2} \right\rangle_{SYM} = e^{iS[\phi_1,\phi_2]}$$

Kruskal Observer and (t, r) Observer



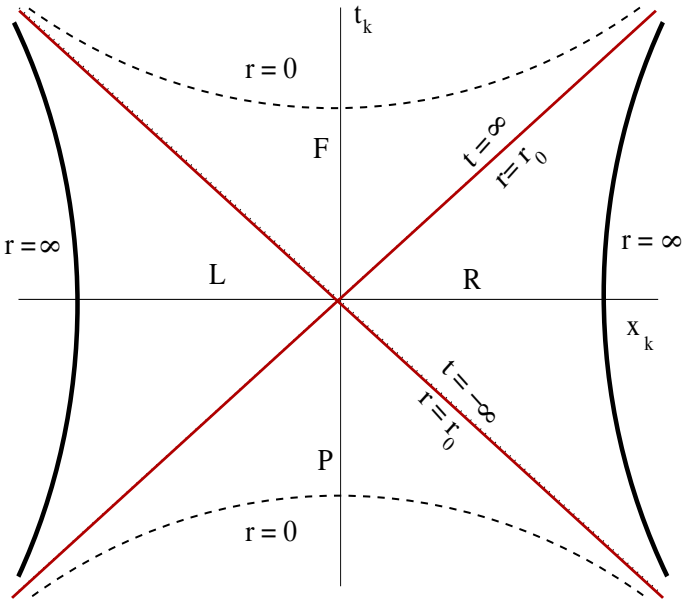
radius

Behind Event Horizon

Integrating out the Bulk

- The real time partition function of string for small fluctuations

$$Z = \int \prod_{t_1} d\mathbf{X}_1(t_1) \prod d\mathbf{X}_2(t_2) \prod_{t,z} d\mathbf{x}_1(t,z) d\mathbf{x}_2(t,z) e^{iS_{NG}}$$



- The integrals over the internal coordinates can be done and yield

$$Z = \int DX_1 DX_2 \, e^{iS_{\text{eff}}[X_{\text{cl}}(X_1(t_1), X_2(t_2))]}$$

Boundary conditions Hawking/Unruh/Israel/Herzog-Son

- Solve the fluctuation eq. in (t, r) coordinates

$$e^{-i\omega t}(r-1)^{+i\frac{\omega}{4}} \qquad \text{out-falling}$$

$$e^{-i\omega t}(r-1)^{-i\frac{\omega}{4}} \qquad \text{in-falling}$$

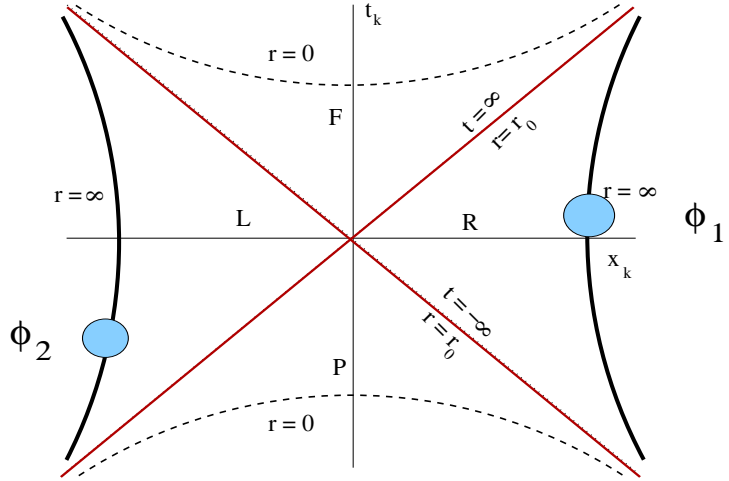
- In Kruskal coordinates these are

$$e^{+i\frac{\omega}{2}\log(-U)} \qquad \text{out-falling at future infinity}$$

$$e^{-i\frac{\omega}{2}\log(V)} \qquad \text{in-falling at past infinity}$$

- Decide how to extend these to the full kruskal plane
  - Negative energy modes are outfalling at the future event horizon
    - Anal. in upper U plane
  - Positive energy modes are infalling at the past event horizon
    - Anal. in lower V plane

Solution in the Kruskal Plane:



- Can now find the solution

$$\mathbf{X}_1 \quad \underbrace{d\mathbf{X}_1/dr}_{\text{fixed by b.c.}} \quad \leftrightarrow \mathbf{X}_2 \quad \underbrace{d\mathbf{X}_2/dr}_{\text{fixed by b.c.}}$$

- Substituting in the action yields the correct result

$$\begin{aligned} iS_{\text{eff}} \quad = \quad & -\frac{1}{2}\int \frac{d\omega}{2\pi}\, X_1(-\omega)\left[-iM_Q^0\omega^2 + iG_{11}(\omega)\right] X_1(\omega) \\ & X_2(-\omega)\left[+iM_Q^0\omega^2 + iG_{22}(\omega)\right] X_2(\omega) \\ & - X_1(-\omega)\left[iG_{12}(\omega)\right] X_2(\omega) - X_2(-\omega)\left[iG_{21}(\omega)\right] X_1(\omega) \end{aligned}$$

Summary of Technical Steps

- Integrate out the bulk. This ammounts to solving the classical equations of motion.
- Fluctuate the end point of a string at $r = \infty$ at very small frequencies: $u = 1/r^2$

$$S_{NG} = \frac{R^2}{2\pi\ell_s^2} \int \underbrace{dt\, dr}_{\text{Area}} \Big[1 - \underbrace{\frac{1}{2}\left(\frac{\dot{\mathbf{x}}_{\parallel}^2}{f} - 4fu\left(\mathbf{x}_{\parallel}'\right)^2\right)}_{\text{Quadratic Fluctuations}}\Big]$$

- Solve for the waves along the string in the full Kruskal plane – Boundary Conditions!
- Compute the classical string action as a quadratic functional of δX_1 and δX_2 .

$$\begin{aligned} \delta X_1 &= \text{Right boundary string endpoint} \\ \delta X_2 &= \text{Left boundary string endpoint} \end{aligned}$$

- Massage the boundary path integral as in the Feynman Vernon procedure

Result

- Find the endpoint of the string obeys the expected Langevin equation

$$M_Q^0 \frac{d^2\mathbf{X}}{dt^2} + \int^t G_R(t-t')\mathbf{X}(t') = \xi$$

- To quadratic order the retarded green function is

$$G_R(\omega) = \underbrace{(\Delta M)}_{\sqrt{\lambda}T/2}\omega^2 - i\omega\frac{\kappa}{2T}$$

- Then find the following effective equation of motion

$$M_{\rm kin}(T)\frac{d^2\mathbf{X}}{dt^2} + \frac{\kappa}{2T}\frac{d\mathbf{X}}{dt} = \xi$$

with the correct kinetic mass (see HKKKY)

$$M_{\rm kin}(T) = M_Q^0 - \frac{\sqrt{\lambda}T}{2} \tag{1}$$