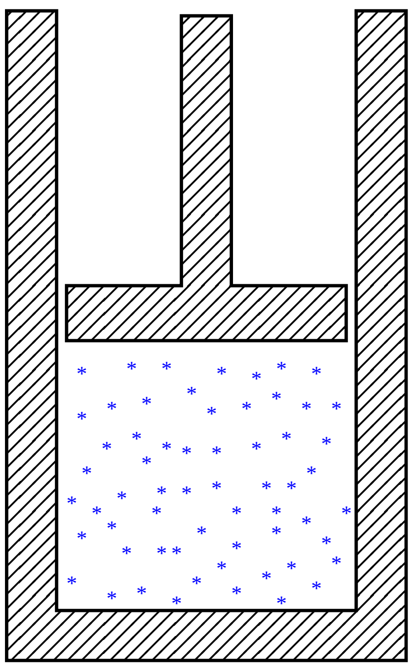
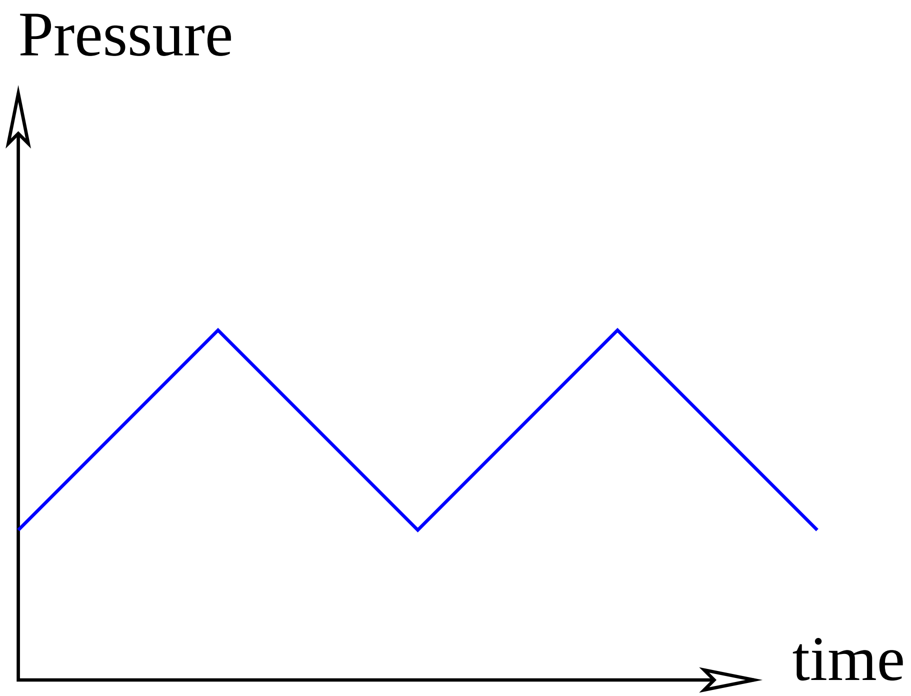
T^μ_μ spectral weight and bulk viscosity Where we can calculate it

Guy D. Moore, Omid Saremi

- Review of bulk viscosity, spectral weight
- Perturbative regime: kinetic theory
  - ∗ High frequency: rising cut
  - ∗ Low frequency: peak
- Near the critical point: universal scaling
  - ∗ Dynamical universality classes: QCD vs. liquid-gas
  - ∗ Critical slowing down and Bulk viscosity
- Summary and conclusions

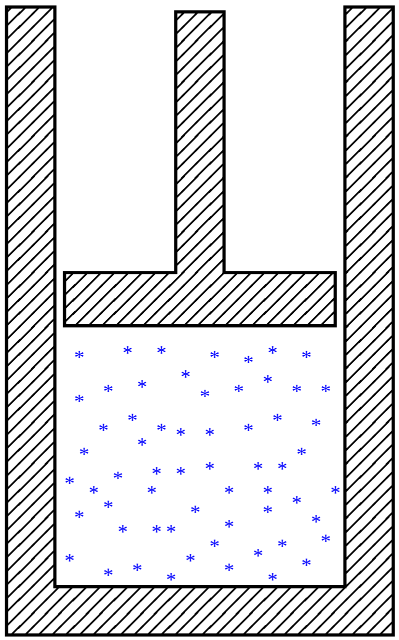
Raise and lower a piston: compress and decompress gas

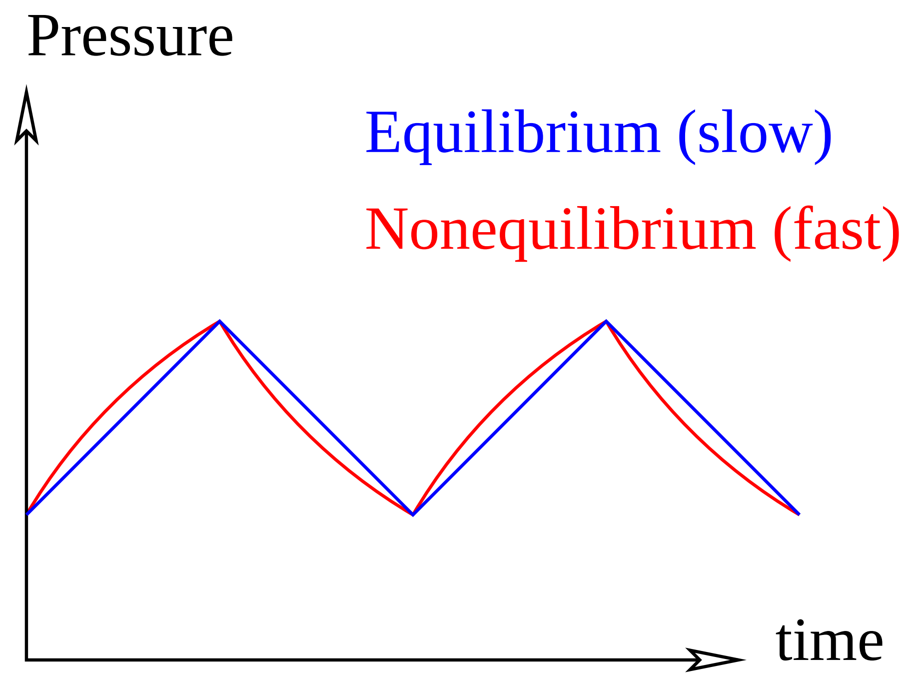




Pressure rises and falls as you compress and decompress.

Compress faster: pressure deviates from equilibrium version



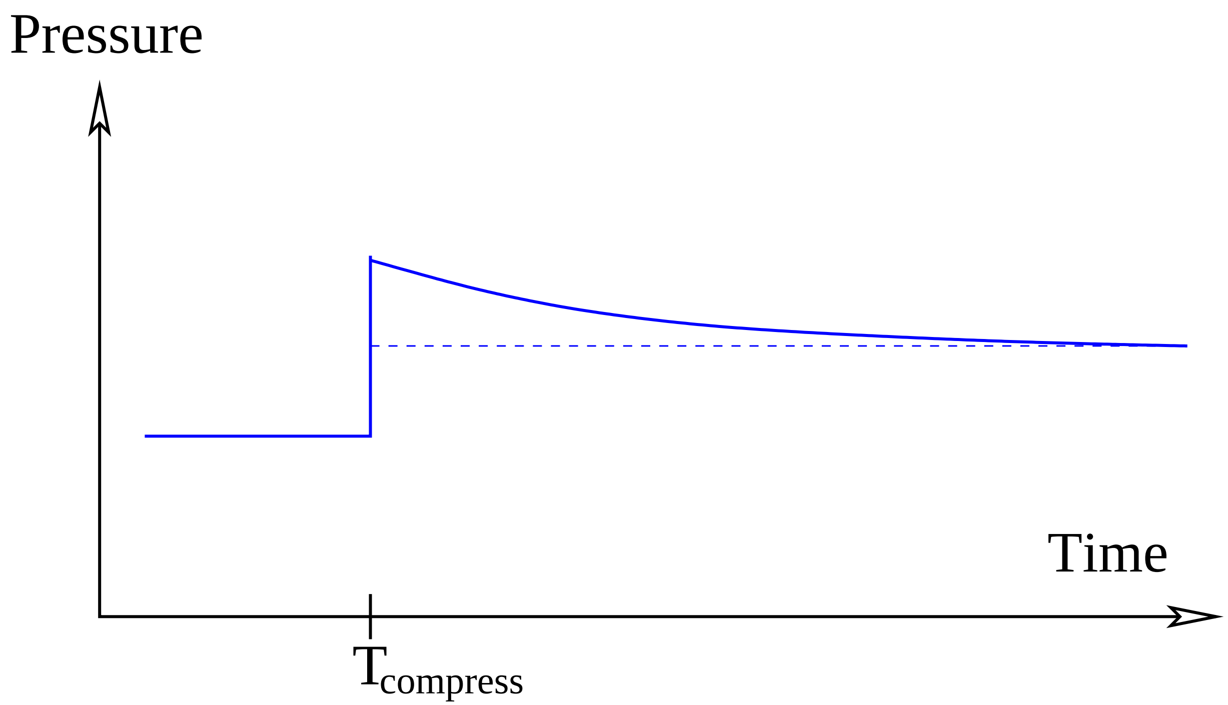


Compression: pressure higher

Decompression: pressure lower Second Law of Thermodynamics

Difference is characterized by Bulk Viscosity

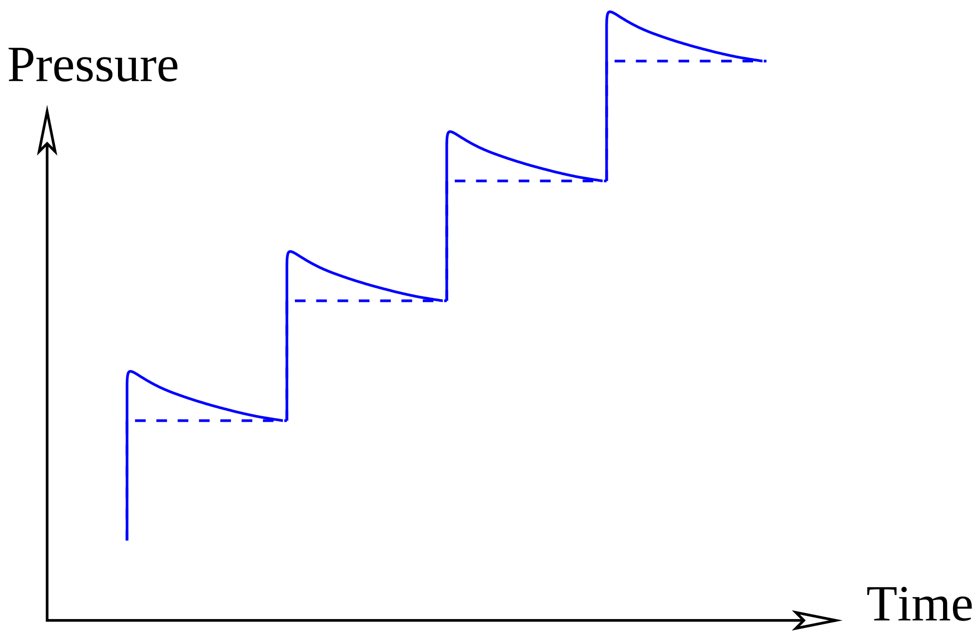
Consider small, sudden compression:



If operator $\mathcal{O}_1$ causes compression, $\mathcal{O}_2$ measures P:

- $\lim_{t\to 0}\langle \mathcal{O}_1(0)\mathcal{O}_2(t)\rangle$ gives height of discontinuity
- $\int_0^\infty dt\,\langle \mathcal{O}_1(0)\mathcal{O}_2(t)\rangle$ gives area under difference curve.

Think of steady compression as many small ones



Integrated extra pressure is

$$\int (P - P_{\mathrm{eq}})dt = (\Delta V_{\mathrm{tot}}) \int_0^\infty dt \, \langle \mathcal{O}_1(0)\mathcal{O}_2(t) \rangle$$

Interesting quantity is integrated extra pressure.

Defined as the bulk viscosity:

$$\int dt(P - P_{\rm eq}) = -\zeta \Delta V = -\zeta \int dt\, \vec{\nabla} \cdot \vec{v}$$

or

$$P - P_{\rm eq} = -\zeta\, \vec{\nabla} \cdot \vec{v}$$

Related to correlator of pressure operator $\mathcal{O}_2 = P = \frac{1}{3} T^i_i$ and expansion operator $\mathcal{O}_1 = \mathcal{O}_2$. Usual arguments:

$$\zeta = \frac{1}{2} \lim_{\omega \to 0} \frac{1}{\omega} \int_{-\infty}^{\infty} dt e^{i\omega t} \int d^3x \left\langle \frac{1}{9} \Big[T^i_i(x,t)\, ,\, T^j_j(0,0) \Big] \right\rangle \, .$$

And small t response described by ω integral.

Is it T^i_i? Or T^μ_μ?

It doesn't matter! $\Big[\,\mathcal{O}_1^\dagger\,,\,\mathcal{O}_1\,\Big] = \Big[\,\mathcal{O}_1^\dagger + c\,,\,\mathcal{O}_1 + c\Big]$.
T^0_0 acts like constant as energy is conserved.

Useful choices:

- T^i_i: intuitively clear
- T^μ_μ: sum rules and exact results
- $T^i_i - \langle T^i_i \rangle \simeq T^i_i + 3c_{\mathrm{s}}^2 T^0_0$: allows KMS

$$\int dt e^{i\omega t}\left\langle \mathcal{O}^\dagger(t)\mathcal{O}(0)\right\rangle = \frac{e^{\omega/T}}{e^{\omega/T}-1}\int dt e^{i\omega t}\left\langle\Big[\,\mathcal{O}(t)\,,\,\mathcal{O}(0)\,\Big]\right\rangle$$

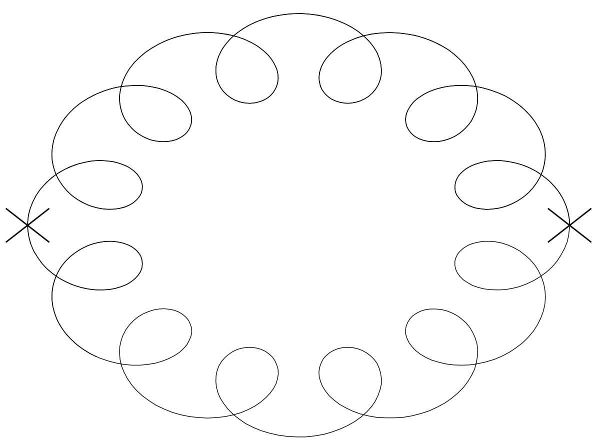
which only holds for op's with vanishing vac value

Perturbative regime

Normalize so $S = \int d^4x \frac{1}{2g^2} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu}$.
Do pure glue for simplicity. Conformal anomaly:

$$T^\mu_\mu = \frac{\beta}{g^4} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} \,, \qquad \beta \equiv \frac{\mu^2 d}{d\mu^2} g^2 \sim g^4 \,.$$

Evaluate Wightman correlator of $(\beta/g^4)G^2 - (1-3c_{\mathrm{s}}^2)T^0_0$.



Leading diagram.
Note $(1-3c_{\mathrm{s}}^2) \sim g^4$ is small;
$(1-3c_{\mathrm{s}}^2)T^0_0$ is g^2 suppressed.

Leading perturbative result

$$G^{>}(Q) = \frac{2\beta^2 d_{\rm A}}{9g^4} \int \frac{d^4P d^4R}{(2\pi)^8} G^{>}_{\mu\alpha}(P) G^{>}_{\nu\beta}(R) (2\pi)^4 \delta^4(Q-P-R)$$
$$\times (g^{\mu\nu} P{\cdot}R - P^\mu R^\nu)(g^{\alpha\beta} P{\cdot}R - P^\alpha R^\beta)$$

Cut propagator

$$G^{>}_{\mu\nu}(P) = [n_b(p^0)+1]\, 2\pi\delta(P^2+m_\infty^2) \sum_\lambda \epsilon_\mu(\lambda)\epsilon^*_\nu(\lambda)\,,$$

Main contribution: large q^0, both lines positive frequency

$$G^{>}_{\rm cut}(\omega,0) = \left[n_b(\tfrac{\omega}{2}) + 1\right]^2 \frac{2\beta^2(g)}{9g^4} \, \frac{2d_{\rm A}\,\omega^4}{32\pi}$$

rising "cut" contrib: order $g^4\omega^4$.

Other contribution

Small q^0, one line positive one negative frequency.

Very naively: $P^2 = 0 = R^2$ and $P + R = 0$ so $P \cdot R = 0$.
Get 0.

Less naive: $P^2 = -m_\infty^2 \sim g^2 T^2$.

Need $(1 - 3c_\mathrm{s}^2) T^0_0$ term (same order).

$$
\begin{aligned}
G^{>}_{\mathrm{pole}}(\omega, 0) &= \delta(\omega) \frac{2}{9}\, 2 d_\mathrm{A} \frac{1}{4\pi} \int_0^\infty n(p)(1 + n(p)) \\
&\qquad \times \left[\left(\frac{1}{3} - c_\mathrm{s}^2 \right) p^2 + \frac{\beta m_\infty^2}{g^2} \right]^2 dp \,.
\end{aligned}
$$

IR singular: Order $g^7 T^4$ area delta function at $\omega = 0$.

Need to know width of peak

Bulk viscosity is $G^{>}(\omega = 0)/T$. Need width of peak

Include imaginary parts on propagators: need ladders as well

Amounts to kinetic treatment. T^{μ}_{μ} in terms of f:

$$(T^i_i + 3c_s^2 T^0_0) = \sum \int \frac{d^3p}{(2\pi)^3} \left[\left(1 - 3c_s^2\right) p^2 + \frac{3\beta m_\infty^2}{g^2} \right] (f_0 + \delta f)$$

Boltzmann equation

$$\mathbf{v} \cdot \boldsymbol{\nabla} f_0 + \partial_t f = -\mathcal{C}[f]$$

becomes

$$\frac{f_0(1+f_0)}{ET} \left(\left[\frac{1}{3} - c_s^2 \right] p^2 - \frac{\beta m_\infty^2}{g^2} \right) = -i\omega \delta f - \mathcal{C}[f]\,.$$

Details of collisions do not change area of peak:

$$\delta f(\omega) = \frac{1}{\mathcal{C} - i\omega}[\text{source}] \quad \rightarrow \quad \int d\omega \delta f(\omega) = [\text{source}]$$

Shape of peak is crudely

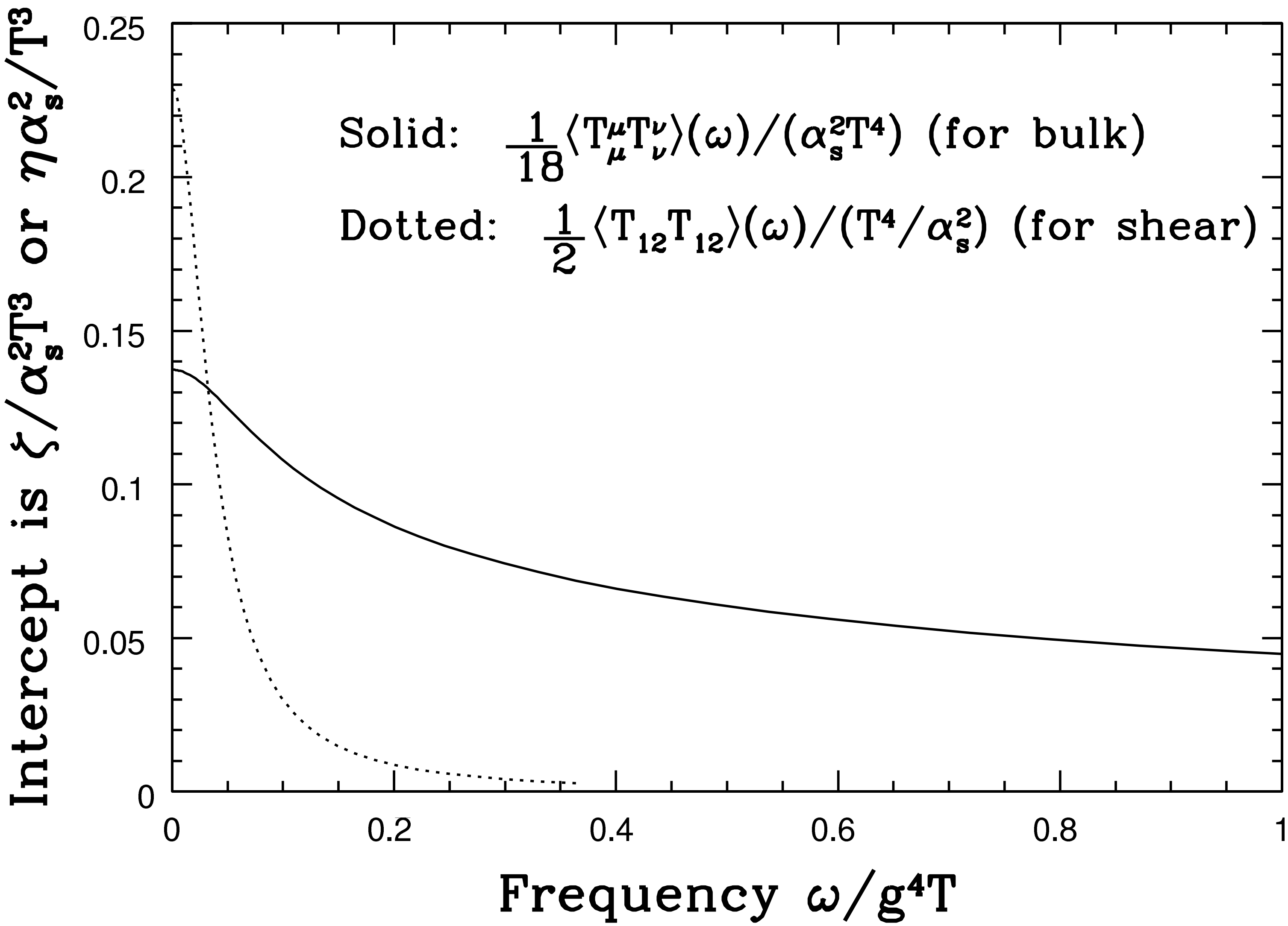
$$G^{>}(\omega) \sim \int \frac{d^3p}{(2\pi)^3} f_0[1+f_0] \left[\left(\frac{1}{3} - c_{\rm s}^2 \right) p^2 + \frac{\beta m_\infty^2}{g^2} \right]^2 \frac{\Gamma[p]}{\omega^2 + \Gamma^2[p]}$$

with $\Gamma[p] \sim g^4 T^3 / p^2$ the large-angle scatt. width

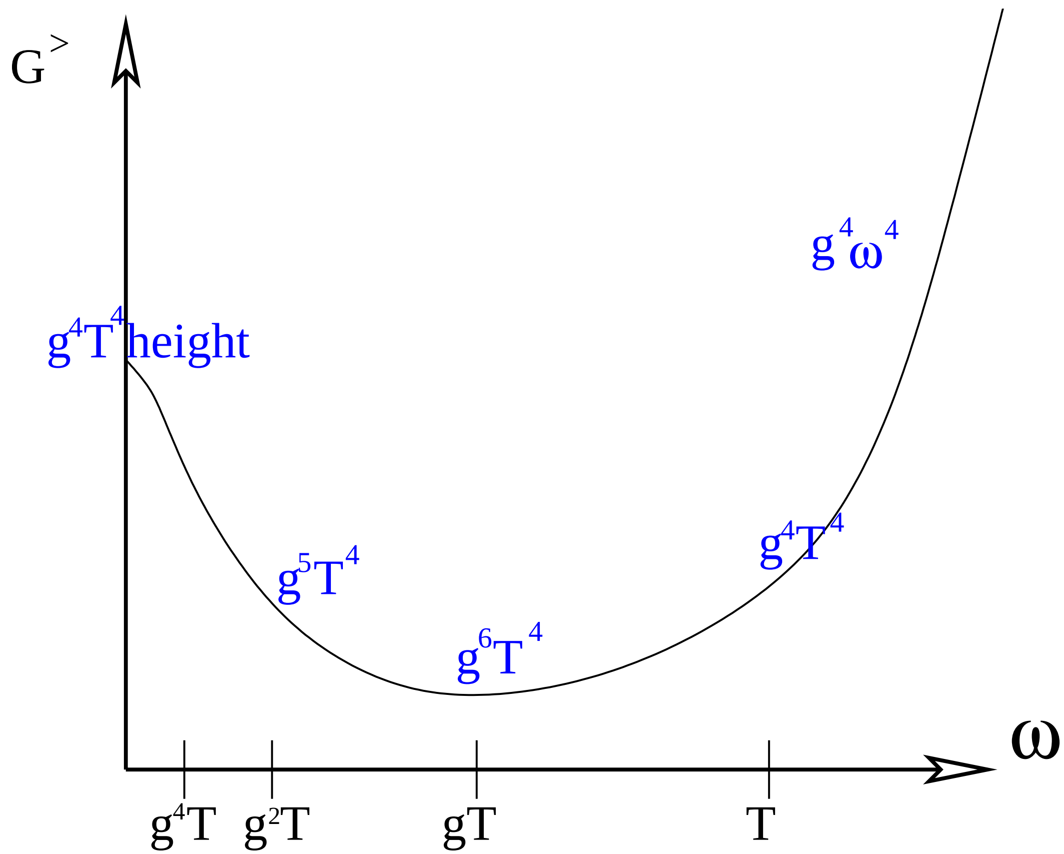
Peak height dominated by hard particles.

"Shoulders" and total area by soft particles.

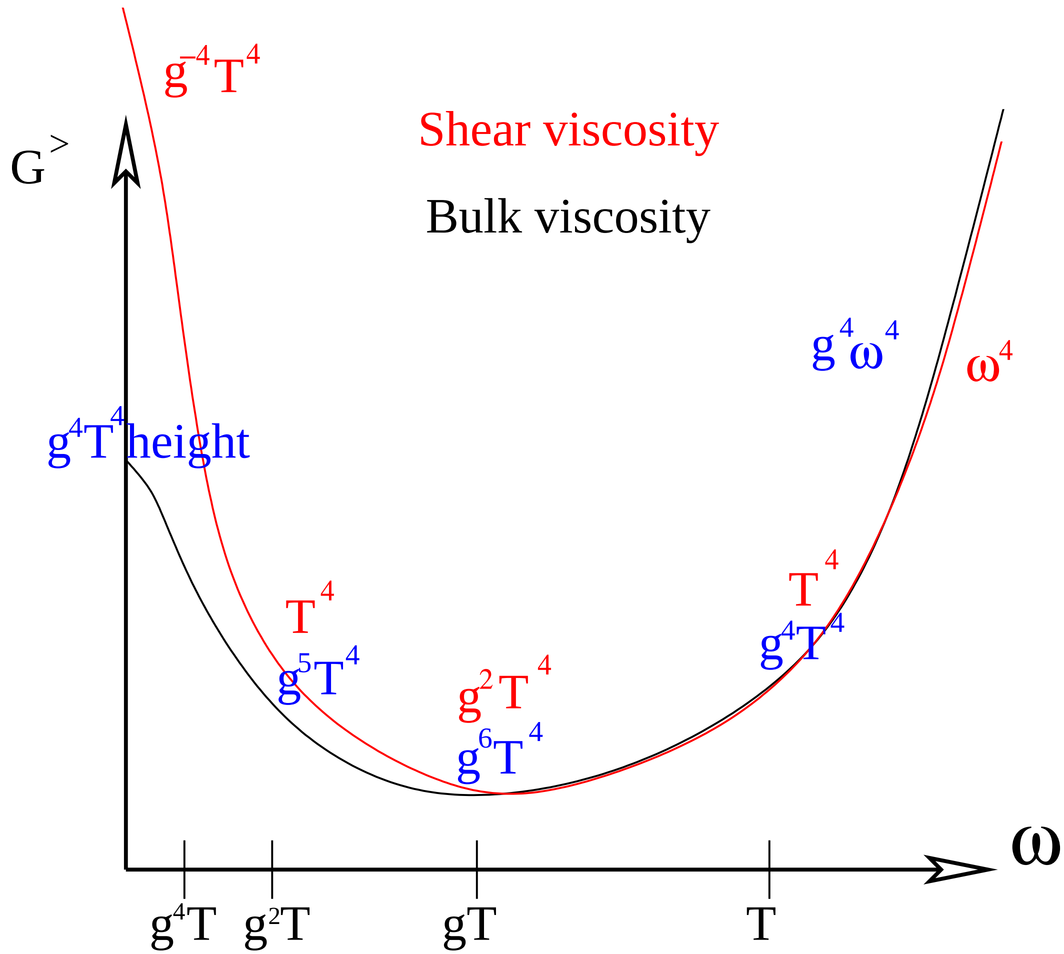
Shape of low frequency peak



Summary:



Summary:



$g^{-4}T^4$
Shear viscosity
Bulk viscosity
$G^>$
$g^4\omega^4$
ω^4
g^4T^4 height
T^4
g^5T^4
g^2T^4
g^6T^4
T^4
g^4T^4
ω
g^4T
g^2T
gT
T

Implications for Euclidean correlators

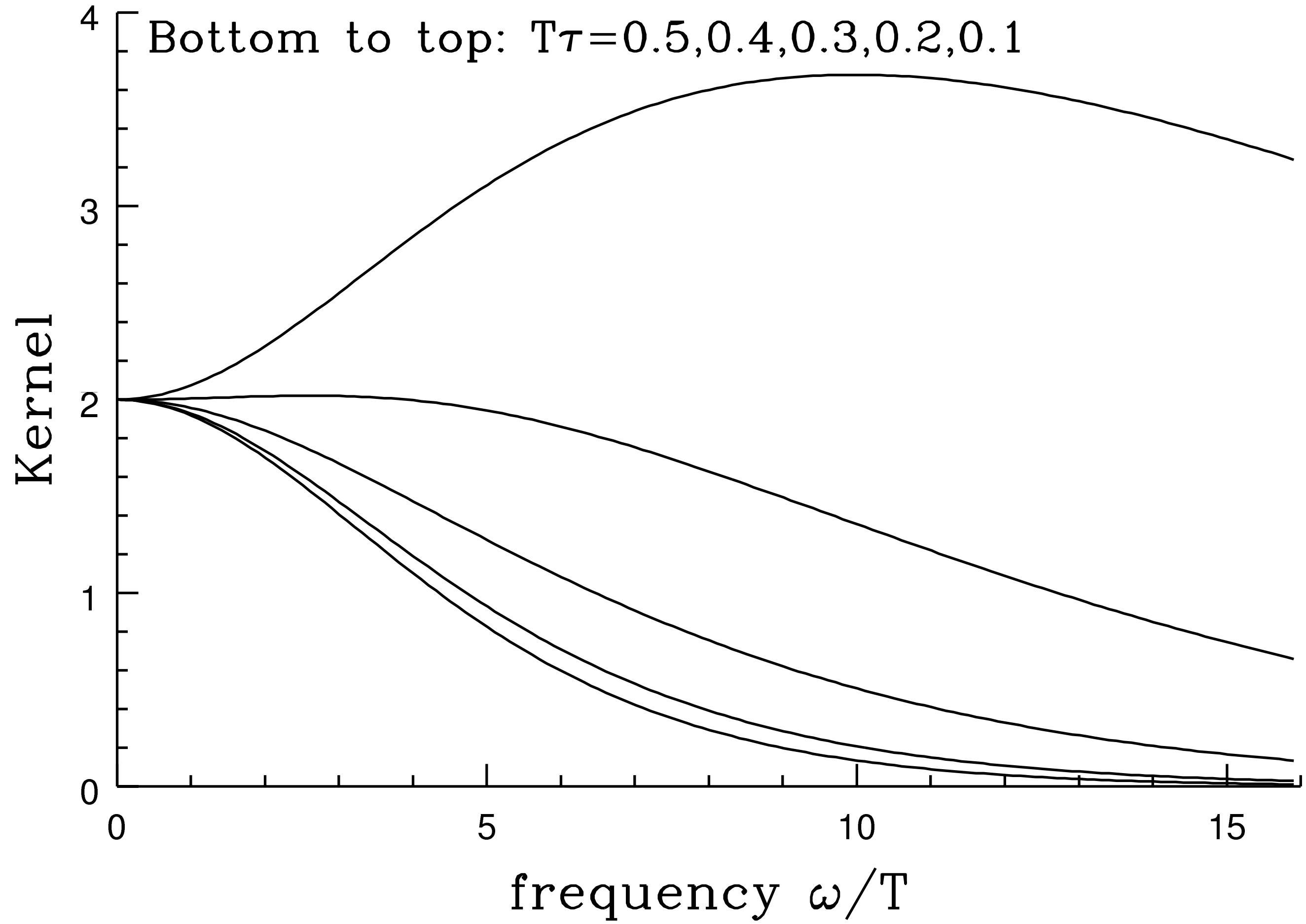
Integral relation between $G_{\mathrm{E}}(\tau)$ and ρ:

$$G_{\mathrm{E}}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\langle \left[\, \mathcal{O}_1 \,,\, \mathcal{O}_1 \,\right] \rangle(\omega)}{\omega} K(\omega, \tau)\,,$$

$$K(\omega, \tau) = \frac{\omega \cosh[\omega(\tau - \beta/2)]}{\sinh(\beta\omega/2)}\,.$$

Do opposite of MEM: compute what spect. wt. implies.

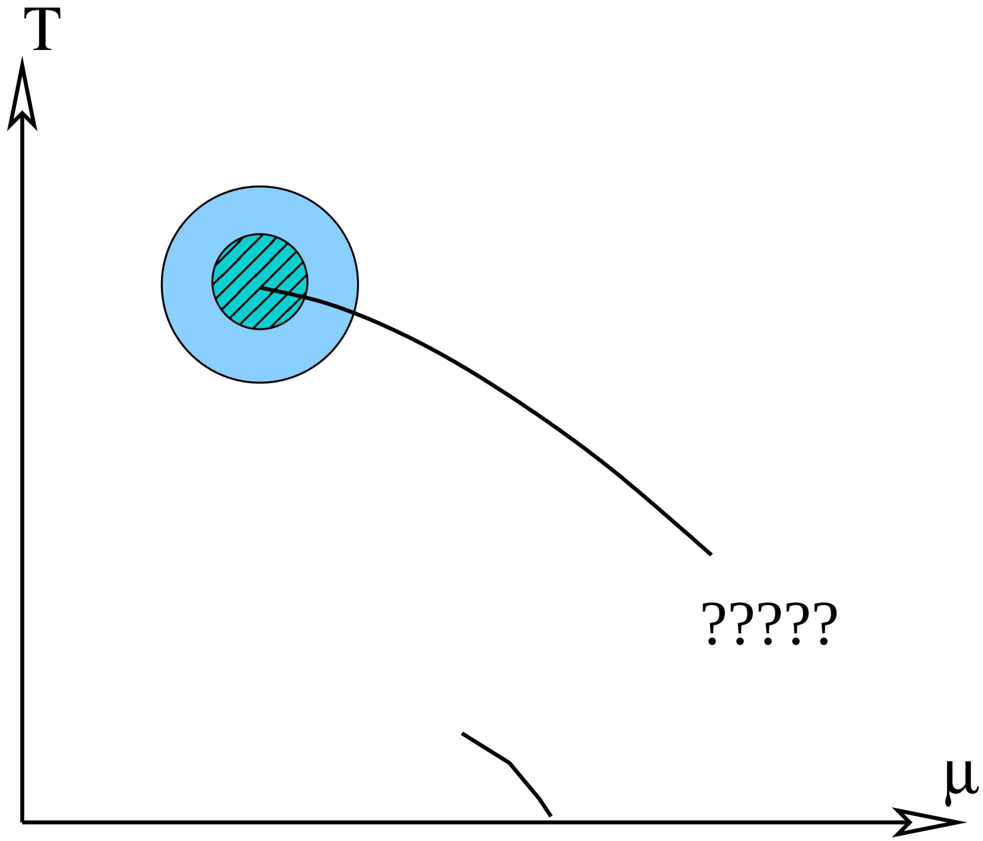
The function $K(\omega, \tau)$



Peak near zero gives common contrib to $G_E(\tau)$, all τ

Another analytically tractible case

Critical region near second-order transition point:



Possible to compute parametric behaviors analytically

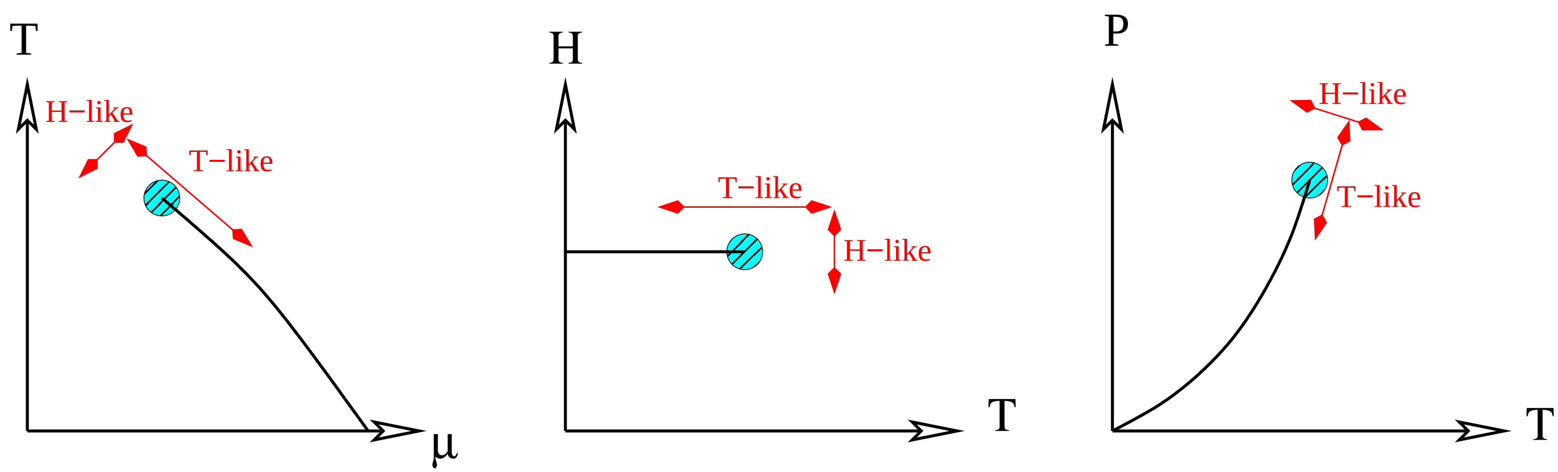
Static universality

"Chiral" phase transition not true symmetry breaking.

Order parameter $\boldsymbol{\psi} = \langle \bar{\psi}\psi \rangle$ same universality as Ising

"Directions" T, μ map linearly into

H, T (Ising) or T, P (liquid-gas)



Correlation length $\xi \sim (T - T_c)^{-\nu}$ $[\nu = 0.630]$

Dynamic universality

Long scale dynamics essentially hydrodynamic
Hohenberg Halperin Rev Mod Phys 49 p435 (1977)

Depend on what quantities are conserved (ψ is not)
Conserved: $T^{0\mu} = (\epsilon, \vec{P})$ and ρ_B
Liquid-gas system: $\epsilon, \vec{P}, \rho$ conserved.

Same dynamic universality as Liquid-Gas

Son Stephanov hep-ph/0401052

Dynamics analyzed to death by CM physics people

Even bulk viscosity analyzed Onuki, Phys Rev E 55 p403 (1997)

Consider Ising system and varying temperature.

Free energy $F[T] = F_{\rm nonsing}[T] + t^{2-\alpha} F_{\rm sing}$ $\qquad t \equiv \dfrac{T - T_c}{T}$

Heat capacity dominated by singular part ($\alpha \simeq 0.11 > 0$)

$$E = T\frac{\partial F}{\partial T} + F = E_{\rm nonsing} + t^{1-\alpha}(2-\alpha)F_{\rm sing} + O(t^{2-\alpha})$$

so

$$C_v \sim \frac{\partial E}{\partial T} \sim t^{-\alpha} + C_{v,{\rm nonsing}}$$

Compression heats: most heat stored by increasing ψ fluct.

Correl. length $\xi \sim t^{-\nu}$. Scaling for $T > k > \xi^{-1}$.

$$C_v[k' > \xi^{-1}] \sim C_{v,{\rm nonsing}} t^{\alpha} \;\Rightarrow\; C_v[k' > k] \sim C_{v,{\rm nonsing}} k^{-\alpha/\nu}$$

Universal dynamic behavior

Fluctuations in $\boldsymbol{\psi}$ on scale $T > k \geq \xi^{-1}$ diffusive,

$$\langle \boldsymbol{\psi}(k,t)\boldsymbol{\psi}(-k,0)\rangle \sim \chi(k)\ \exp(-t/\tau)\,, \qquad \tau \sim k^{-z}$$

Here z dynamic critical exponent: $z \simeq 3$ for liquid-gas

Abrupt compression: $\boldsymbol{\psi}$ has no time to respond
Compressed system initially has same $\boldsymbol{\psi}$ fluct.

Equilibration: $\boldsymbol{\psi}$ must readjust for all $k \geq \xi^{-1}$

Time scale for equilibration $\xi^z \sim t^{z\nu}$.

Almost true: $\zeta \sim T^3(\xi T)^z = T^3 t^{z\nu}$.

Small correction: P mostly relaxes sooner.

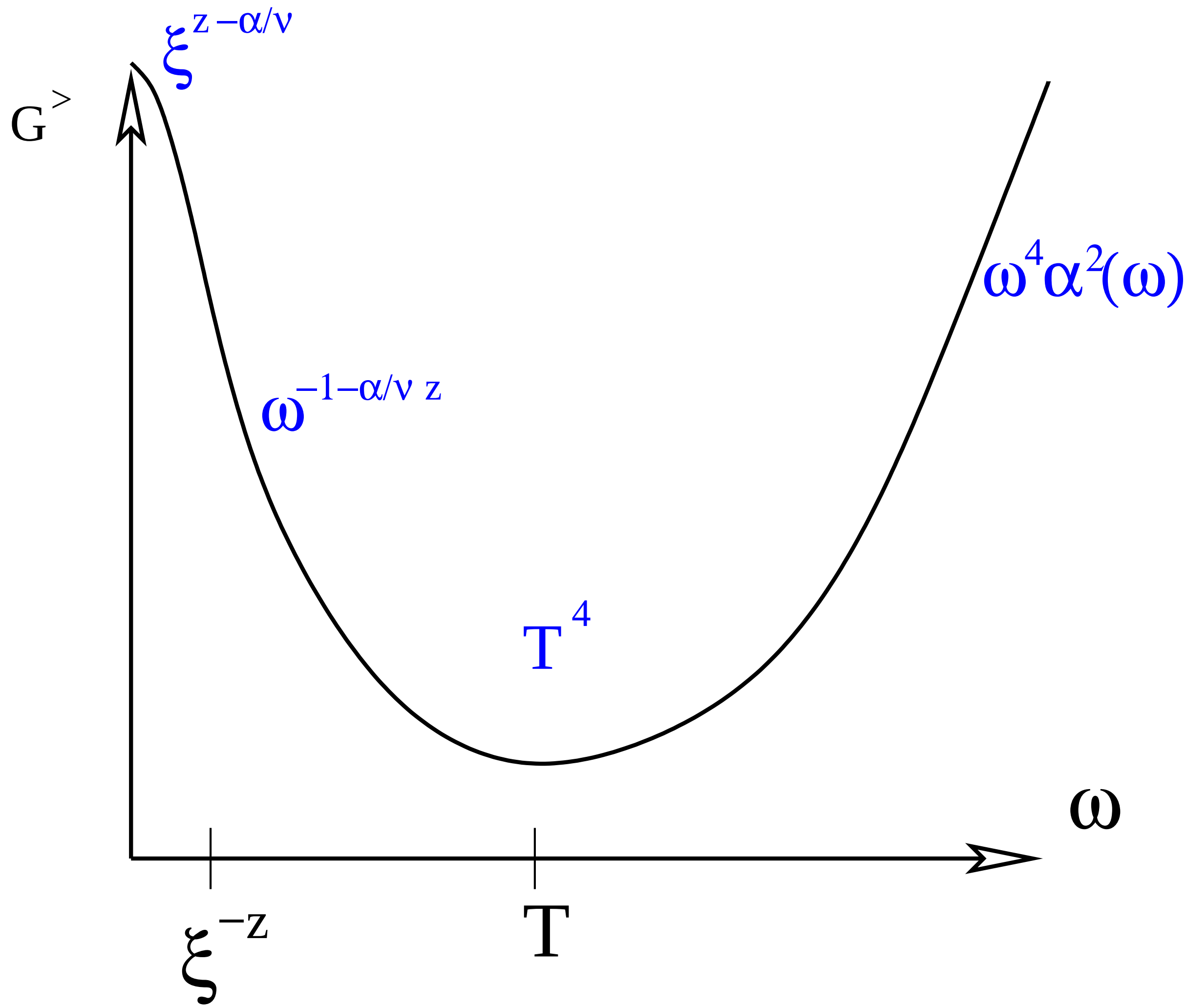
Consider abrupt compression.

- Nonsingular fields change: Pressure rises $\delta P \sim \delta\epsilon$
- Time τ later: modes with $k > \tau^{-1/z}$ thermalized
- These singular modes absorb most $\delta\epsilon$, have no P.
- Energy fraction in nonsingular modes: $(k/T)^{\alpha/\nu}$.

$\delta P(\tau) \sim \delta\epsilon\, \tau^{-\alpha/\nu z}$ until $\tau \sim t^{z/\nu}$.

Hence, $\zeta = \int d\tau \delta P(\tau)/\delta\epsilon \sim \xi^{z-\alpha/\nu} \sim t^{-z\nu+\alpha}$

Slow dynamics: another low ω peak!



Implications for spectral weight

Again, low frequency $\omega < T$ peak in spectral weight.

All $G_{\rm E}(\tau)$ raised by common amount: Area under peak.

- Determined by static universality
- Does not diverge as $T \to T_c$

Shape of peak essential to finding $\zeta =$height.
Very hard to determine from $G_{\rm E}(\tau)$.

Conclusions

- Slow equilibration $\Rightarrow$ Peak in spectral function
- Euclidean Green function cannot find shape of sharp peak
- Slow equilibration at weak coupling: "Wide shouldered" peak
- Slow equilibration near critical point: nearly ω^{-1} shaped peak, $\zeta \sim \xi^{z-\alpha/\nu}$

Euclidean methods probably fail unless crossover very "smooth"

Comment on Kharzeev and Tuchin

They claim spectral weight

$$\int d\omega \frac{\rho}{\omega} \sim \frac{T^5 \partial}{\partial T} \frac{\epsilon - 3P}{T^4} \sim T^5 \partial_T (1 - 3c_{\rm s}^2)$$

Weak coupling: Since $(1 - c_{\rm s}^2) \sim g^4$ this is $\mathcal{O}(g^6 T^4)$.

Contradicts our peak of area $g^7 T^4$ and cut $g^4 \omega^4$

Their estimates of shape of peak also very naive

Critical behavior: they miss ξ^z time scale, resulting peak

Their results depend on Kramers-Kronig, fail if $\rho \geq \omega^0$ at large ω. Actual behavior $g^4 \omega^4 + g^6 T^4 \omega^0$ up to logs