Financial Modeling
&
Its Discontents

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Outline

The syntax of theoretical finance superficially resembles that of theoretical physics, but the semantics is very different. In this talk I compare modeling in the two fields, and discuss the appropriate way to use valuation models in financial markets.

- Introduction
- Models in Physics
- Models in Finance
- The One Law of Financial Modeling
- Using the Law
- A Poor Man's Short Guide to Quantitative Finance
- The Right Way To Use Models in Finance

Models in Physics

- **Fundamental Models or Theories**

Fundamental models attribute effects to deep dynamical causes.

Kepler's laws of planetary motion – not quite a theory:

  – planets move about the sun in elliptical orbits;

  – the line from the sun to the planet sweeps out equal areas in equal times; and

  – $(\text{period})^2 \sim (\text{radius})^3$.

Nevertheless, the laws do provide profound insight.
Newton adds dynamics, provides a fundamental theory.

- **Phenomenological Models**

A toy or analogy to help visualize something unobservable.

  – Liquid drop model of nucleus;

Calibrate to known phenomena, then use it to predict the unknown.

- **Statistical Models**

Models in Finance

There are no fundamental theories in finance. There are no laws.
That's why the textbooks are so mathematically rigorous.

- **A Prototypical Financial Model**

  Find the value of a Park Ave penthouse given the price of a Battery Park studio.

  – (implied) price per square foot is the model variable

- **The Aims of Financial Models**

  – Models are used to rank securities by value

  – Models are used to interpolate from liquid prices to illiquid ones

  – Models translate linear intuitive quantities into nonlinear dollar values

  Implied price per square foot to apartment value

  Future yield to bond price

  Future volatility to option price

- **[Statistical Models]**
  Physicists use statistics to test theories. Economists use it to find relationships.
  Mostly regression without explicit dynamics. Correlation isn't causality.

The Scientific Foundations of Finance

- Science: reductive
- Engineering: constructive

Mechanical engineering : Newtonian mechanics :: Electrical engineering : Maxwells equations

Mechanical engineering : Newtonian mechanics :: Financial engineering: What?

What is the science behind financial engineering?

The One Law of Financial Modeling

God's laws while standing on one leg:

> *Do not do unto others as you would not have them do unto you. All the rest is commentary. Go and learn.*

The law of financial modeling while standing on one leg:

> *If you want to know the value of a security, use the price of another security that's similar to it. All the rest is strategy. Go and build.*

Financial economists call their version *the law of one price:*

> *Any two securities with identical future payoffs,* ***no matter how the future turns out****, should have identical current prices.*

Using the Law of One Price

To value a *target* security, find some other *replicating portfolio* of *liquid securities* with the same future payoffs in all states of he world.

The value of the target is the value of the replicating portfolio.

- **The Role of Models**

Models are used to prove the identity of the future payoffs:

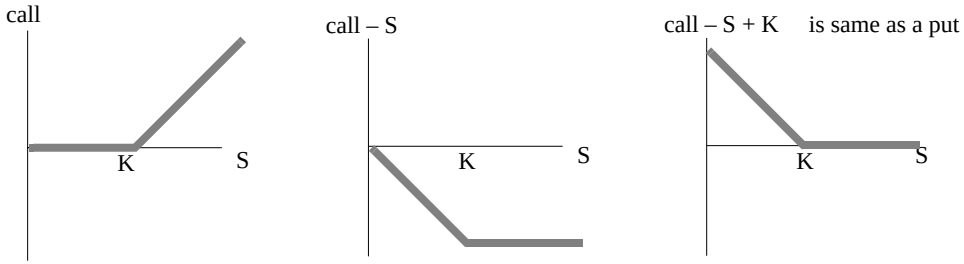
1. Since the future is uncertain, you must model that uncertainty by specifying the range and probability of future scenarios for the prices of all relevant securities.

2. You need a strategy for creating a replicating portfolio that, in each of these future scenarios, will have identical payoffs to those of the target security.

   Replication can be *static* or *dynamic*.

Styles of Replication: Static, Dynamic

A *static replicating portfolio* is a collection of securities that, once defined and never altered, reproduces the payoff of a target security under all future scenarios. Static replication the simplest and most comprehensive method of valuation, but is feasible only in the rare cases when the target security closely resembles the liquid securities available. For example, put-call parity.



For more complex or non-linear securities, like stock options, static replicating portfolios are unavailable, but sometimes you can find a *dynamic replicating portfolio*, a mixture of liquid securities whose proportions, continually adjusted by switching from one security into another, will replicate the payoff of the target.

It's a bit like a recipe for making fruit salad. Black-Scholes-Merton found a recipe for making an option out of stocks and riskless bonds.

Static or dynamic, the initial price of the replicating portfolio is the estimated value of the target.

Weaker Version of the Law of One Price

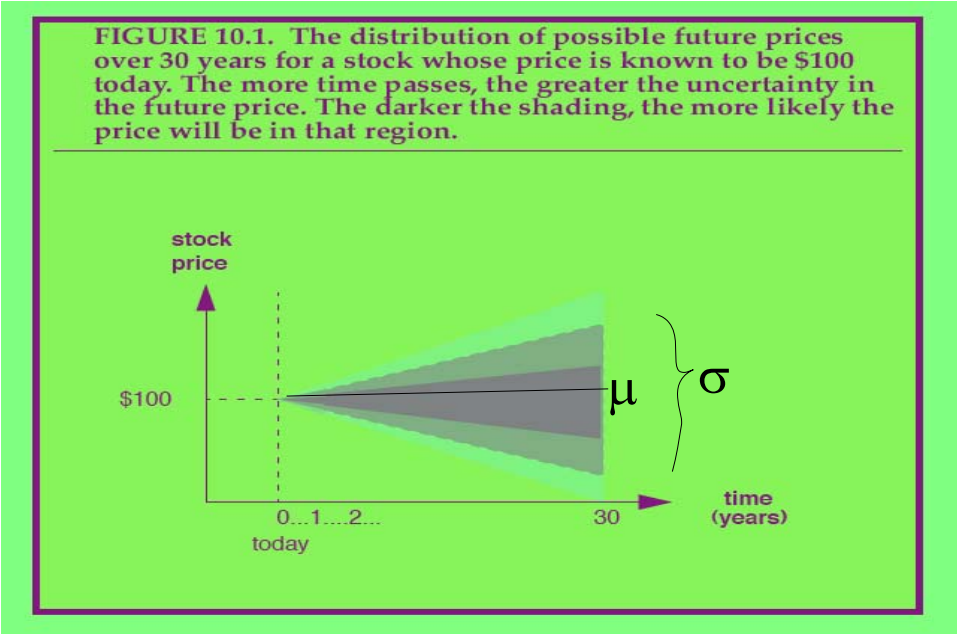
Any two securities with identical future ***unavoidable*** *risk, no matter how the future turns out, should have identical current prices/expected returns.*

Ways of Avoiding Risk

- Diversification: many uncorrelated securities together will have zero risk
- Hedging: cancelling one security's risk by trading a correlated security

What's left is *unavoidable* risk.

Another Assumption: Geometric Brownian Motion for Stocks



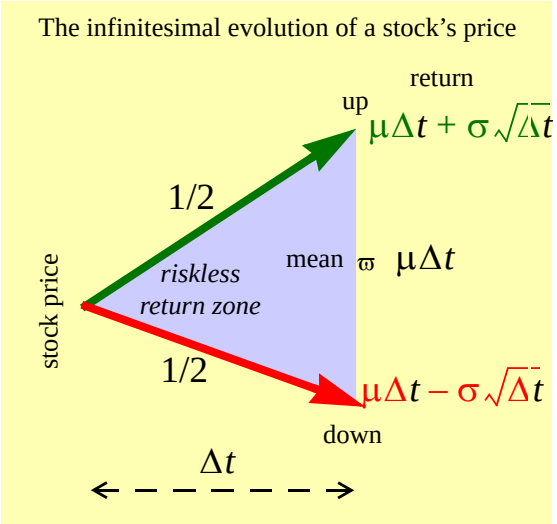
Stock prices diffuse:

Not strictly true, but people like to start somewhere.

Notation:
Riskless rate r

Mean expected return μ

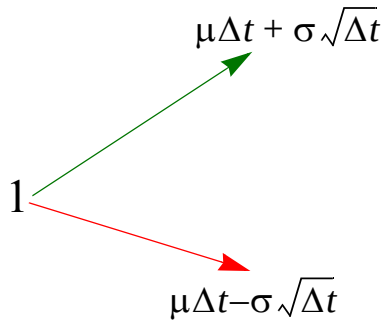
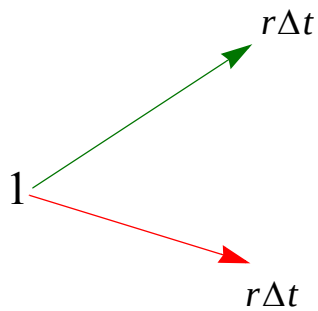
Risk σ is the annual standard deviation of annual returns.



The Fundamental Question: Expected Return vs. Risk?

- Use the weak law of one price:

  Two portfolios with the same perceived instantaneous unavoidable risk should have the same expected return.

Instrument	Evolution	Risk	Mean Return
risky stock		σ	$\mu = r + (\mu - r)$
riskless bond		0	r

A fifty-fifty mix of stock and riskless bond relates risk to return

Instrument	Evolution	Risk	Mean Return
50-50 mix of stock and risk-less bond	$1 \to \frac{1}{2}(\mu + r)\Delta t + \frac{\sigma}{2}\sqrt{\Delta t}$ $1 \to \frac{1}{2}(\mu + r)\Delta t - \frac{\sigma}{2}\sqrt{\Delta t}$	$\sigma/2$	$(\mu + r)/2 = r + \frac{(\mu - r)}{2}$

Note: the excess return $\mu - r$ over the riskless rate is proportional to the volatility σ

$$\frac{\mu - r}{\sigma} = \lambda$$

Equal Sharpe ratios λ: Excess return per unit of risk is the same for all stocks.

- More risk, more expected return.
- What is the value of λ?

Risk Reduction By Diversification Shows λ is Zero

- The removal of risk by the law of large numbers when stocks are uncorrelated.

If

you can buy a very large portfolio of stocks,

and their returns are uncorrelated

Then

asymptotically the portfolio risk $\sigma \to 0$

and so its return $\mu \to r$

Therefore the return of each stock must be the riskless rate, and therefore

$\lambda = 0$ Zero risk premium, expect riskless growth

- You are not paid to take on diversifiable risk.

6. Risk Reduction By Hedging

- The removal of risk by cancellation of common factors.
- Example: all stocks are correlated with the "market" M. You can remove this component of risk from any stock by combining it with a short position in the market.
- $\mu_i - \beta\mu_M$ carries no market risk, only residual risk, where $\beta = \rho_{iM}\dfrac{\sigma_i}{\sigma_M}$

You can diversify over all stocks and their residual risk. Since you are not paid to take on diversifiable risk,

$$(\mu_S - r) = \beta_{SM}(\mu_M - r)$$ Capital Asset Pricing Model

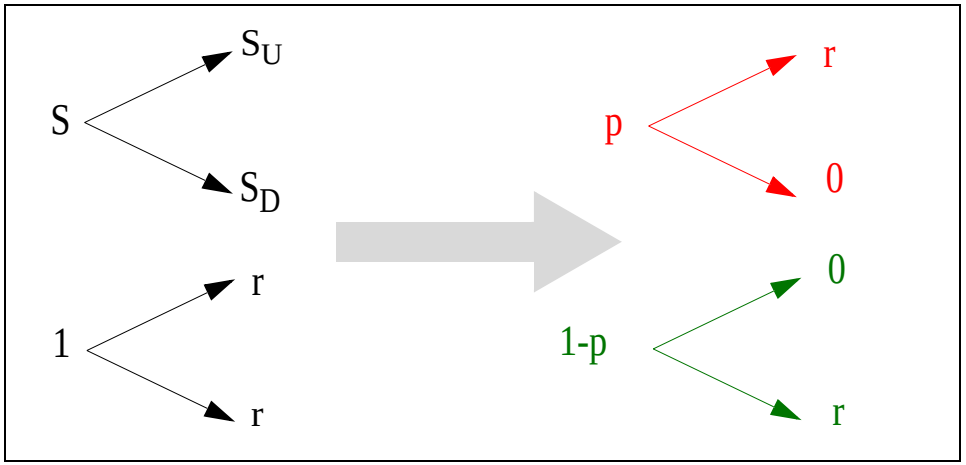
The expected return of a stock is proportional to the market's return times it co-movement with the market. More risk, more expected return.

Does this really seem to be true? Not in great detail.

Options Are Not Independent Securities: Dynamic Replication

Options/derivatives have terminal payoffs which are non-linear functions of stock price: $C_T = C(S,T)$.

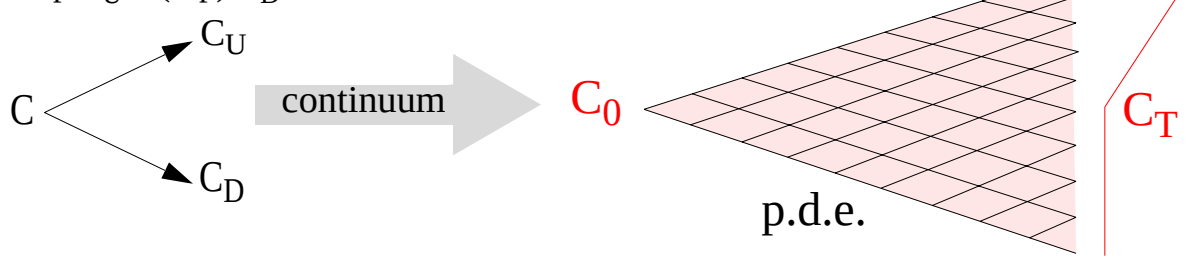
A stock and a bond can be decomposed into Arrow-Debreu securities *p* and *1-p* that span the price space for a short time Δt:



p depends on the volatility of the stock price.

With these one-state securities p and 1-p, you can dynamically replicate the payoff of the non-linear option C(S) at each instant:

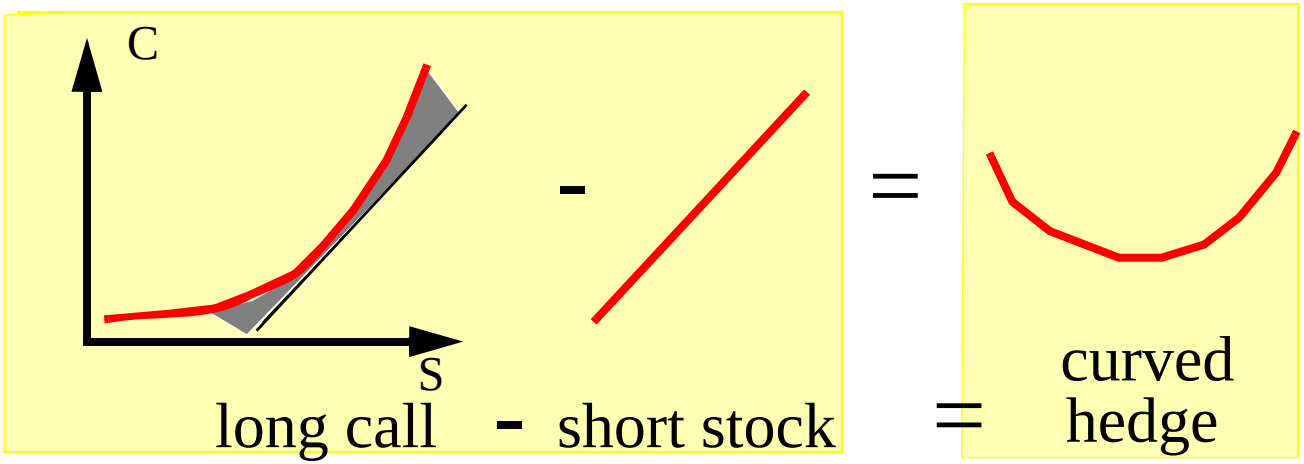
$$rC = p\,C_U + (1-p)\,C_D$$



Since you can replicate the payoff, you don't care about path of stock, only its volatility.
Options traders bet on volatility.

Options and Dynamic Replication

What happens when you buy a call C and hedge its exposure to the underlying stock?



You get a payoff profile which is parabolic: you make money whether the stock goes up or down, and you know how much you make if you know the future volatility.

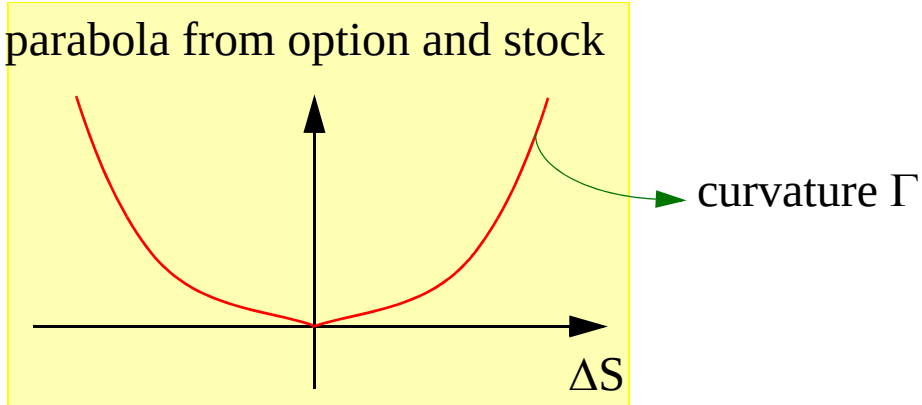
You've replicated, out of an option and stock, a riskless investment, a riskless bond.

Option – stock = bond

or

Option = stock + bond

The Black-Scholes Equation



You make money proportional to the squared actual move in the index in any direction.

$$\text{Quadratic gain from curvature} = \frac{1}{2}\Gamma(\Delta S)^2 = \frac{1}{2}\Gamma\sigma^2 S^2 \Delta t$$

You lose money for an option bought at implied (expected) volatility Σ as the time to profit from a move diminishes (Black-Scholes time decay):

- Loss from time decay $= \Theta(\Delta t)$ where $\Theta = \partial C / \partial t$
- This position is riskless and so the hedged position must behave like a riskless bond.

$$\frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = 0$$

- This Black-Scholes equation determines the value of the option.
  (The Sharpe ratio of option and stock are equal.)

The P&L from Hedging A Long Option Position

Net P&L of a hedged position during time Δt

$$\frac{1}{2}\Gamma S^2(\sigma^2 - \Sigma^2)\Delta t$$

gain from curvature with actual vol σ

loss from time decay expecting vol Σ

If you guess the future volatility right all the time, then $\sigma = \Sigma$ and so the profit is exactly zero because you've replicated the option perfectly. But, if you guess wrong, or if stocks don't evolve smoothly and more than the curvature is important, then …

8. Extensions of replication (the past 25 years)

- Extension of the Black-Scholes-Merton replication method of pricing derivatives on currencies, commodities, interest-rate-sensitive securities, mortgages, credit-derivatives, etc.

  Strategy:

  – build a realistic model of the stochastic behavior of the underlying security;

  – Calibrate it to the current prices of the underlying security;

  – Figure out how to replicate the derivative security;

  – Find the value of the derivative by backward induction or Monte Carlo simulation of the replication process.

- Traders have become more analytical as they realize that they are trading volatility, and develop both an understanding and a feel for the model.

- Markets are able to *estimate* the value of exotic and hybrid options.

- Perturbation of the idealized Black-Scholes model to take account of the real world and *behavioral and perceptual issues*:

  – illiquidity, transactions cost, noncontinuous trading, skew, non-normal distributions, market participants' behavior, etc.

The Right Way To Use Financial Models

- Sophisticated Vulgarity
  In physics it pays to drop down deep; in finance you're getting out of your depth, so stay shallow and move in straight lines from known to unknown.
- A good model is easily embraceable because it incorporates concepts that allow you to stress-test the world in your imagination. It asks just enough of you, but not too much.
- Building trading systems that make it easy to use models is as hard as building the models themselves.
- Avoid axiomatization.
- Think of models as gedanken experiments.
- *If the fool would persist in his folly, he would become wise* -- William Blake
- The dangers of idolatry.