The Lifshitz regime in QCD

RDP, VV Skokov & A Tsvelik, 1801.08156

Chiral spirals and their fluctuations

1. Standard phase diagram in $T$ & $\mu$: critical end-point (CEP)

   Not seen from lattice at small $\mu$

2. Quarkyonic phase at large $N_c$ (analytic) and $N_c = 2$ (lattice)

3. Chiral Spirals in Quarkyonic matter: sigma models, SU(N) and U(1)

4. Phase diagram: just a 1st order line,
   with large fluctuations in the Lifshitz regime
“Standard” phase diagram for QCD in T & μ: CEP?

Lattice: at quark chemical potential $\mu = 0$, crossover at $T_{ch} \sim 154$ MeV

At $\mu \neq 0$, quarks *might* change scalar 4-pt coupling $< 0$, so transition 1st order

Must meet at a Critical End Point (CEP), *true* 2nd order phase transition

Asakawa & Yazaki ‘89, Stephanov, Rajagopal & Shuryak ‘98 & ‘99
The Phases of QCD

Early Universe
Future LHC Experiments

Current RHIC Experiments

Quark-Gluon Plasma

Crossover

1st order phase transition

Hadron Gas

Critical Point

Vacuum

Baryon Chemical Potential

Future FAIR Experiments

Color Superconductor

Neutron Stars

Nuclear Matter

900 MeV

0 MeV
Lifshitz phase diagram for QCD

Instead: “Lifshitz regime”: strongly coupled, large fluctuations
Unbroken 1st order line to spatially inhomogeneous phases = “chiral spirals”
Hints in heavy ion data?
Fundamental problem in field theory: analogies to phase diagram for polymers
Could be CEP as well...

Quark-Gluon Plasma

hadronic

$T \uparrow$

$\mu \rightarrow$

Lifshitz regime

1st order line

Chiral spirals

Quark matter
Lattice, hot QCD: no CEP at small $\mu$

Lattice: Hot QCD, 1701.04325

Expand about $\mu = 0$, power series in $\mu^{2n}$, $n = 1, 2, 3$.

Estimate radius of convergence. *No sign of CEP by $\mu_{qk} \sim T$*

![Plot](diagram.png)

"disfavored region for location of a critical point"

No LQCD data
So if there is no critical endpoint, what could be going on?
Lattice for $T = 0$, $\mu \neq 0$, \textit{two} colors


Heavy pions, $m_\pi \sim 740$ MeV. $\sqrt{\sigma} = 470$ MeV. $32^4$ lattice, $a \sim .04$ fm

Confined until very high $\mu_q k \sim 1$ GeV. 

\textit{Bare} Polyakov loop:

\[ \langle \text{loop} \rangle \uparrow \]
Lattice for $T = 0$, $\mu \neq 0$, two colors

Lattice: Bornyakov et al, 1711.01869.

String tension in time: nonzero up to $\mu_{qk} \sim 750$ MeV
Phases for $N_c = 2$, $T \sim 0$, $\mu \neq 0$

Braguta, Ilgenfritz, Kotov, Molochkov, & Nikolaev, 1605.04090 (earlier: Hands, Skellerud + …)

Lattice: $N_c = 2$, $N_f = 2$. $m_\pi \sim 400$ MeV, fixed $T \sim 50$ MeV, vary $\mu_{qk}$.

Hadronic phase: $0 \leq \mu_{qk} < m_\pi / 2 \sim 200$ MeV. Confined, independent of $\mu$

Dilute baryons: $200 < \mu_{qk} < 350$. Bose-Einstein condensate (BEC) of diquarks.

Dense Baryons: $350 < \mu_{qk} < 600$. Pressure not perturbative, BEC

Quarkyonic: $600 < \mu_{qk} < 1100$: pressure ~ perturbative, but excitations confined
(Wilson loop ~ area)

Perturbative: $1100 < \mu_{qk}$, but $\mu a$ too large.
Quarkyonic matter

McLerran & RDP 0706.2191

At large $N_c$, $g^2 N_c \sim 1$, $g^2 N_f \sim 1/N_c$, so need to go to large $\mu \sim N_c^{1/2}$.

$$m_{Debye}^2 = g^2 ((N_c + N_f/2)T^2/3 + N_f \mu^2/(2\pi^2))$$

Doubt large $N_c$ applicable at $N_c = 2$.

When does perturbation theory work?

$T = \mu = 0$: scattering processes computable for momentum $p > 1$ GeV

$T \neq 0$: $p > 2 \pi T$, lowest Matsubara energy

$\mu \neq 0, T = 0$: $\mu$ is like a scattering scale, so perhaps $\mu_{\text{pert}} \sim 1$ GeV.

At least for the pressure. Excitations determined by region near Fermi surface
Possible phases of cold, dense quarks

**Confined:** $0 \leq \mu_{qk} < \frac{m_{\text{baryon}}}{3}$. $\mu$ doesn’t matter

**Dilute baryons:** $m_{\text{baryon}} \frac{3}{3} < \mu_{qk} < \mu_{\text{dilute}}$. Effective models of baryons, pions

**Dense baryons:** $\mu_{\text{dilute}} < \mu_{qk} < \mu_{\text{dense}}$. Pion/kaon condensates.

**Quarkyonic:** $\mu_{\text{dilute}} < \mu_{qk} < \mu_{\text{perturbative}}$. 1-dim. chiral spirals.

**Perturbative:** $\mu_{\text{perturbative}} < \mu_{qk}$. Color superconductivity

$\mu_{\text{perturbative}} \sim 1 \text{ GeV}$?

Dense baryons and quarkyonic *continuously* related.

*U(1) order parameter in both.*
Relevance for neutron stars

Fraga, Kurkela, & Vuorinen 1402.6618.

Maximum $\mu_{qk}$ may reach quarkyonic (for pressure), but true perturbative?

Ghisoiu, Gorda, Kurkela, Romatschke, Säppi, & Vuorinen, 1609.04339: pressure($\mu_{qk}$) $\sim g^6$.

Will be able to compute $\Lambda_{\text{pert}} = \# \mu_{qk} \# \sim 1$?
Kojo, Hidaka, McLerran & RDP 0912.3800: as toy model, assume confining potential

\[ \Delta_{00} = \frac{\sigma_0}{(\vec{p}^2)^2}, \quad \Delta_{ij} \sim \frac{1}{p^2} \]

Near the Fermi surface, reduces to effectively 1-dim. problem in patches. For either massless or massive quarks, excitations have zero energy about Fermi surface; just Fermi velocity \( v_F < 1 \) if \( m \neq 0 \).

Spin in 4-dim. -> “flavor” in 1-dim., so extended \( 2N_f \) flavor symmetry, \( \text{SU}(N_f)_L \times \text{SU}(N_f)_R \rightarrow \text{SU}(2N_f)_L \times \text{SU}(2N_f)_R \). Similar to Glozman,1511.05857.

Extended \( 2N_f \) flavor sym. broken by transverse fluctuations, only approximate.

Number of patches \( N_{\text{patch}} \sim \mu/\sigma_0 \), so spherical Fermi surface recovered as \( \sigma_0 \rightarrow 0 \).
Transitions with # patches

Minimal number of patches = 6.

Probably occurs in dense baryonic phase.

In quarkyonic, presumably weak 1st order transitions as # patches changes.

Like Keplers....
Chiral spirals in 1+1 dimensions

In 1+1 dim., can eliminate $\mu$ by chiral rotation:

$$q' = e^{i\mu z\Gamma_5} q, \quad \bar{q}(\slashed{D} + i\mu\Gamma_0)q = \bar{q}'\slashed{D}q', \quad \Gamma_5\Gamma_z = \Gamma_0$$

Thus a constant chiral condensate automatically becomes a chiral spiral:

$$\bar{q}'q' = \cos(2\mu z)\bar{q}q + i\sin(2\mu z)\bar{q}\gamma_5 q$$

Argument is only suggestive.

N.B.: anomaly ok, gives quark number: $\langle \bar{q}\Gamma_0 q \rangle = \mu/\pi$

Pairing is between quark & quark-hole, both at edge of Fermi sea. Thus chiral condensate varies in $z$ as $\sim 2\mu$. 
Bosonization in 1+1 dimensions

Do not need detailed form of chiral spiral to determine excitations. Use bosonization. For one fermion,

\[ \overline{\psi} \phi \psi \leftrightarrow (\partial_i \phi)^2 \]

\( \phi \) corresponds to U(1) of baryon number. In general, non-Abelian bosonization. For flavor modes,

\[ S_{\text{eff}}^{\text{flavor}} = \int dt \int dz \frac{1}{16\pi} \text{tr}(\partial_\mu U^\dagger)(\partial_\mu U) + \ldots \]

where \( U \) is a SU(2 N_f) matrix.

Do not show Wess-Zumino-Witten terms for level 3 = \# colors. Also effects of transverse fluctuations, reduce SU(2 N_f) -> SU(N_f); quark mass

Lastly, SU(3) + level 2 N_f sigma model. Modes are gapped by confinement.
Pion/kaon condensates & U(1) phonon

Overhauser ‘60, Migdal ‘71....Kaplan & Nelson ‘86...

Pion/kaon condensate:

\[ \langle \bar{q}_L q_R \rangle \sim \langle \Phi \rangle \sim \Phi_0 \exp(i(qz + \phi)t_3) \]

Condensate along \( \sigma \) and \( \pi^0 \) \( \Rightarrow \) \( t_3 \). Kaon condensate \( \sigma \) and \( K \), etc.

Excitations are the \( SU(N_f) \) Goldstone bosons \textit{and} a “phonon”, \( \varphi \).

Phases with pion/kaon condensates and quarkyonic Chiral Spirals both spontaneously break U(1), have associated massless field.

\textit{Continuously connected}: \( SU(N_f) \) of \( \pi/K \) condensate \( \Rightarrow \sim SU(2N_f) \) of CS’s.

Fluctuations same in both.

Perhaps WZW terms for \( \pi/K \) condensates?
Anisotropic fluctuations in Chiral Spirals

Spontaneous breaking of global symmetry =>
Goldstone Bosons have derivative interactions, $\sim \partial^2$

$\pi/K$ condensates and CS’s break both global and rotational symmetries

Interactions along condensate direction usual quadratic, $\sim \partial_z^2$

Those quadratic in transverse momenta, $\sim \partial_\perp^2$, cancel, leaving quartic, $\sim \partial_\perp^4$.

$$\mathcal{L}_{\text{eff}} = f_\pi^2 |(\partial_z - i k_0)U|^2 + \kappa |\partial_\perp^2 U|^2 + \ldots$$

Valid for both the U(1) phonon $\varphi$ and Goldstone bosons U

Hidaka, Kamikado, Kanazawa & Noumi 1505.00848;
Lee, Nakano, Tsue, Tatsumi & Friman, 1504.03185; Nitta, Sasaki & Yokokura 1706.02938
No long range order in Chiral Spirals

Consider tadpole diagram with anisotropic propagator

\[
\int d^2 k_\perp \, dk_z \, \frac{1}{(k_z - k_0)^2 + (k^2_\perp)^2} \sim \int d^2 k_\perp \, \frac{1}{k^2_\perp} \sim \log \Lambda_{\text{IR}}
\]

Old story for \(\pi/K\) condensates: Kleinert ‘81; Baym, Friman, & Grinstein, ‘82.

Similar to smectic-C liquid crystals: ordering in one direction, liquid in transverse. Hence anisotropic propagator

[Images of nematic, smectic, and cholesteric phases with increasing opacity]
Chiral Spirals in 1+1 dimensions

Overhauser/Migdal’s pion condensate:  \((\sigma, \pi^0) = f_\pi(\cos(k_0 z), \sin(k_0 z))\)

*Ubiquitous* in 1+1 dimensions: Basar, Dunne & Thies, 0903.1868; Dunne & Thies 1309.2443+ ...

*Wealth* of exact solutions, phase diagrams at *infinite* \(N_f\).

Usual Gross-Neveu model:
- Phase diagram
- Chiral spiral:

\[ \langle \bar{q}q \rangle \neq 0 \]
\[ \downarrow \langle \bar{q}q \rangle_{CS} \neq 0 \]

\(T \uparrow \quad \mu \rightarrow \)
Chiral Spirals in 3+1 dimensions

In 3+1, common in NJL models: Nickel, 0902.1778 + ....Buballa & Carignano 1406.1367 + ...

In reduction to 1-dim, $\Gamma_{5}^{1-dim} = \gamma_{0}\gamma_{z}$, so chiral spiral between $\bar{q}q$ & $\bar{q}\gamma_{0}\gamma_{z}\gamma_{5}q$
Both of these phase diagrams are dramatically affected by fluctuations:

no Lifshitz point in 1+1 or 3+1 dimensions at finite N

there is a *Lifshitz regime*
Standard phase diagram

\[ \mathcal{L} = (\partial_\mu \phi)^2 + m^2 \phi^2 + \lambda \phi^4 + \kappa \phi^6 \]

Negative quartic coupling, \( \lambda \), turns a 2nd order transition into 1st order. Two phases.

\[ \langle \phi \rangle = 0 \]

\[ \langle \phi \rangle \neq 0 \]

\[ m^2 \rightarrow \]

\[ X = \text{tri-critical point, } m^2 = \lambda = 0 \]
Lifshitz phase diagram \textit{(in mean field theory)}

\[ \mathcal{L}_{\text{Lifshitz}} = (\partial_0 \phi)^2 + Z (\partial_i \phi)^2 + \frac{1}{M^2} (\partial_i \phi)^2 + m^2 \phi^2 + \lambda \phi^4 \]

Negative kinetic term, \(Z < 0\), generates spatially inhomogeneous phase, CS. Three phases.

\[ \langle \phi \rangle \neq 0 \quad \langle \phi \rangle = 0 \]

\[ m^2 \rightarrow \]

\[ X = \text{Lifshitz point, } m^2 = Z = 0 \]

\[ \langle \phi \rangle_{CS} \neq 0 \]
No massless modes in too few dimensions

No massless modes in $d \leq 2$ dimensions:

$$\int d^2 k \frac{1}{k^2} \sim \log \Lambda_{\text{IR}}$$

Cannot break a continuous symmetry in $d \leq 2$ dimensions: instead of Goldstone bosons, generate a mass non-perturbatively.

*Lifshitz point:* $Z = m^2 = 0$, so propagator just $\sim 1/k^4$:

$$\int d^4 k \frac{1}{k^4} \sim \log \Lambda_{\text{IR}}$$

Hence *no* Lifshitz point in $d \leq 4$ (spatial) dimensions.

*Must* generate either a mass $m^2$, or term $\sim Z \ p^2 \neq 0$, non-perturbatively.
Lifshitz regime (shaded):

$Z$ and/or $m^2$ are $\neq 0$ everywhere

strongly coupled, non-perturbative

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle_{CS} \neq 0$

Brazovski 1st

$Z \uparrow$

2nd

$\langle \phi \rangle = 0$

$m^2 \rightarrow$

Brazovski 1st
Example: inhomogenous polymers

Like mixing oil & water: polymers A & B, with AB diblock copolymer ("co-AB")

Three phases: high temperature, A & B mix, symmetric phase

low temperature, little co-AB: A & B separate, broken phase

co-AB tends to decrease interface tension between A & B phases, can turn it negative. Like $Z < 0$

Low temperature, high concentration co-AB: "lamellar" phase, stripes of A & B. Like smectic.
Lifshitz point in inhomogenous polymers: mean field

Three phases, symmetric, broken, & spatially inhomogenous

Mean field predicts Lifshitz point at given T & concentration of co-AB
Lifshitz regime in inhomogenous polymers

Instead of Lifshitz point predicted by mean field theory, find

**Bicontinuous microemulsion: Z ≠ 0, m² = 0: Lifshitz regime**

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Table:

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<tr>
<th>Jones &amp; Lodge</th>
<th>2012</th>
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<td>Polymer Jour.</td>
<td>131 (44)</td>
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Bicontinuous microemulsion: $Z \approx 0$

**Experiment**

Jones & Lodge,
Polymer Jour. 131 (44) 2012

**Self-consistent field theory**

Fredrickson, “The equilibrium theory of inhomogenous polymers”
Phase diagram for QCD in $T$ & $\mu$: usual picture

Two phases, one Critical End Point (CEP) between crossover and line of 1\textsuperscript{st} order transitions
Ising fixed point, dominated by \textit{massless} fluctuations at CEP
Lifshitz phase diagram for QCD

Lifshitz regime: strongly coupled, large fluctuations

Unbroken 1st order line to spatially inhomogeneous phases = “chiral spirals”

Heavy ions: could go through two 1st order transitions

$T_0$: maximum $T$, point of equal concentrations (unequal entropy)
Fluctuations at 7 GeV

Beam Energy Scan, down to 7 GeV.
Fluctuations *MUCH* larger when up to 2 GeV than to 0.8 GeV.
Trivial multiplicity scaling? ... or Chiral Spiral?
But fluctuations in nucleons, not pions.

X. Luo & N. Xu, 1701.02105, fig. 37; Jowazee, 1708.03364

\[ c_n = \frac{\partial^n}{\partial \mu^n} p(T, \mu) \]
Experimentally

For *any* sort of periodic structure (1D, 2D, 3D...),

Fluctuations concentrated about some characteristic momentum $k_0$

So “slice and dice”: bin in intervals, 0 to .5 GeV, .5 to 1., etc.

*If* peak in fluctuations in a bin not including zero, *may* be evidence for $k_0 \neq 0$.

*Signals for Lifshitz regime?*

*Must* measure fluctuations in pions, kaons...
Gertrude Stein about Oakland, California, ~ 1890:

“There’s no there, there.”
Heavy ion collisions at low energy:

There is a there, there

*But what is it?*
NJL models and Lifshitz points

Consider Nambu-Jona-Lasino models.
Nickel, 0902.1778 & 0906.5295 + .... + Buballa & Carignano 1406.1367

\[ \mathcal{L}_{\text{NJL}} = \overline{\psi} (\partial + g\sigma) \psi + \sigma^2 \]

Integrating over \( \psi \),

\[ \text{tr} \log(\partial + g\sigma) \sim \ldots + \kappa_1 ((\partial\sigma)^2 + \sigma^4) + \ldots \]

Due to scaling, \( \partial \rightarrow \lambda\partial, \sigma \rightarrow \lambda\sigma \).
Consequently, in NJL @ 1-loop, tricritical = Lifshitz point.

Special to including only \( \sigma \) at one loop.
Not generic: violated by the inclusion of more fields, to two loop order, etc.

Improved gradient expansion near critical point:
Carignano, Anzuni, Benhar, & Mannarelli, 1711.08607.
Consider $m^2 > 0, Z < 0$: minimum in propagator at nonzero momentum

Brazovski ‘75; Hohenberg & Swift ‘95 + ... ;
Lee, Nakano, Tsue, Tatsumi & Friman, 1504.03185; Yoshiike, Lee & Tatsumi 1702.01511

\[
\Delta^{-1} = m^2 + Z k^2 + k^4 / M^2 \\
= m_{\text{eff}}^2 - 2 Z k_z^2 + O(k_z^3, k_z k_{\perp}^2)
\]

$k=(k_{\perp}, k_z-k_0)$: no terms in $k_{\perp}^2$, only $(k_{\perp}^2)^2$.

Due to spon. breaking of rotational sym.

1-loop tadpole diagram:

\[
\int d^3k \frac{1}{k_z^2 + m_{\text{eff}}^2 + \ldots} \sim M^2 \int \frac{dk_z}{k_z^2 + m_{\text{eff}}^2} \sim \frac{M^2}{m_{\text{eff}}}
\]

Effective reduction to 1-dim for any spatial dimension $d$, any global symmetry
1\textsuperscript{st} order transition in 1-dim.

*Strong* infrared fluctuations in 1-dim., both in the mass:

$$\Delta m^2 \sim \lambda \int d^3k \frac{1}{k_z^2 + m_{\text{eff}}^2 + \ldots} \sim \lambda \frac{M}{m_{\text{eff}}}$$

and for the coupling constant:

$$\Delta \lambda \sim -\lambda^2 \int \frac{d^3k}{(k_z^2 + m_{\text{eff}}^2 + \ldots)^2} \sim -\lambda^2 M^3 \int_{m_{\text{eff}}} \frac{dk_z}{k_z^4} \sim -\lambda \frac{M^3}{m_{\text{eff}}^3}$$

Cannot tune $m_{\text{eff}}^2$ to 0: $\lambda_{\text{eff}}$ goes negative, 1\textsuperscript{st} order trans. induced by fluctuations

*Not* like other 1st order fluc-ind’ed trans’s: just that in 1-d, $m_{\text{eff}}^2 \neq 0$ always