Skyrmions and Nuclei

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Outline

1. Chiral Symmetry and EFT
2. Classical Skyrmions
3. Quantised Skyrmions
4. Carbon-12 and Oxygen-16
5. Summary
1. Chiral Symmetry and EFT

- SO(3) Isospin symmetry is the visible symmetry in strong interaction physics of particles and nuclei. Pions form a triplet, an isovector. Proton-neutron and up-down quarks are isospin doublets, so isospin is effectively SU(2) (like spin).

- \((u, d)\) mass difference and Coulomb effects break isospin symmetry. Isospin still useful for light and medium-mass nuclei. E.g. Lithium-7/Beryllium-7 is an isospin doublet, with similar nuclear spectra.

- There is a larger, spontaneously broken, Chiral Symmetry \(SO(4) \simeq SU(2) \times SU(2)\) for massless quarks. Isospin is the unbroken subgroup. Pions are (approximate) Goldstone bosons.
Effective Field Theory (EFT)

- An EFT of hadrons dispenses with quarks. Chiral symmetry acts by SO(4) rotations on fields $\sigma, \pi_1, \pi_2, \pi_3$. There is also a nonlinear constraint

$$\sigma^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1$$

so three fields are dynamical (the pions). The combined field lies on this 3-sphere.

- EFT is chirally symmetric if it only involves derivatives of the fields in an SO(4) invariant way.

- The vacuum is

$$\sigma = 1 \quad \text{and} \quad \pi = 0.$$ 

SO(3) is the unbroken subgroup and acts by (iso)rotations on $\pi$.

- An additional SO(3) invariant potential gives pions a small mass.
Skyrme Model

- The Skyrme model is a simple EFT with three parameters, but not carefully tuned to hadronic physics. Perturbatively, it describes massive, interacting pions.
- Skyrme's key idea: No explicit nucleon fields. Nucleons are identified with solitons (Skyrmions) of the pion theory. These are smooth, localised solutions of the field equations.
- Large $N_c$ limit of QCD → Nucleon becomes heavy and classical. QCD symmetries and anomalies are consistent with a topological baryon number (Witten).
- Chiral bag model of nucleons has constituent quarks inside and pion fields outside. Boundary condition makes the pion field similar to Skyrmion structure. As bag radius goes to zero, just a Skyrmion remains – the Cheshire Cat principle (Goldstone and Jaffe, Rho).
SU(2) Skyrme field

\[ U(x) = \sigma(x) \mathbf{1}_2 + i\pi(x) \cdot \tau \]

with \( \sigma^2 + \pi \cdot \pi = 1 \). Boundary condition \( U \to \mathbf{1}_2 \) as \( |x| \to \infty \).

Current (gradient of \( U \)) is \( R_\mu = (\partial_\mu U)U^{-1} \).

Skyrme Lagrangian \( L \) (Skyrme units) is

\[
\int \left\{ -\frac{1}{2} \text{Tr}(R_\mu R_\mu) - \frac{1}{16} \text{Tr}([R_\mu, R_\nu][R_\mu, R_\nu]) - m_\pi^2 \text{Tr}(\mathbf{1}_2 - U) \right\} d^3x
\]
2. Classical Skyrmions

- Skyrmions occur naturally, because the field $U$ in ordinary space $\mathbb{R}^3$ can wind round the (target) 3-sphere, with an integer winding number:

  \[
  \text{Winding number} = \text{Degree of } U = \text{Baryon number } B .
  \]

- The integral formula for Baryon number $B$ is

  \[
  B = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} \text{Tr}(R_i R_j R_k) \, d^3x .
  \]

- Dominant Skyrmions are minimal energy, static solutions for each $B$, and are interpreted as intrinsic structures of nuclei.

- Visualise Skyrmions using Runge colour sphere: Colours recording the normalised pion field $\hat{\pi}(\mathbf{x})$ are superposed on a constant energy density surface.
Constant energy density surfaces of $B = 1$ to $B = 8$ Skyrmions (with $m_\pi = 0$) [R. Battye and P. Sutcliffe]
$B = 1$ Skyrmion (two different orientations)
$B = 2$ Skyrmion [D. Feist]
Scattering $B = 1$ Skyrmions [D. Foster and S. Krusch]
$B = 3$ Skyrmion
$B = 4$ Skyrmion – Alpha particle
$B = 6$ Skyrmion (two different orientations)
\( B = 8 \) Skyrmion \((m_\pi = 1)\)
How Skyrmions are Constructed

- Basic $B = 1$ Skyrmion is based on spherical “hedgehog” ansatz. Radial profile is solution of ODE.

- To construct larger Skyrmions: Put $B = 1$ Skyrmions in attractive relative orientations. Best arrangements are subclusters of FCC lattice. Four Skyrmion orientations occur, on four sublattices. Optimal cluster symmetries are tetrahedral or cubic.

- Or: Build Skyrmion fields $\mathbb{R}^3 \to S^3$ using rational maps $S^2 \to S^2$ and radial profiles [C. Houghton, NSM and P. Sutcliffe]. This gives approximate solutions, whose symmetries help in the quantisation programme.

- Or: Relate Skyrmions to other solitons, e.g. monopoles, instantons – ansatz of M. Atiyah and NSM.
3. Quantised Skyrmions

- Skyrmions can be quantised as (coloured) rigid bodies: They acquire spin and isospin. Skyrme field topology and Skyrmion symmetries constrain the allowed states [Finkelstein-Rubinstein].

- Quantised vibrational deformations of Skyrmions can also be considered.

- The principal test of the Skyrme model: Compare quantised Skyrmions to nuclei and their excited states. Several energy spectra and a few EM transitions have been calculated so far.

- The results are closer to collective models and cluster models of nuclei, than to shell model physics.
Lowest-energy quantum states for small $B$ were found by Adkins, Nappi and Witten; Braaten and Carson; Walhout:

- $B = 1$: Proton and neutron, with spin $J = \frac{1}{2}$ and isospin $I = \frac{1}{2}$. Excited states (Delta-resonances) have $J = I = \frac{3}{2}$.
- $B = 2$: $^2\text{H}$ (Deuteron as spinning torus), with $J = 1$ and $I = 0$.
- $B = 3$: $^3\text{H}$ and $^3\text{He}$, with $J = \frac{1}{2}$ and $I = \frac{1}{2}$.
- $B = 4$: $^4\text{He}$ (Alpha particle), with $J = I = 0$. 
Classically spinning $B = 1$ Skyrmions, modelling $p$ and $n$ states
[D. Foster and NSM]
- $B = 6$ Skyrmion has rotational states like stack of three $B = 2$ Skyrmions (d+d+d or $\alpha$+d). Isospin 0 states represent Lithium-6, and have spins $J^P = 1^+, 3^+$.

- $B = 7$ dodecahedral Skyrmion has isospin $\frac{1}{2}$ states with minimal spin $\frac{7}{2}$. A vibrational mode allows $B = 4 / B = 3$ clusters, and states of spin $\frac{3}{2}$, $\frac{1}{2}$ and $\frac{5}{2}$ (C.J. Halcrow).

- Therefore, the spin $\frac{7}{2}$ state of Lithium-7/Beryllium-7 predicted to have smallest radius.

- Beryllium-8’s clear rotational band with resonance states $J^P = 0^+, 2^+, 4^+$ reproduced by the quantised “double-cube” Skyrmion, with energies close to $\frac{1}{2V}J(J + 1)$.
Energy level diagram for nuclei with $B = 8$
Skyrmions of Larger Baryon Number

- $B > 8$ Skyrmions have multi-layer geometry. New methods needed to construct configurations to relax numerically. Skyrmions are more compact when $m_\pi \approx 1$ (its physical value).

- Gluing together $B = 4$ cubes with colours touching on faces works for $B = 12, 16, 24, 32$ [Feist].

- One can also cut chunks from the Skyrme crystal, a cubic array of half-Skyrmions, with exceptionally low energy per Skyrmion [Castillejo et al., Kugler and Shtrikman]. $B = 32$ and $B = 108$ illustrate this. One can cut single Skyrmions off the corners of these chunks, to obtain, e.g. $B = 31$ and $B = 100$.

- The Coulomb energy for large $B$ shifts Skyrmion energies but has little effect on their shapes or quantum wavefunctions.
$B = 12$ Skyrmion with $D_{3h}$ symmetry
$B = 12$ Skyrmion with $D_{4h}$ symmetry
$B = 32$ Skyrmion
$B = 31$ Skyrmion ($B=32$ with corner cut off)
$B = 108$ Skyrmion [P.H.C. Lau]
4. Carbon-12 and Oxygen-16

- Two $B = 12$ Skyrmions have very similar energies – both are apparently stable. One is a triangle of $B = 4$ cubes with $D_{3h}$ symmetry, the other a linear chain with $D_{4h}$ symmetry.

- P.H.C. Lau and NSM have quantised their rotational motion as rigid bodies. The relevant inertia coefficients $V_{11} = V_{22}$ and $V_{33}$ have been calculated for each Skyrmion.

- Allowed states for the triangular Skyrmion have spin/parity $J^P = 0^+, 2^+, 3^-, 4^-, 4^+, 5^-, 6^+, 6^-, 6^+$ in rotational bands with $K = 0, 3, 6$. Their energies are

$$E(J, K) = \frac{1}{2V_{11}} J(J + 1) + \left( \frac{1}{2V_{33}} - \frac{1}{2V_{11}} \right) K^2.$$ 

These match well the experimental rotational band of the ground state.
The Hoyle State of Carbon-12

- Our model suggests that the $0^+$ Hoyle state corresponds to the linear chain Skyrmion.
- The $J^P = 0^+, 2^+, 4^+$ rotational band of Hoyle state excitations [M. Freer et al.] has much smaller slope than the ground state band.
- The ratio of slopes, and the ratio of the root mean square radii of the Carbon-12 ground state and Hoyle state, are well fitted by the Skyrmions.
- The Skyrme model makes predictions for several spin 6 states.
Rotational energy spectrum of the two $B = 12$ Skyrmions, compared with data
States of Oxygen-16

- Four $B = 4$ cubes can be arranged as a tetrahedron or flat square. Rigid-body quantisation of these gives rotational bands, but misses states associated with vibrations.
- C. Halcrow, C. King and NSM have considered a 2-parameter family of configurations of four cubes, and have quantised these parameters together with rotations. This incorporates both the tetrahedron and flat square.
- The configurations all have $D_2$ symmetry so miss the $1^-$ vibrational states.
Scattering channel of four $B = 4$ Skyrmions (alpha particles)
Model of $B = 16$ Tetrahedral Skyrmion [A. Aitta and NSM]
Model of $B = 16$ Flat Square Skyrmion
Energy level diagram for $^{16}\text{O}$ states. Solid $\equiv$ Skyrme model. Circle/Triangle $\equiv$ $+/-$ parity. Hollow $\equiv$ Experiment.
5. Summary

- Fundamental to the Skyrmion model of nuclei is the chiral field \( U(x) \) of an EFT. Skyrmions are topological solitons. Protons and neutrons are \( B = 1 \) Skyrmions with quantised spin and isospin.

- Toroidal and cubic Skyrmions, with \( B = 2 \) and \( B = 4 \), appear as substructures in all Skyrmions, and are easily seen. They correspond to deuteron and alpha particle constituents of larger nuclei.

- If Skyrmions are treated as rigid bodies, the binding energies are too large, and charge densities have too much spatial inhomogeneity.

- Recent work on Skyrmion quantisation allows for vibrations, and relative motion of sub-clusters. This gives improved models for \(^7\text{Li}/^7\text{Be}\) and \(^{16}\text{O}\).
Supplementary Material
Quantisation Constraints for $B = 6$ Skyrmion

- The $B = 6$ Skyrmion has $D_{4d}$ symmetry. Its two rotational generators give Finkelstein-Rubinstein constraints

$$e^{i \frac{\pi}{2} L_3} e^{i \pi K_3} |\psi\rangle = |\psi\rangle$$
$$e^{i \pi L_1} e^{i \pi K_1} |\psi\rangle = - |\psi\rangle.$$

(Note: $L_i, K_i$ are spin and isospin operators w.r.t. body-fixed axes. There are no constraints on the spin and isospin projections w.r.t. space-fixed axes.)

- Allowed states have Isospin 0 ($^6\text{Li}$), with spin/parity

$$J^P = 1^+, 3^+, 4^-, 5^+, 5^-, \cdots,$$

and Isospin 1 ($^6\text{He}, ^6\text{Li}, ^6\text{Be}$), with spin/parity

$$J^P = 0^+, 2^+, 2^-, \cdots.$$

- The energy spectrum depends on rotational and isorotational moments of inertia, which we can calculate.
$B = 8$ States

- For isospin 1 (Lithium-8 and Boron-8) the Skyrme model predicts low-energy $J^P = 0^-, 2^-$ states in addition to the known $J^P = 2^+, 3^+$ states. These have not been seen, but may be hard to produce and observe.
- The isospin 2 quintet (Helium-8 etc.) has correct $J^P = 0^+$ ground states.
- We are currently studying the circumstances where isospin excitations can be treated as collective, with spectra $I(I + X)$. 