Skyrmions and Nuclei

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- 1. Chiral Symmetry and EFT
- 2. Classical Skyrmions
- 3. Quantised Skyrmions
- 4. Carbon-12 and Oxygen-16
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1. Chiral Symmetry and EFT

- SO(3) Isospin symmetry is the visible symmetry in strong interaction physics of particles and nuclei. Pions form a triplet, an isovector. Proton-neutron and up-down quarks are isospin doublets, so isospin is effectively SU(2) (like spin).
- (u, d) mass difference and Coulomb effects break isospin symmetry. Isospin still useful for light and medium-mass nuclei. E.g. Lithium-7/Beryllium-7 is an isospin doublet, with similar nuclear spectra.
- There is a larger, spontaneously broken, Chiral Symmetry SO(4) $\simeq$ SU(2) $\times$ SU(2) for massless quarks. Isospin is the unbroken subgroup. Pions are (approximate) Goldstone bosons.

Effective Field Theory (EFT)

- An EFT of hadrons dispenses with quarks. Chiral symmetry acts by SO(4) rotations on fields $\sigma, \pi_1, \pi_2, \pi_3$. There is also a nonlinear constraint

$$\sigma^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = 1$$

  so three fields are dynamical (the pions). The combined field lies on this 3-sphere.
- EFT is chirally symmetric if it only involves derivatives of the fields in an SO(4) invariant way.
- The vacuum is

$$\sigma = 1 \quad \text{and} \quad \boldsymbol{\pi} = 0\,.$$

  SO(3) is the unbroken subgroup and acts by (iso)rotations on $\boldsymbol{\pi}$.
- An additional SO(3) invariant potential gives pions a small mass.

Skyrme Model

- The Skyrme model is a simple EFT with three parameters, but not carefully tuned to hadronic physics. Perturbatively, it describes massive, interacting pions.
- Skyrme's key idea: No explicit nucleon fields. Nucleons are identified with solitons (Skyrmions) of the pion theory. These are smooth, localised solutions of the field equations.
- Large N_c limit of QCD $\rightarrow$ Nucleon becomes heavy and classical. QCD symmetries and anomalies are consistent with a topological baryon number (Witten).
- Chiral bag model of nucleons has constituent quarks inside and pion fields outside. Boundary condition makes the pion field similar to Skyrmion structure. As bag radius goes to zero, just a Skyrmion remains – the Cheshire Cat principle (Goldstone and Jaffe, Rho).

- SU(2) Skyrme field

$$U(x) = \sigma(x)\,\mathbf{1}_2 + i\boldsymbol{\pi}(x)\cdot\boldsymbol{\tau}$$

with $\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi} = 1$. Boundary condition $U \to \mathbf{1}_2$ as $|\mathbf{x}| \to \infty$.
- Current (gradient of U) is $R_\mu = (\partial_\mu U)U^{-1}$.
- Skyrme Lagrangian L (Skyrme units) is

$$\int \left\{ -\frac{1}{2}\mathrm{Tr}(R_\mu R_\mu) - \frac{1}{16}\mathrm{Tr}([R_\mu, R_\nu][R_\mu, R_\nu]) - m_\pi^2 \mathrm{Tr}(\mathbf{1}_2 - U) \right\} d^3x$$

2. Classical Skyrmions

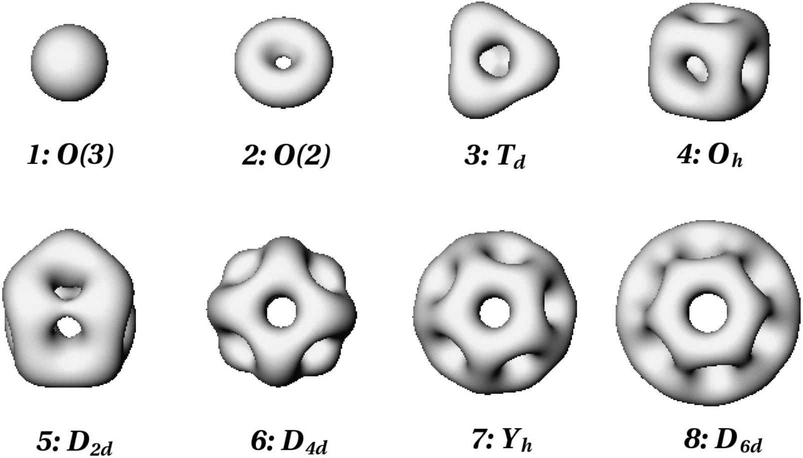
- Skyrmions occur naturally, because the field U in ordinary space $\mathbb{R}^3$ can wind round the (target) 3-sphere, with an integer winding number:

$$\text{Winding number} = \text{Degree of } U = \text{Baryon number } B\,.$$

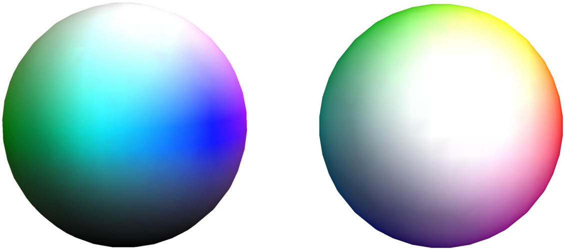
- The integral formula for Baryon number B is

$$B = -\frac{1}{24\pi^2}\int \varepsilon_{ijk}\mathrm{Tr}(R_i R_j R_k)\, d^3x\,.$$

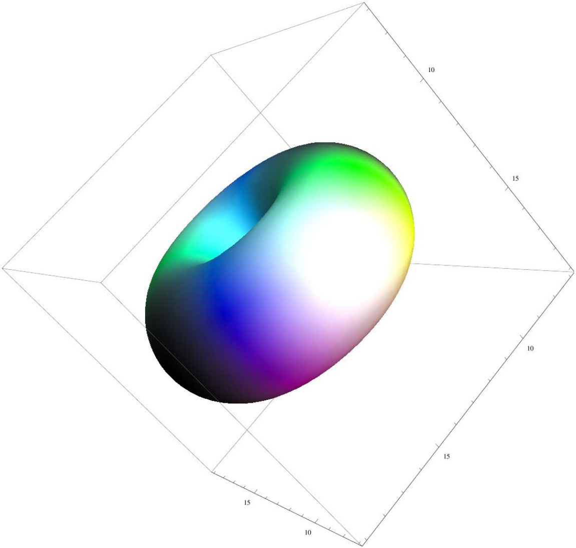
- Dominant Skyrmions are minimal energy, static solutions for each B, and are interpreted as intrinsic structures of nuclei.
- Visualise Skyrmions using Runge colour sphere: Colours recording the normalised pion field $\hat{\pi}(\mathbf{x})$ are superposed on a constant energy density surface.



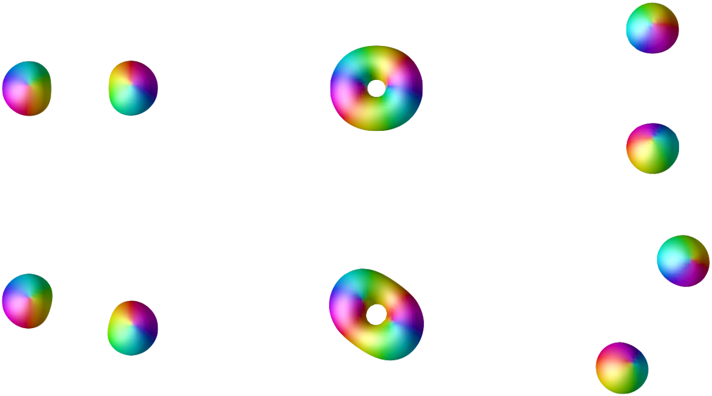
Constant energy density surfaces of $B = 1$ to $B = 8$ Skyrmions (with $m_\pi = 0$) [R. Battye and P. Sutcliffe]



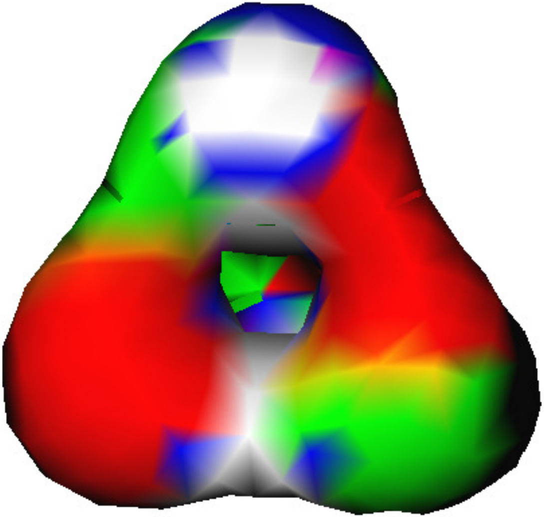
$B = 1$ Skyrmion (two different orientations)



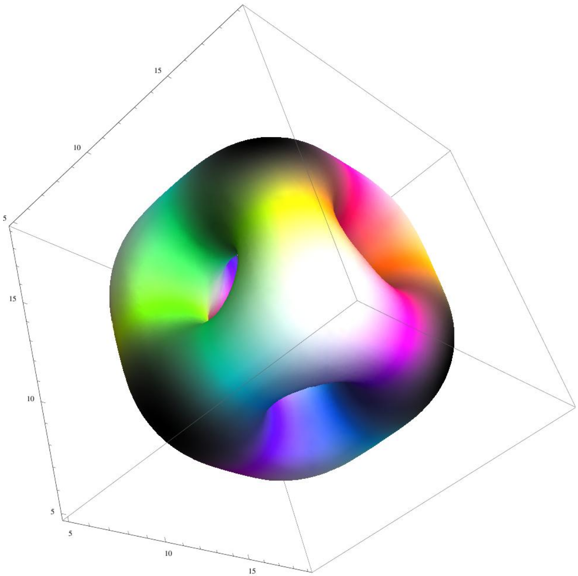
$B = 2$ Skyrmion [D. Feist]



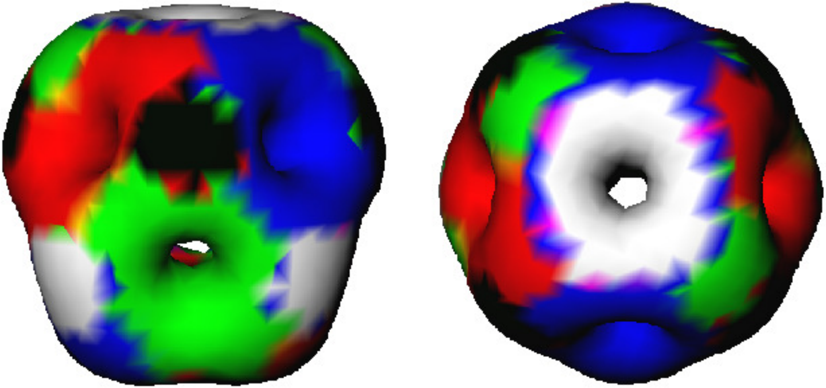
Scattering $B = 1$ Skyrmions [D. Foster and S. Krusch]



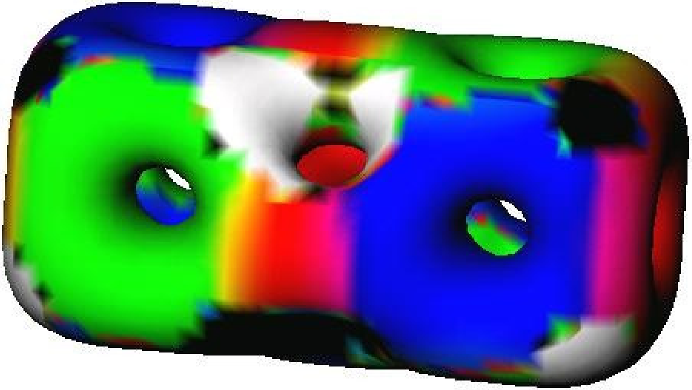
$B = 3$ Skyrmion



$B = 4$ Skyrmion – Alpha particle



$B = 6$ Skyrmion (two different orientations)



$B = 8$ Skyrmion ($m_\pi = 1$)

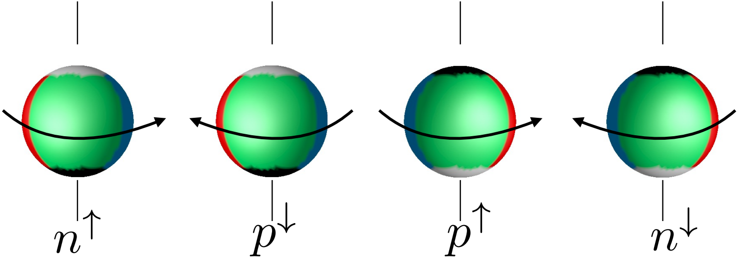
How Skyrmions are Constructed

- Basic $B = 1$ Skyrmion is based on spherical “hedgehog” ansatz. Radial profile is solution of ODE.
- To construct larger Skyrmions: Put $B = 1$ Skyrmions in attractive relative orientations. Best arrangements are subclusters of FCC lattice. Four Skyrmion orientations occur, on four sublattices. Optimal cluster symmetries are tetrahedral or cubic.
- Or: Build Skyrmion fields $\mathbb{R}^3 \to S^3$ using rational maps $S^2 \to S^2$ and radial profiles [C. Houghton, NSM and P. Sutcliffe]. This gives approximate solutions, whose symmetries help in the quantisation programme.
- Or: Relate Skyrmions to other solitons, e.g. monopoles, instantons – ansatz of M. Atiyah and NSM.

3. Quantised Skyrmions

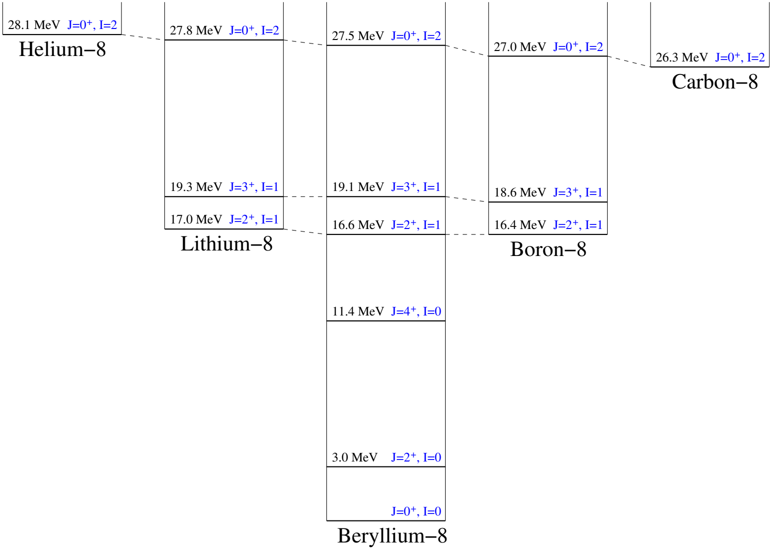
- Skyrmions can be quantised as (coloured) rigid bodies: They acquire spin and isospin. Skyrme field topology and Skyrmion symmetries constrain the allowed states [Finkelstein-Rubinstein].
- Quantised vibrational deformations of Skyrmions can also be considered.
- The principal test of the Skyrme model: Compare quantised Skyrmions to nuclei and their excited states. Several energy spectra and a few EM transitions have been calculated so far.
- The results are closer to collective models and cluster models of nuclei, than to shell model physics.

- Lowest-energy quantum states for small B were found by Adkins, Nappi and Witten; Braaten and Carson; Walhout:
- $B = 1$: Proton and neutron, with spin $J = \frac{1}{2}$ and isospin $I = \frac{1}{2}$. Excited states (Delta-resonances) have $J = I = \frac{3}{2}$.
- $B = 2$: ^{2}H (Deuteron as spinning torus), with $J = 1$ and $I = 0$.
- $B = 3$: ^{3}H and ^{3}He, with $J = \frac{1}{2}$ and $I = \frac{1}{2}$.
- $B = 4$: ^{4}He (Alpha particle), with $J = I = 0$.



Classically spinning $B = 1$ Skyrmions, modelling p and n states
[D. Foster and NSM]

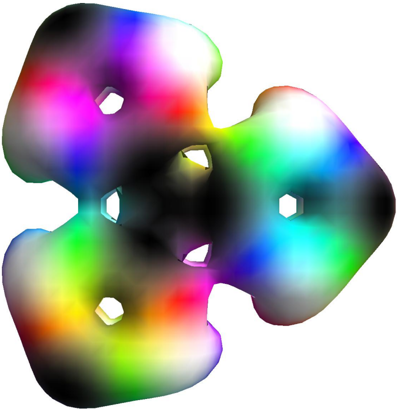
- $B = 6$ Skyrmion has rotational states like stack of three $B = 2$ Skyrmions (d+d+d or α+d). Isospin 0 states represent Lithium-6, and have spins $J^P = 1^+, 3^+$.
- $B = 7$ dodecahedral Skyrmion has isospin $\frac{1}{2}$ states with minimal spin $\frac{7}{2}$. A vibrational mode allows $B = 4$ / $B = 3$ clusters, and states of spin $\frac{3}{2}$, $\frac{1}{2}$ and $\frac{5}{2}$ (C.J. Halcrow).
- Therefore, the spin $\frac{7}{2}$ state of Lithium-7/Beryllium-7 predicted to have smallest radius.
- Beryllium-8's clear rotational band with resonance states $J^P = 0^+, 2^+, 4^+$ reproduced by the quantised "double-cube" Skyrmion, with energies close to $\frac{1}{2V} J(J+1)$.



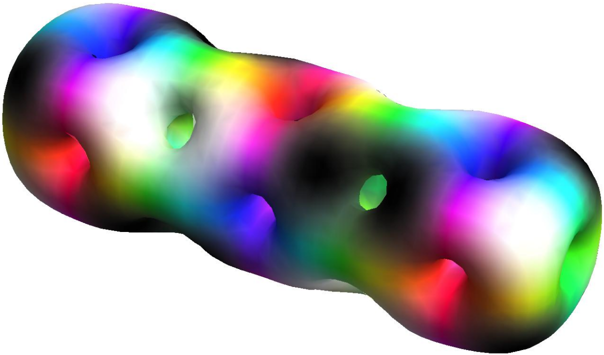
Energy level diagram for nuclei with $B = 8$

Skyrmions of Larger Baryon Number

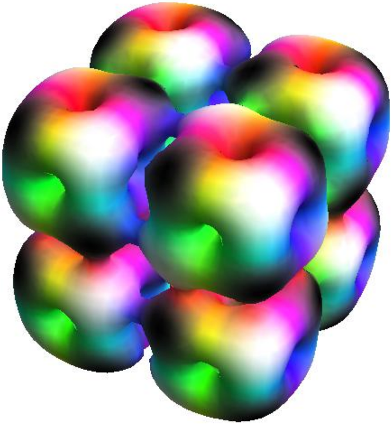
- $B > 8$ Skyrmions have multi-layer geometry. New methods needed to construct configurations to relax numerically. Skyrmions are more compact when $m_\pi \simeq 1$ (its physical value).
- Gluing together $B = 4$ cubes with colours touching on faces works for $B = 12, 16, 24, 32$ [Feist].
- One can also cut chunks from the Skyrme crystal, a cubic array of half-Skyrmions, with exceptionally low energy per Skyrmion [Castillejo et al., Kugler and Shtrikman]. $B = 32$ and $B = 108$ illustrate this. One can cut single Skyrmions off the corners of these chunks, to obtain, e.g. $B = 31$ and $B = 100$.
- The Coulomb energy for large B shifts Skyrmion energies but has little effect on their shapes or quantum wavefunctions.



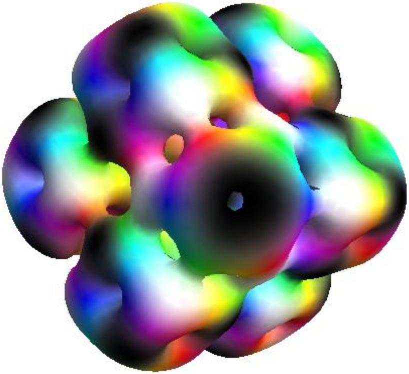
$B = 12$ Skyrmion with D_{3h} symmetry



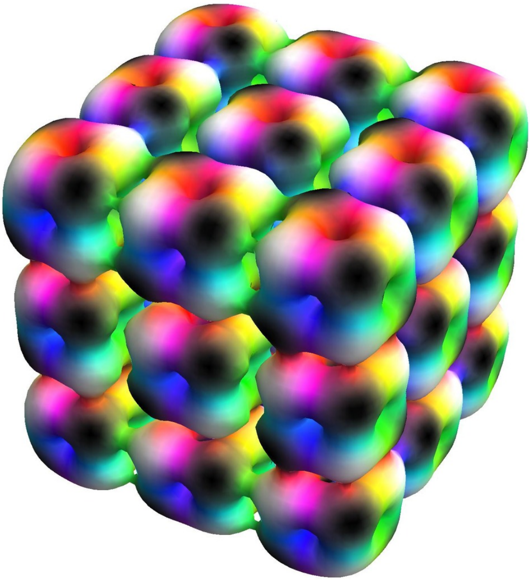
$B = 12$ Skyrmion with D_{4h} symmetry



$B = 32$ Skyrmion



$B = 31$ Skyrmion (B=32 with corner cut off)



$B = 108$ Skyrmion [P.H.C. Lau]

4. Carbon-12 and Oxygen-16

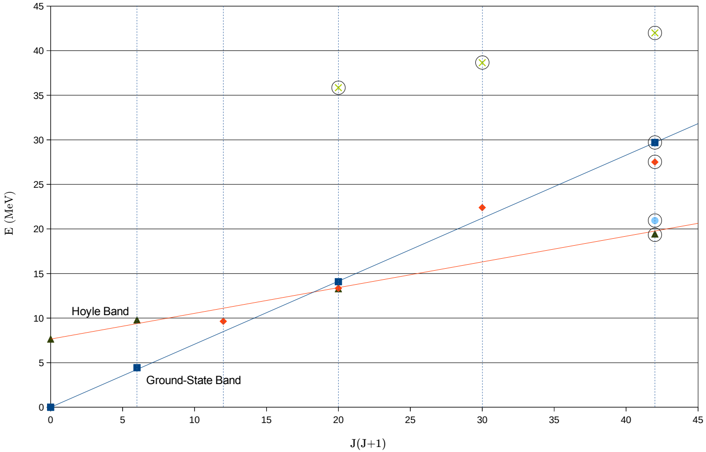
- Two $B = 12$ Skyrmions have very similar energies – both are apparently stable. One is a triangle of $B = 4$ cubes with D_{3h} symmetry, the other a linear chain with D_{4h} symmetry.
- P.H.C. Lau and NSM have quantised their rotational motion as rigid bodies. The relevant inertia coefficients $V_{11} = V_{22}$ and V_{33} have been calculated for each Skyrmion.
- Allowed states for the triangular Skyrmion have spin/parity $J^P = 0^+, 2^+, 3^-, 4^-, 4^+, 5^-, 6^+, 6^-, 6^+$ in rotational bands with $K = 0, 3, 6$. Their energies are

$$E(J,K) = \frac{1}{2V_{11}} J(J+1) + \left(\frac{1}{2V_{33}} - \frac{1}{2V_{11}} \right) K^2 \,.$$

These match well the experimental rotational band of the ground state.

The Hoyle State of Carbon-12

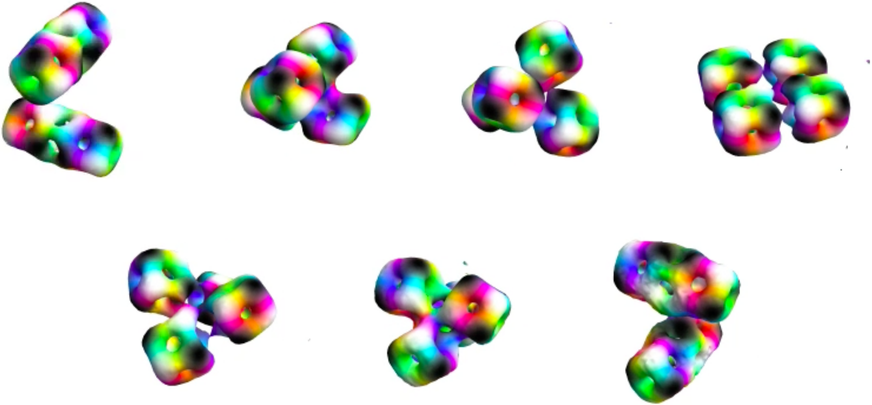
- Our model suggests that the 0^+ Hoyle state corresponds to the linear chain Skyrmion.
- The $J^P = 0^+, 2^+, 4^+$ rotational band of Hoyle state excitations [M. Freer et al.] has much smaller slope than the ground state band.
- The ratio of slopes, and the ratio of the root mean square radii of the Carbon-12 ground state and Hoyle state, are well fitted by the Skyrmions.
- The Skyrme model makes predictions for several spin 6 states.



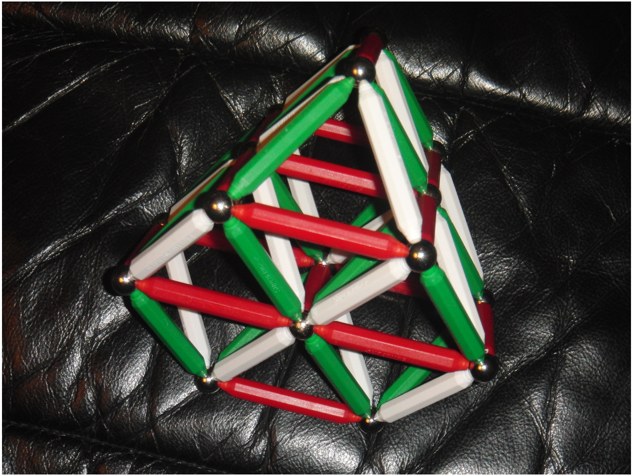
Rotational energy spectrum of the two $B = 12$ Skyrmions, compared with data

States of Oxygen-16

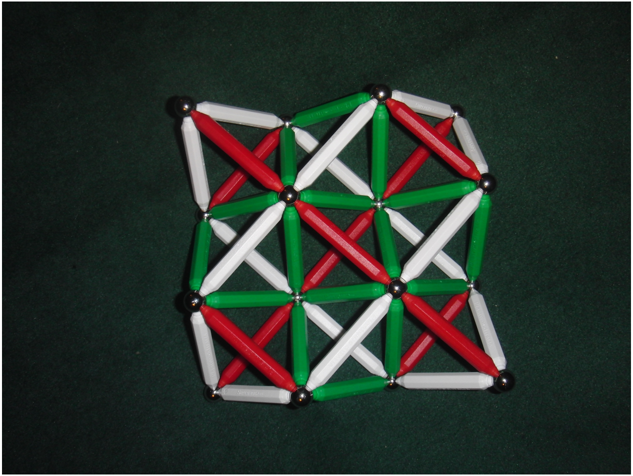
- Four $B = 4$ cubes can be arranged as a tetrahedron or flat square. Rigid-body quantisation of these gives rotational bands, but misses states associated with vibrations.
- C. Halcrow, C. King and NSM have considered a 2-parameter family of configurations of four cubes, and have quantised these parameters together with rotations. This incorporates both the tetrahedron and flat square.
- The configurations all have D_2 symmetry so miss the 1^- vibrational states.



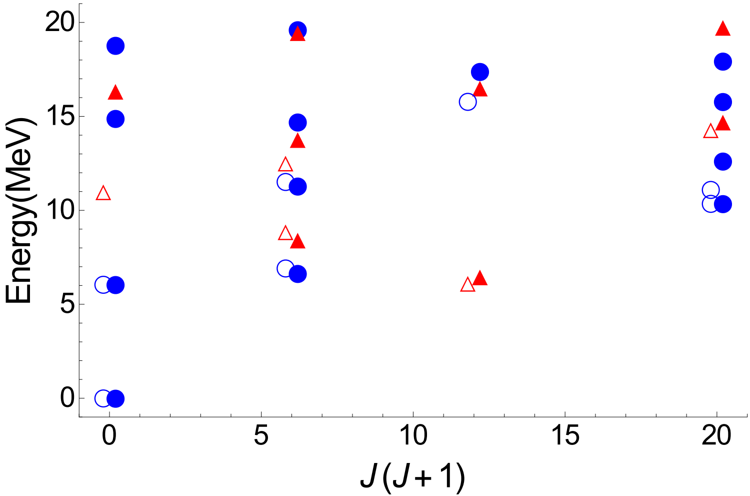
Scattering channel of four $B = 4$ Skyrmions (alpha particles)



Model of $B = 16$ Tetrahedral Skyrmion [A. Aitta and NSM]



Model of $B = 16$ Flat Square Skyrmion



Energy level diagram for ^{16}O states. Solid $\equiv$ Skyrme model. Circle/Triangle $\equiv +/-$ parity. Hollow $\equiv$ Experiment.

5. Summary

- Fundamental to the Skyrmion model of nuclei is the chiral field $U(x)$ of an EFT. Skyrmions are topological solitons. Protons and neutrons are $B = 1$ Skyrmions with quantised spin and isospin.
- Toroidal and cubic Skyrmions, with $B = 2$ and $B = 4$, appear as substructures in all Skyrmions, and are easily seen. They correspond to deuteron and alpha particle constituents of larger nuclei.
- If Skyrmions are treated as rigid bodies, the binding energies are too large, and charge densities have too much spatial inhomogeneity.
- Recent work on Skyrmion quantisation allows for vibrations, and relative motion of sub-clusters. This gives improved models for ^{7}Li/^{7}Be and ^{16}O.

Supplementary Material

Quantisation Constraints for $B = 6$ Skyrmion

- The $B = 6$ Skyrmion has D_{4d} symmetry. Its two rotational generators give Finkelstein-Rubinstein constraints

$$
\begin{aligned}
e^{i\frac{\pi}{2}L_3} e^{i\pi K_3} |\Psi\rangle &= |\Psi\rangle \\
e^{i\pi L_1} e^{i\pi K_1} |\Psi\rangle &= -|\Psi\rangle \,.
\end{aligned}
$$

  (Note: L_i, K_i are spin and isospin operators w.r.t. body-fixed axes. There are no constraints on the spin and isospin projections w.r.t. space-fixed axes.)

- Allowed states have Isospin 0 (^{6}Li), with spin/parity

$$J^P = 1^+, 3^+, 4^-, 5^+, 5^-, \cdots ,$$

  and Isospin 1 (^{6}He, ^{6}Li, ^{6}Be), with spin/parity

$$J^P = 0^+, 2^+, 2^-, \cdots .$$

- The energy spectrum depends on rotational and isorotational moments of inertia, which we can calculate.

$B = 8$ **States**

- For isospin 1 (Lithium-8 and Boron-8) the Skyrme model predicts low-energy $J^P = 0^-, 2^-$ states in addition to the known $J^P = 2^+, 3^+$ states. These have not been seen, but may be hard to produce and observe.
- The isospin 2 quintet (Helium-8 etc.) has correct $J^P = 0^+$ ground states.
- We are currently studying the circumstances where isospin excitations can be treated as collective, with spectra $I(I + X)$.