QCD phase diagram at large $N_c$

The standard lore:

QCD Phase Diagram vs temperature, $T$, and quark chemical potential, $\mu$

One transition, chiral = deconfined, “semicircle”

Large $N_c$:

Two transitions, chiral $\neq$ deconfinement

Not just a critical end point, but a new “quarkyonic” phase:

Confined, chirally symmetric baryons: massive, parity doubled.

Work exclusively in rotating arm approximation...

McLerran & RDP, 0706.2191, to appear in NPA.
The first semicircle

Cabibbo and Parisi ‘75: Exponential (Hagedorn) spectrum limiting temperature, or transition to new, “unconfined” phase. One transition.

Punchline today: below for chiral transition, deconfinement splits off at finite $\mu$.

Fig. 1. Schematic phase diagram of hadronic matter. $\rho_B$ is the density of baryonic number. Quarks are confined in phase I and unconfined in phase II.
Phase diagram, ~ ‘06

Lattice, \( T \neq 0, \mu = 0 \): two possible transitions; one crossover, same \( T \). Karsch ’06

Remains crossover for \( \mu \neq 0 \)? Stephanov, Rajagopal, & Shuryak ‘98:
Critical end point where crossover turns into first order transition
Experiment: freezeout line

Cleymans & Redlich ‘99: Line for chemical equilibration at freezeout ~ semicircle.

N.B.: for $T = 0$, goes down to ~ nucleon mass.
Lattice “transition” appears above freezeout line? Schmidt ‘07

N.B.: small change in $T_c$ with $\mu$?
Lattice $T_c$, vs $\mu$

Rather small change in $T_c$ vs $\mu$? Depends where $\mu_c$ is at $T = 0$. Fodor & Katz ‘06
EoS of nuclear matter

Akmal, Panharipande, & Ravenhall ‘98: Equation of State for nuclear matter, T=0
E/A = energy/nucleon. Fits to various nuclear potentials

Anomalously small: binding energy of nuclear matter 15 MeV!
Calc’s reliable to ~ twice nuclear matter density.
Expansion in large $N_c$

‘t Hooft ’74: let $N_c \to \infty$, with $\lambda = g^2 N_c$ fixed.

$\sim N_c^2$ gluons in adjoint representation, vs $\sim N_c$ quarks in fundamental rep. ⇒

large $N_c$ dominated by gluons (iff $N_f = \#$ quark flavors small)

“Double line” notation. Useful even at small $N_c$ (Yoshimasa Hidaka & RDP)
Large $N_c$: “planar” diagrams

$\sim g^2 N_c = \lambda$

Planar diagram, $\sim \lambda^2$

Non-planar diagram, $\sim \lambda^2 / N_c$

Suppressed by $1 / N_c$
Quark loops are suppressed at large $N_c$ if $N_f$, # quark flavors, is held fixed.

Thus: limit of: large $N_c$, small $N_f$

Quarks introduced as external sources.

Analogous to “quenched” approximation, expansion about $N_f = 0$.

Veneziano ‘78: take both $N_c$ and $N_f$ large. Even more difficult.
Form factors at large $N_c$

$J \sim$ (gauge invariant) mesonic current

$$< J(x)J(0) > \sim N_c$$

Infinite # of planar diagrams for $< J J >$:

Confinement $\Rightarrow$ sum over mesons, form factors $\sim N_c^{1/2}$

$$< J(x)J(0) > \sim \int d^4 p \ e^{ip \cdot x} \sum_n < 0|J|n > \frac{1}{p^2 + m_n^2} < n|J|0 >$$

$$< J(x)J(0) > \sim N_c \Rightarrow < 0|J|n >\sim \sqrt{N_c} \text{ if } m_n \sim 1$$
Mesons & glueballs free at $N_c = \infty$

With form factors $\sim N_c^{1/2}$, 3-meson couplings $\sim 1/N_c^{1/2}$; 4-meson, $\sim 1/N_c$

For glueballs, 3-glueball couplings $\sim 1/N_c$, 4-glueball $\sim 1/N_c^2$

Mesons and glueballs don’t interact at $N_c = \infty$.

Large N limit always (some) classical mechanics Yaffe ‘82
Witten ‘79: Baryons have $N_c$ quarks, so nucleon mass $M_N \sim N_c \Lambda_{QCD}$.

Baryons like “solitons” of large $N_c$ limit ($\sim$ Skyrmion)

Leading correction to baryon mass:

\[ g^2 \times N_c \times N_c \sim \lambda N_c \]

Appears $\sim g^4 N_c^4 \sim \lambda^2 N_c^2$?

No, iteration of average potential, mass still $\sim N_c$. 

\[ g^2 \times N_c \times N_c \sim \lambda N_c \]
Baryons are *not* free at $N_c = \infty$

Baryons interact strongly. Two baryon scattering $\sim N_c$:

\[ g^2 \times N_c \times N_c \sim \lambda N_c \]

Scattering of three, four... baryons also $\sim N_c$

Mesons also interact strongly with baryons, $\sim N_c^0 \sim 1$

\[ g^2 \times N_c \sim \lambda \]
Skyrmions and $N_c = \infty$ baryons

Witten ‘83; Adkins, Nappi, Witten ‘83: Skyrme model for baryons

$$\mathcal{L} = f_\pi^2 \text{tr}|V_\mu|^2 + \kappa \text{tr}[V_\mu, V_\nu]^2, \ V_\mu = U^\dagger \partial_\mu U, \ U = e^{i\pi/2f_\pi}$$

Baryon soliton of pion Lagrangian: $f_\pi \sim N_c^{1/2}, \ \kappa \sim N_c, \ \text{mass} \sim f_\pi^2 \sim \kappa \sim N_c$.

Single baryon: at $r = \infty, \pi^a = 0, \ U = 1$. At $r = 0, \pi^a = \pi \frac{r^a}{r}$.

Baryon number topological: Wess & Zumino ’71; Witten ’83.

Huge degeneracy of baryons: multiplets of isospin and spin, $I = J$: $1/2 \ldots N_c/2$.

Obvious as collective coordinates of soliton, coupling spin & isospin

Dashen & Manohar ’93, Dashen, Jenkins, & Manohar ‘94:

Baryon-meson coupling $\sim N_c^{1/2},$

Cancellations from extended SU(2 $N_f$) symmetry.
Towards the phase diagram at $N_c = \infty$

As example, consider gluon polarization tensor at zero momentum. (at leading order, $\sim$ Debye mass$^2$, gauge invariant)

$$\Pi^{\mu\mu}(0) = g^2 \left( \left( N_c + \frac{N_f}{2} \right) \frac{T^2}{3} + \frac{N_f \mu^2}{2\pi^2} \right) = \lambda \frac{T^2}{3} , \quad N_c = \infty$$

For $\mu \sim N_c^0 \sim 1$, at $N_c = \infty$ the gluons are blind to quarks.

When $\mu \sim 1$, deconfining transition temperature $T_d(\mu) = T_d(0)$

Chemical potential only matters when larger than mass:

$\mu_{\text{Baryon}} > M_{\text{Baryon}}$. Define $m_{\text{quark}} = M_{\text{Baryon}}/N_c$; so $\mu > m_{\text{quark}}$.

“Box” for $T < T_c$; $\mu < m_{\text{quark}}$: confined phase baryon free, since their mass $\sim N_c$.

Thermal excitation $\sim \exp(-m_B/T) \sim \exp(-N_c) = 0$ at large $N_c$.

So hadronic phase in “box” = mesons & glueballs only, no baryons.
At least three phases. At large $N_c$, can use pressure, $P$, as order parameter. Hadronic (confined): $P \sim 1$. Deconfined, $P \sim N_c^2$. Thorn ’81; RDP ’84...

$P \sim N_c$: quarks or baryons = “quark-yonic”. Chiral symmetry restoration? N.B.: mass threshold at $m_q$ neglects (possible) nuclear binding, Son.

Phase diagram at $N_c = \infty$, I

Deconfined

Hadronic

Quarkyonic

1st order
Nuclear matter at large $N_c$

$\mu_{\text{Baryon}} = \sqrt{k_F^2 + M^2}$, $k_F$ = Fermi momentum of baryons.

Pressure of ideal baryons density times energy of non-relativistic baryons:

$$P_{\text{ideal baryons}} \sim n(k_F) \frac{k_F^2}{M} \sim \frac{1}{N_c} \frac{k_F^5}{\Lambda_{QCD}}$$

This is small, $\sim 1/N_c$. The pressure of the $I = J$ tower of resonances is as small:

$$\delta P_{\text{resonances}} \sim \frac{1}{M} \frac{k_F^8}{\Lambda_{QCD}^3} \sim \frac{1}{N_c} \frac{k_F^8}{\Lambda_{QCD}^4}$$

Two body interactions are huge, $\sim N_c$ in pressure.

$$\delta P_{\text{two body int.'s}} \sim N_c \frac{n(k_F)^2}{\Lambda_{QCD}^2} \sim N_c \frac{k_F^6}{\Lambda_{QCD}^2}$$

At large $N_c$, nuclear matter is dominated by potential, not kinetic terms!

Two body, three body... interactions all contribute $\sim N_c$. 
Window of nuclear matter

Balancing $P_{\text{ideal baryons}} \sim P_{\text{two body int.'s}}$, interactions important very quickly,

$$k_F \sim \frac{1}{N_c^2} \Lambda_{QCD}$$

For such momenta, only two body interactions contribute.

By the time $k_F \sim 1$, all interactions terms contribute $\sim N_c$ to the pressure.

But this is very close to the mass threshold,

$$\mu - m_q = \frac{\mu_B - M}{N_c} = \frac{k_F^2}{2MN_c} \sim \frac{1}{N_c^2} k_F^2$$

Hence “ordinary” nuclear matter is only in a very narrow window.

One quickly goes to a phase with pressure $P \sim N_c$.

So are they baryons, or quarks?
At high density, $\mu \gg \Lambda_{\text{QCD}}$, compute $P(\mu)$ in QCD perturbation theory.

To $\sim g^4$, Freedman & McLerran ('77); Ipp, Kajantie, Rebhan, & Vuorinen '06

$$P_{\text{pert.}}(\mu) \sim N_c N_f \mu^4 F_0(g^2(\mu/\Lambda_{\text{QCD}}), N_f)$$

At $\mu \neq 0$, only diagrams with at least one quark loop contribute. Still...

For $\mu >> \Lambda_{\text{QCD}}$, but $\mu \sim N_c^0 \sim 1$, calculation reliable.

Compute $P(\mu)$ to $\sim g^6, g^8...$? No “magnetic mass” at $\mu \neq 0$, well defined $\forall (g^2)^n$. 
“Quarkyonic” phase at large $N_c$

As gluons blind to quarks at large $N_c$, for $\mu \sim N_c^0 \sim 1$, *confined* phase for $T < T_d$

This includes $\mu >> \Lambda_{QCD}$! **Central puzzle.** We suggest:

To left: Fermi sea.

Deep in the Fermi sea, $k << \mu$, looks like quarks.

But: within $\sim \Lambda_{QCD}$ of the Fermi surface, confinement $\Rightarrow$ *baryons*

We term combination “quark-yonic”

OK for $\mu >> \Lambda_{QCD}$. When $\mu \sim \Lambda_{QCD}$, baryonic “skin” entire Fermi sea.

But what about chiral symmetry breaking?
Skyrmion crystals

Skyrmion crystal: soliton periodic in space.
Kutschera, Pethick & Ravenhall (KPR) ’84; Klebanov ’85 + ... 
Lee, Park, Min, Rho & Vento, hep-ph/0302019

At low density, chiral symmetry broken by Skyrme crystal, as in vacuum.
Chiral symmetry *restored* at nonzero density: \(< U > = 0* in each cell.

Goldhaber & Manton ’87: due to “half” Skyrmion symmetry in each cell.
Forkel, Jackson et al, ’89: excitations *are* chirally symmetric.

Easiest to understand with “spherical” crystal, KPR ’84, Manton ’87.
Take same boundary conditions as a single baryon, but for sphere of radius R:
At \( r = R \): \( \pi^a = 0 \). At \( r = 0 \), \( \pi^a = \pi r^a/r \). Density one baryon/(4 \( \pi R^3/3 \)).

At high density, term \( \sim \kappa \) dominates, so energy density \( \sim \) baryon density\(^{4/3}\).
Like perturbative QCD! Accident of simplest Skyrme Lagrangian.
Schwinger-Dyson equations at large $N_c$: 1+1 dim.’s

‘t Hooft ‘74: as gluons blind to quarks at large $N_c$, S-D eqs. simple for quark: Gluon propagator, and gluon quark anti-quark vertex unchanged. To leading order in $1/N_c$, only quark propagator changes:

\[
g^2 D \int \frac{d^2k}{k^2} \sim g^2 D r
\]

‘t Hooft ‘74: in 1+1 dimensions, single gluon exchange generates linear potential,

In vacuum, Regge trajectories of confined mesons. Baryons?

Solution at $\mu \neq 0$? Should be possible, not yet solved.

Thies et al ’00...06: Gross-Neveu model has crystalline structure at $\mu \neq 0$
Schwinger-Dyson eqs. at large $N_c$: 3+1 dim.’s

Glozman & Wagenbrunn 0709.3080: in 3+1 dimensions, confining gluon propagator, $1/(k^2)^2$ as $k^2 \to 0$:

\[ g^2 \int d^3k \frac{e^{ikr}}{k^2} \left(1 + \frac{\sigma}{k^2}\right) \sim g^2 \sigma r, \ r \to \infty \]

Involves mass parameter, $\sigma$. At $\mu = 0$, \( \langle \bar{\psi}\psi \rangle = (.23\sqrt{\sigma})^3 \)

Take S-D eq. at large $N_c$, so confinement unchanged by $\mu \neq 0$.

Find chiral symmetry restoration at $\mu = .11\sqrt{\sigma}$

Hence: in two models at $\mu \neq 0$, chiral symmetry restoration in confined phase
Asymptotically large $\mu$

For $\mu \sim (N_c)^p$, $p > 0$, gluons no longer blind to quarks. Perturbatively,

$$P_{\text{pert.}}(\mu, T) \sim N_c N_f \mu^4 F_0, \ N_c N_f \mu^2 T^2 F_1, \ N_c^2 T^4 F_2.$$  

First two terms from quarks & gluons, last only from gluons. Two regimes:

$\mu \sim N_c^{1/4} \Lambda_{\text{QCD}}$:

- $N_c \mu^4 F_0 \sim N_c^2 F_2 \sim N_c^2 >> N_c \mu^2 F_1 \sim N_c^{3/2}$.  
  - Gluons & quarks contribute equally to pressure; quark cont. T-independent.

$\mu \sim N_c^{1/2} \Lambda_{\text{QCD}}$:

- New regime: $m_{\text{Debye}}^2 \sim g^2 \mu^2 \sim 1$, so gluons feel quarks.

  - $N_c \mu^4 F_0 \sim N_c^3 >> N_c \mu^2 F_1, \ N_c^2 F_2 \sim N_c^2$.  
  - Quarks dominate pressure, T-independent.

Eventually, first order deconfining transition can either:
- end in a critical point, or bend over to $T = 0$: ?
We suggest: quarkyonic phase includes chiral trans. Order by usual arguments.

Mocsy, Sannino & Tuominen ‘03: splitting of transitions in effective models

But: quarkyonic phase confined. Chirally symmetric baryons?
Chirally symmetric baryons

B. Lee, ‘72; DeTar & Kunihiro ’89; Jido, Oka & Hosaka, hep-ph/0110005; Zschiesche et al nucl-th/0608044. Consider two baryon multiplets. One usual nucleon, other parity partner, transforming opposite under chiral transformations:

\[ \psi_{L,R} \rightarrow U_{L,R} \psi_{L,R} ; \; \chi_{L,R} \rightarrow U_{R,L} \chi_{L,R} \]

With two multiplets, can form chirally symmetric (parity even) mass term:

\[ \psi_L \chi_R - \psi_R \chi_L + \chi_R \psi_L - \chi_L \psi_R \]

Also: usual sigma field, \[ \Phi \rightarrow U_L \Phi U_R^\dagger \], couplings for linear sigma model:

\[ g_1 \psi_L \Phi \psi_R + g_2 \chi_R \Phi \chi_L \]

Generalized model at \[ \mu \neq 0 \]: D. Fernandez-Fraile & RDP ’07...
Anomalies?

‘t Hooft, ‘80: anomalies rule out massive, parity doubled baryons in vacuum:
   No massless modes to saturate anomaly condition

Itoyama & Mueller’83; RDP, Trueman & Tytgat ‘97:
At $T \neq 0$, $\mu \neq 0$, anomaly constraints far less restrictive (many more amplitudes)
   E.g.: anomaly unchanged at $T \neq 0$, $\mu \neq 0$, but Sutherland-Veltman theorem fails

Must do: show parity doubled baryons consistent with anomalies at $\mu \neq 0$.
   At $T \neq 0$, $\mu = 0$, no massless modes. Anomalies probably rule out model(s).
   But at $\mu \neq 0$, always have massless modes near the Fermi surface.

Casher ‘79: heuristically, confinement => chiral sym. breaking in vacuum
   Especially at large $N_c$, carries over to $T \neq 0$, $\mu = 0$.
   Does not apply at $\mu \neq 0$: baryons strongly interacting at large $N_c$.

Banks & Casher ’80: chiral sym. breaking from eigenvalue density at origin.
Splittorff & Verbaarschot ‘07: at $\mu \neq 0$, eigenvalues spread in complex plane.
   (Another) heuristic argument for chiral sym. restoration in quarkyonic phase.
Guess for phase diagram in QCD

*Pure* guesswork: deconfining & chiral transitions split apart at critical end-point?
Line for deconfining transition first order to the right of the critical end-point?
Critical end-point for deconfinement, or continues down to T=0?

Diagram: