# Generalities of the QCD phase diagram: $T \neq 0$, $\mu = 0$

1. Phase transitions in mean field theory, and not:
   four dimensions
   first order transitions from negative coupling
   fluctuation induced transitions

2. Chiral phase transition

3. Deconfining phase transition: (bare) Polyakov loop

4. Possible phase transitions:
   deconfinement versus chiral symmetry restoration:
   *many* possible scenarios!

5. Is the QCD coupling big at $T_c$?
   Perturbative computations at $T \neq 0$
   "Helsinki" resummation

6. 't Hooft basis for SU(N): "birdtrack" diagrams

1. Phase transitions in mean field theory, and not

four dimensions

first order transitions from negative coupling

fluctuation induced transitions

# Mean field theory: phase transitions at T = 0

Consider scalar field $\phi$ invariant under global Z(2) symmetry, $\phi \to -\phi$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\phi^4$$

In four dimensions, most general renormalizeable Lagrangian.
For stability at large $\phi$, $\lambda > 0$.
Only phase transition possible is $m^2 \to 0$

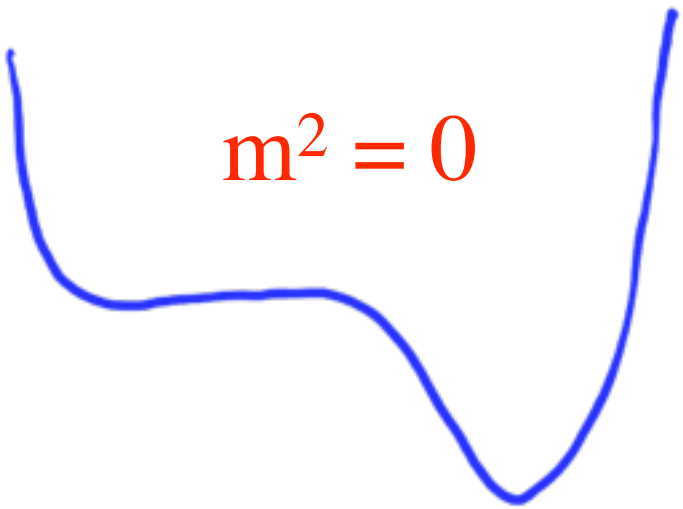
In four dim.'s, *only* a second order transition is possible.
Mean field theory works, up to logarithmic corrections:
$\lambda$ is dimensionless, "runs" log.'y in 4D.

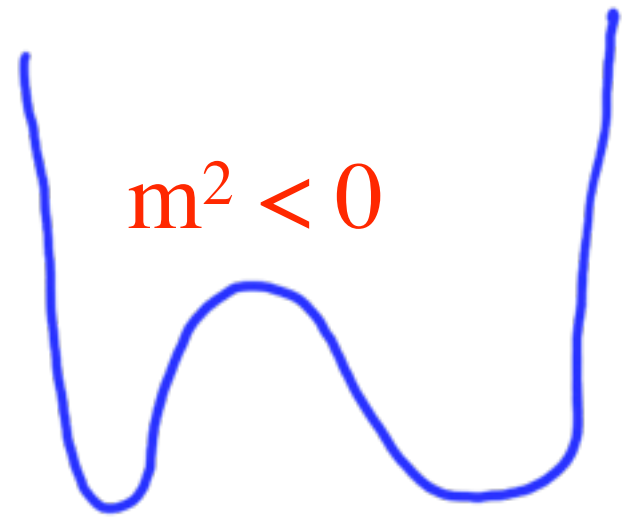
# First order phase transition at T = 0

Now drop the global Z(2) symmetry, so all powers of $\phi$ are allowed:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + h\,\phi + \frac{1}{2}\,m^2\,\phi^2 + \kappa\,\phi^3 + \lambda\,\phi^4$$

The linear term, ~ h, is like a background field, so $< \phi >$ is always $\neq 0$
Because of the cubic term, cannot "zero" the potential:




With no "flat" potential, always have barrier between degenerate, but distinct, vacua. Masses, etc, are different in two vacua, so:
cubic invariants generate *first* order transitions

# First order phase transitions at T ≠ 0

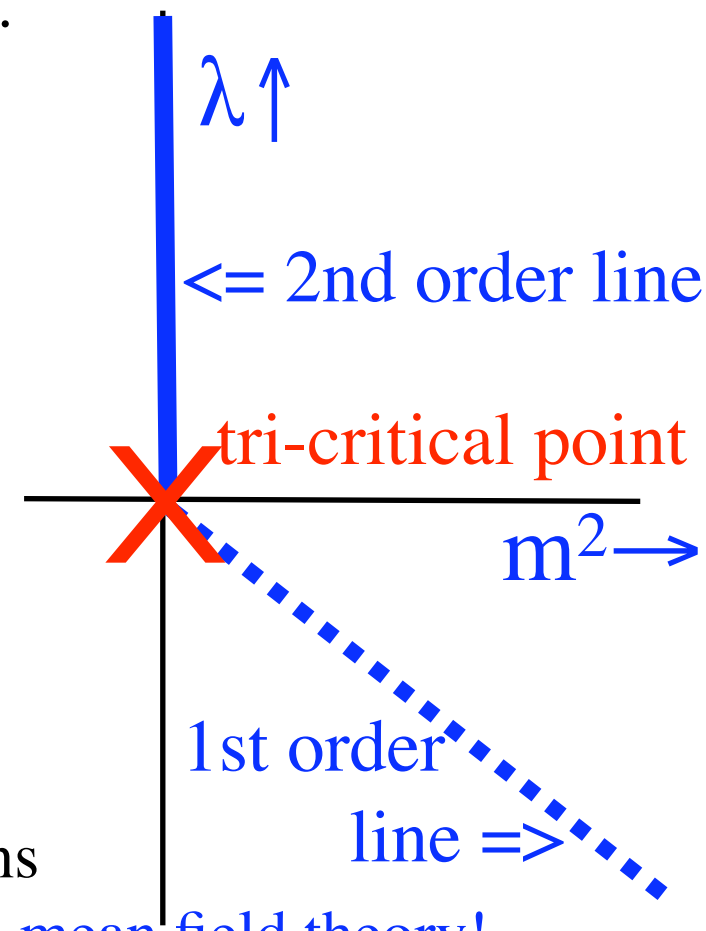
Imaginary time τ: 1 → 1/T. So theory 3-dimensional for distances >> 1/T.
Again, take global Z(2) symmetry, $\phi \to -\phi$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\, m^2\, \phi^2 + \lambda\, \phi^4 + \xi\, \phi^6$$

Renormalizability => $\phi^6$ : ξ dimensionless, λ ~ mass.
Stability => ξ > 0, so λ can have *either* sign.

First order transition from *negative* quartic coupling.

General phase diagram in plane of $m^2$ and λ:




$m^2 = 0, \lambda > 0$ : second order transitions
$m^2 > 0, \lambda > 0$ : first order transitions
$m^2 = \lambda = 0$ : "tri-"critical point.

Tri-critical point: mean field + logarithmic corrections
Critical point *very* difficult: power law corrections to mean field theory!

# Renormalization group in 4 - ε dimensions

Consider theory with Z(2) symmetry, interaction ~ $\lambda\,\phi^4$

Wilson-Fischer '72: in 4 - ε dimensions, λ has dimensions of $(\text{mass})^{\varepsilon}$.

Construct β-function for the "running" of the coupling:

$$\beta(\lambda) = \frac{\partial \lambda}{\partial \mu} = -\epsilon\,\lambda + \#\,\lambda^2 + \ldots$$

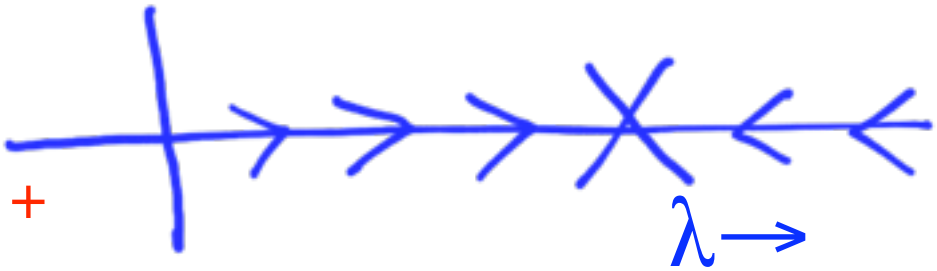
First term from mass dimension of λ.

Second term from one loop diagram,

and *positive*: λ → 0 in infrared limit.

*Why?* Euclidean action $\exp(-\lambda\,\phi^4)$, $(-)^2 = +$

For small ε, power law behavior of critical point calculable.



# Renormalization group: two couplings

Now consider a theory with O(4) x O(2) symmetry:
$\phi^i$ = complex valued field with four components.
There are two possible couplings, first O(8) invariant, second just O(4) x O(2):

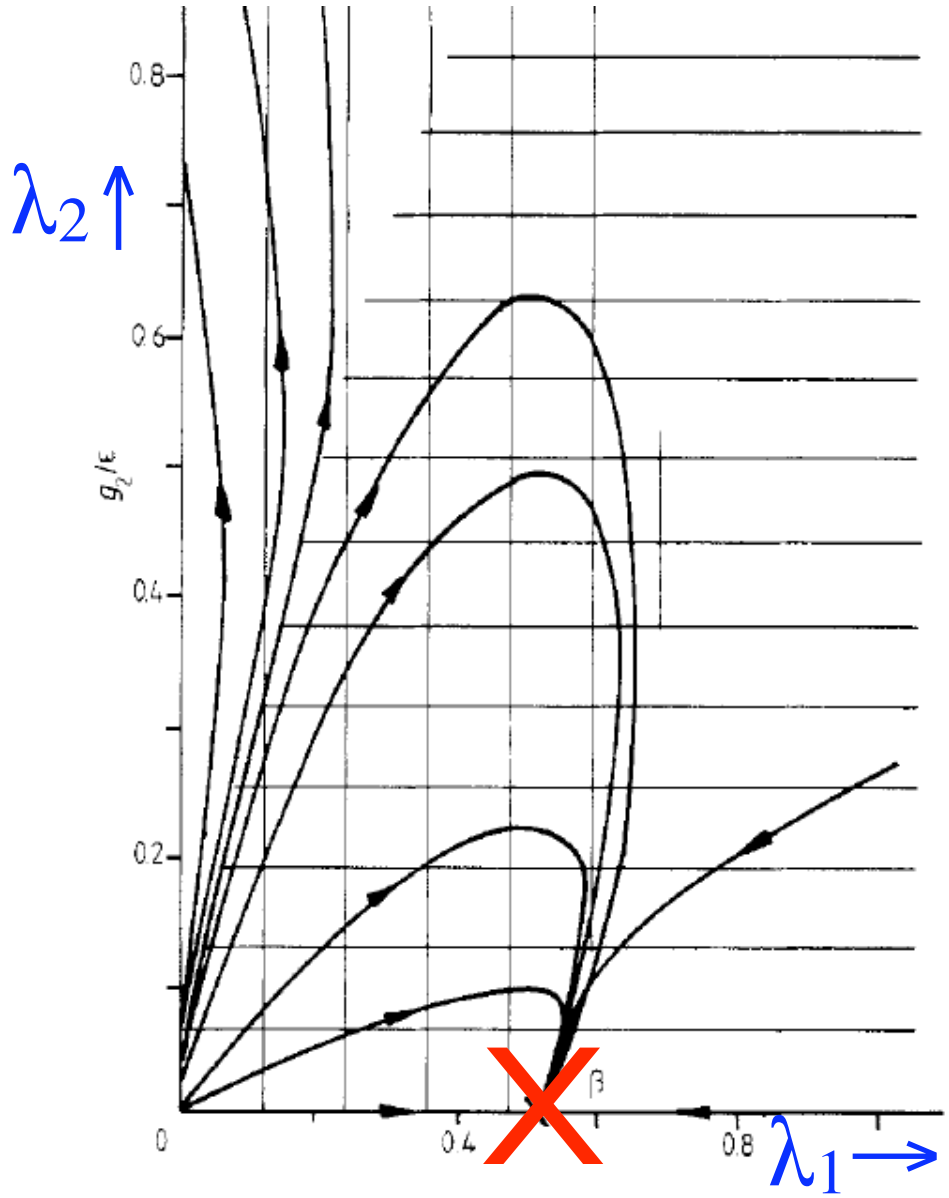
$$\mathcal{L} = \frac{1}{2}\,|\partial_\mu \vec{\phi}|^2 + \frac{1}{2}\,m^2\,|\vec{\phi}|^2 + \lambda_1\,(\vec{\phi}^* \cdot \vec{\phi})^2 + \lambda_2\,(\vec{\phi}^* \cdot \vec{\phi}^*)(\vec{\phi} \cdot \vec{\phi})$$

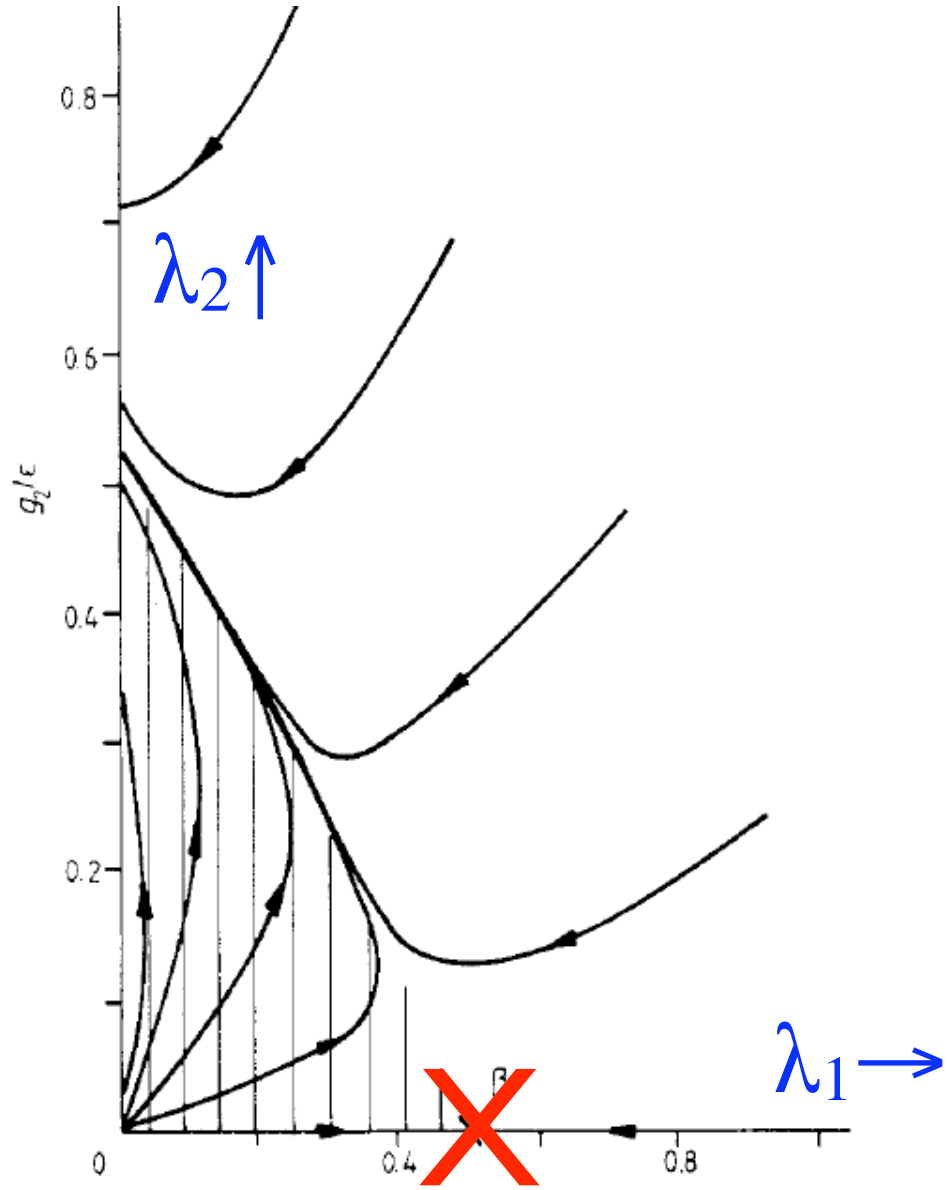
β-functions:

$$\beta(\lambda_1) = -\epsilon\,\lambda_1 \quad + 16\,\lambda_1^2 \quad + 14\,\lambda_1\,\lambda_2 + \;3\,\lambda_2^2$$
$$\beta(\lambda_2) = -\epsilon\,\lambda_2 \qquad\qquad\qquad + 12\,\lambda_1\,\lambda_2 + \;10\,\lambda_2^2$$

Yawn. All signs positive, as according to previous argument.
But now couplings run to *negative* values!
"Fluctuation induced" first order transition = Coleman-Weinberg phenomenon.

# Running couplings: 2nd order vs fluc. ind'd 1st order




O(-2) x O(-2): IR stable fixed point, 2nd order transition

O(4) x O(2): IR *un*stable fixed point, fluctuation induced first order

# 2. Chiral phase transition

# Chiral Symmetries of Massless Quarks

For quarks in QCD:

$$\begin{aligned}\mathcal{L}_{\text{quark}} &= \overline{\psi}^a \not{D}\, \psi^a + m\, \overline{\psi}^a \psi^a \\ &= \overline{\psi}^a_L \not{D}\, \psi^a_L + \overline{\psi}^a_R \not{D}\, \psi^a_R + m\, (\overline{\psi}^a_L \psi^a_R + \overline{\psi}^a_R \psi^a_L)\end{aligned}$$

where $\not{D} = \gamma^\mu(\partial_\mu - igA_\mu)$ , $\psi_{L,R} = (1 \pm \gamma_5)\psi/\sqrt{2}$ . Thus massless quarks are invariant under a global symmetry,

$$\psi^a_{L,R} \to U^{ab}_{L,R}\, \mathrm{e}^{i\theta\gamma_5} \psi^b_{L,R}$$

which is a classical symmetry of $G^{cl}_f = SU_L(N_f) \times SU_R(N_f) \times U_A(1)$

Because of quantum mechanical effects: the axial anomaly, and instantons, the axial U(1) symmetry is only Z($N_f$):

$$\mathcal{L}_{\text{inst.}} = \epsilon^{a_1 a_2 \ldots a_{N_f}}\, \overline{\psi}^{a_1}_L \psi^{a_1}_R \ldots \overline{\psi}^{a_{N_f}}_L \psi^{a_{N_f}}_R$$

Thus the quantum symmetry is $G^{qu}_f = SU_L(N_f) \times SU_R(N_f) \times Z(N_f)$

This becomes $G_f = SU_V(N_f)$ when all quark masses are equal.

# Evaporation of Chiral Symmetry

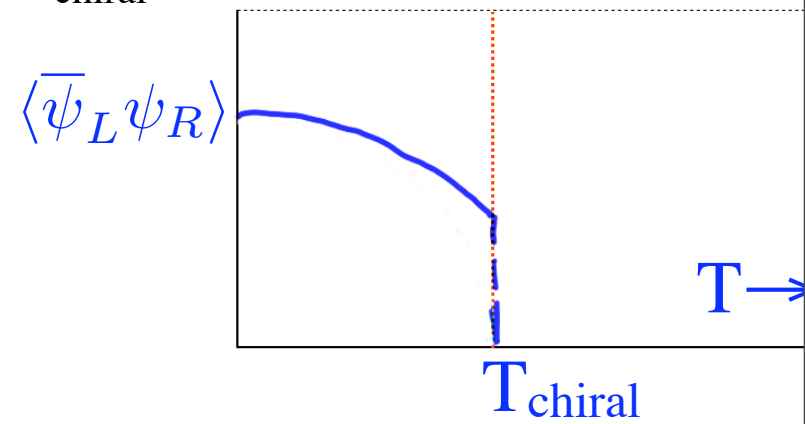
Assume chiral symmetry broken at T=0, restored at $T_{chiral}$ :

Construct $$\Phi^{ab} = \overline{\psi}^a_L \; \psi^b_R$$

where

$$\Phi \to e^{i\alpha} \; U^\dagger_L \; \Phi \; U_R$$

The invariant (interaction) Lagrangian is




$$\mathcal{L}^{int} = \lambda_1 (\mathrm{tr}\, \Phi^\dagger \Phi)^2 + \lambda_2 \, \mathrm{tr}(\Phi^\dagger \Phi)^2 + c_{inst} \;\; \det \Phi$$

$\det \Phi \sim \Phi^{N_f}$ For three flavors, IF $c_{inst}$ is nonzero at $T_{chiral}$, *first* order trans.!

What if instantons evaporate before $T_{chiral}$ ? $c_{inst}(T_{chiral}) << c_{inst}(0)$?

Calculation in 4 - ε dim.'s shows: RDP & Stein, '81; RDP & Wilczek '84

fluctuation induced *first* order transition for $N_f = \sqrt{2} + O(\varepsilon)$.

QCD: for two flavors, chiral transition either first or second order.

Three or more flavors: first order. IF one can ignore deconfinement!

N.B.: quark (~ pion) masses weaken any 1st order trans.

# Effective theory for two flavors

For two flavors, can rewrite $SU_L(2) \times SU_R(2) \sim O(4)$:

$$\Phi = \begin{pmatrix} \phi_0 + \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & \phi_0 - \phi_3 \end{pmatrix}$$

Similarly, axial $U(1) \sim O(2)$.

Effective theory, including instanton interaction, is

$$\mathcal{L} = |\partial_\mu \phi|^2 + \frac{1}{2} m^2 |\vec{\phi}|^2 + c_{inst}(\vec{\phi^*}^2 + \vec{\phi}^2)$$
$$+\lambda_1 (\vec{\phi}^* \cdot \vec{\phi})^2 + \lambda_2 (\vec{\phi}^* \cdot \vec{\phi}^*)(\vec{\phi} \cdot \vec{\phi})$$

For real $\phi$, chiral symmetry breaking by $m^2 < 0$: massive $\sigma$ plus massless $\pi$'s.

With complex $\phi$, when $c_{inst}$ nonzero, also have massive $\eta$ plus three massive f's.

# 3. Deconfining phase transition

# Hidden Z(N) symmetry in SU(N)

Start with SU(N) gauge theory *without* quarks.
Work in imaginary time, τ: 0 -> 1/T. Bosons periodic, fermions anti-periodic.
't Hooft: "hidden" global symmetry: consider gauge transf.

$$\Omega(\tau) = \mathrm{e}^{2\pi i \tau T\, t^{NN}} \quad , \quad t^{NN} = \frac{1}{N} \begin{pmatrix} \mathbf{1}_{N-1} & 0 \\ 0 & -(N-1) \end{pmatrix}$$

$t^{NN}$: $\sigma^3$ for two colors, $\lambda^8$ for three colors.
Independent of x, so *global*. Further,
at τ =1/T pure phase, proportional to unit matrix!

$$\Omega(1/T) = \mathrm{e}^{2\pi i/N}\, \mathbf{1}_N$$

Under a gauge transformation,
this Ω ok for gluons, not for quarks:

$$A_\mu \to \frac{1}{-ig}\, \Omega^\dagger D_\mu\, \Omega \quad ; \quad \psi \to \Omega\, \psi$$

The Z(N) is related to the center of SU(N) (commute with all group elements).
Symmetry *non*-local in τ. How to probe it?

Z(N) symmetry *could* be *completely* washed out by quarks. Dynamical question.

# Z(N) symmetry and Polyakov loops

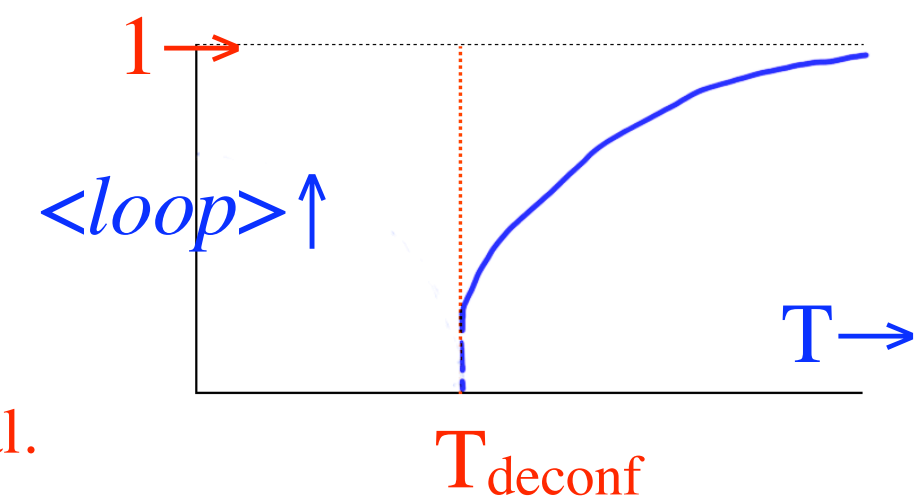
Wilson loop that wraps all the way around in τ = Polyakov(-Susskind) loop:

$$\ell(x) = \frac{1}{N}\,\mathrm{tr}\,\mathcal{P}\exp(ig\int_0^{1/T} A_0(x,\tau)\,d\tau)$$

*Not* invariant under all gauge transf.'s: $\ell \longrightarrow \mathrm{e}^{2\pi i/N}\,\ell$

Use Polyakov loops to test response to Z(N) charge.
Loop like propagator of infinitely massive test quark.

Confinement => <*loop*> = 0 at low temperature.
Deconfinement => <*loop*> ≠ 0 at high temp.
above $T_{deconf}$ . *Opposite* to chiral symmetry.




Classify transitions by change in <*loop*>.
Quarks act like background Z(N) field.
*Assume* overall normalization of loop physical.

# Z(N) symmetry and deconfinement

Classification of possible deconfining phase transition:
construct Z(N) invariant theory,

$$\ell \to e^{2\pi i/N}\,\ell$$

$$\mathcal{L} = |\partial_\mu \ell|^2 + m^2|\ell|^2 + \lambda(|\ell|^2)^2 + \kappa(\ell^N + (\ell^*)^N)$$

Blue terms inv. under U(1); red terms, under Z(N).

For N=2: symmetry Z(2) = Ising. Can be second order.

N=3 (QCD!): cubic invariant, so *always* first order.

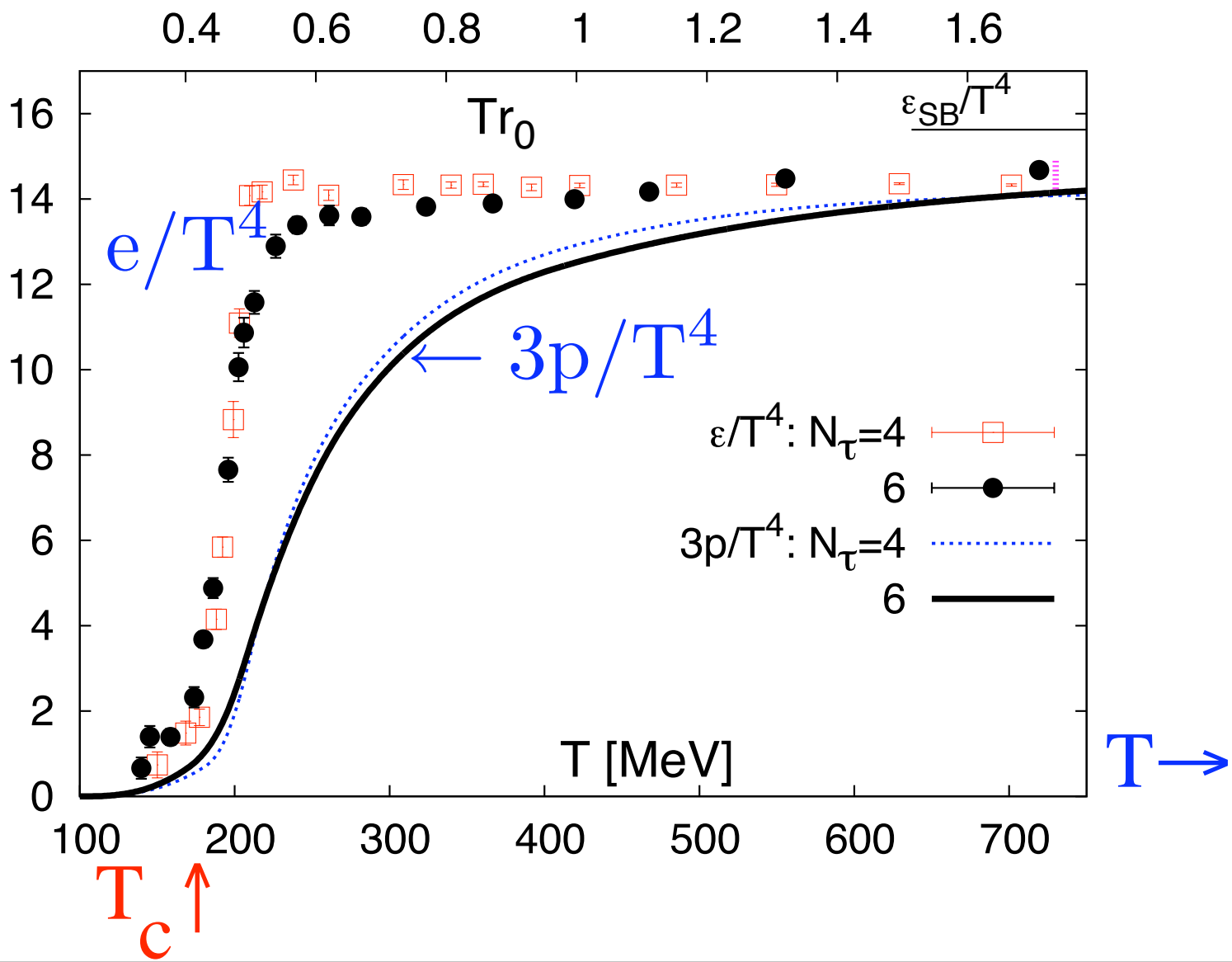
N ≥ 4: last term marginal or irrelevant: can be either first or second order.

Svetitsky & Yaffe, '82

Lattice: deconfining transition *second* order for N=2,
*first* order for N ≥ 3.

# Lattice: SU(3) glue, with quarks

Cheng et al, 0719.0354: e = energy density, p = pressure. $N_t$ = # time steps: 4, 6.
Three flavors of dynamical quarks: *close* to QCD.
Crossover, no true phase transition. (Ideal gas: e = 3p)

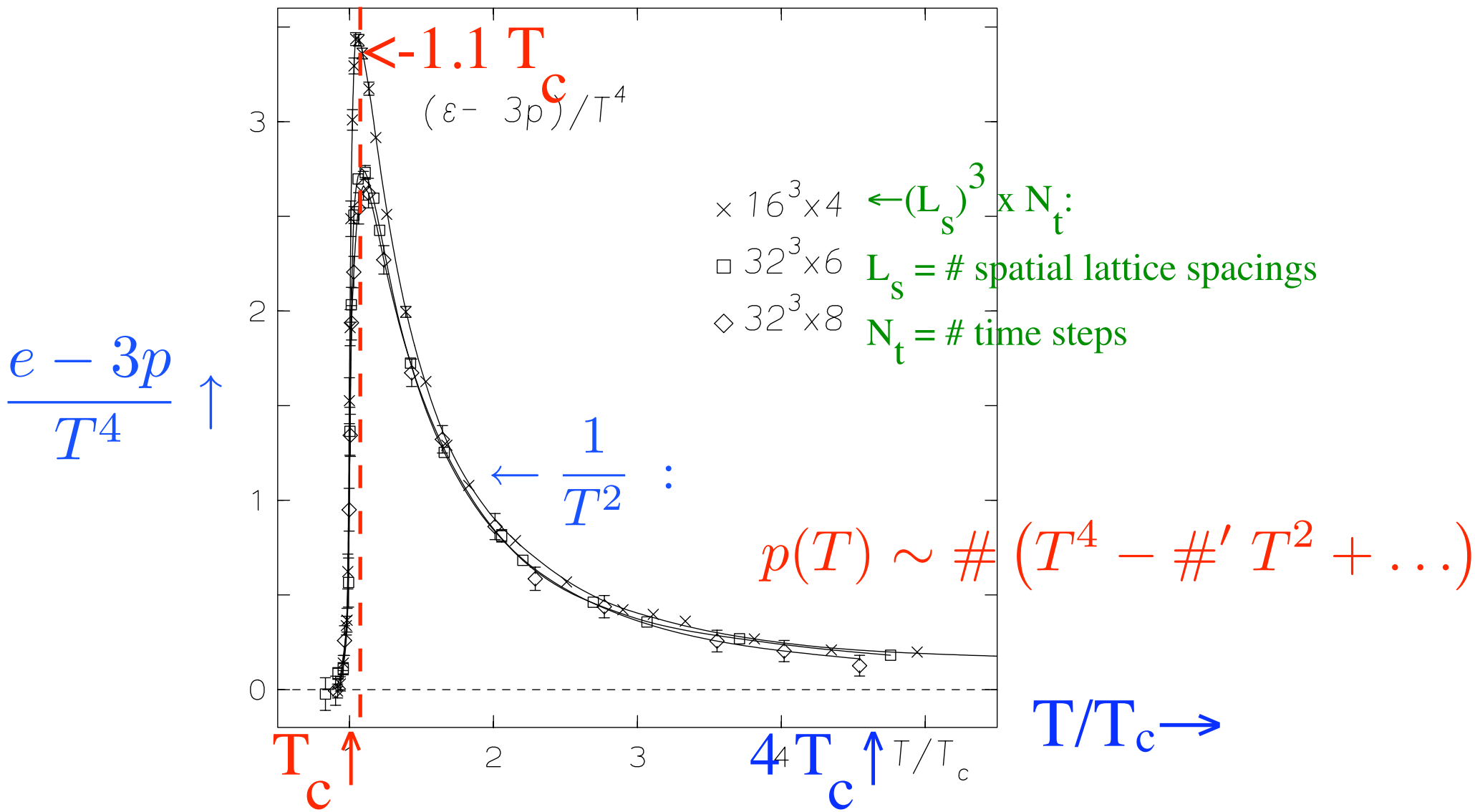


# Lattice: SU(3) glue, no quarks: weakly 1st order

More sensitive than pressure: $(e-3p)/T^4$, e = energy density, p = pressure

Bielefeld, hep-lat/9602007. $N_t$ = # time steps: 6, 8 near continuum limit?

Pressure: sum of ideal gas, $T^4$, plus $T^2$ , *then* "MIT bag constant", $T^0$.



# Lattice, no quarks: SU(3) close to SU(∞)?

Bringoltz & Teper, hep-lat/0506034 & 0508021:

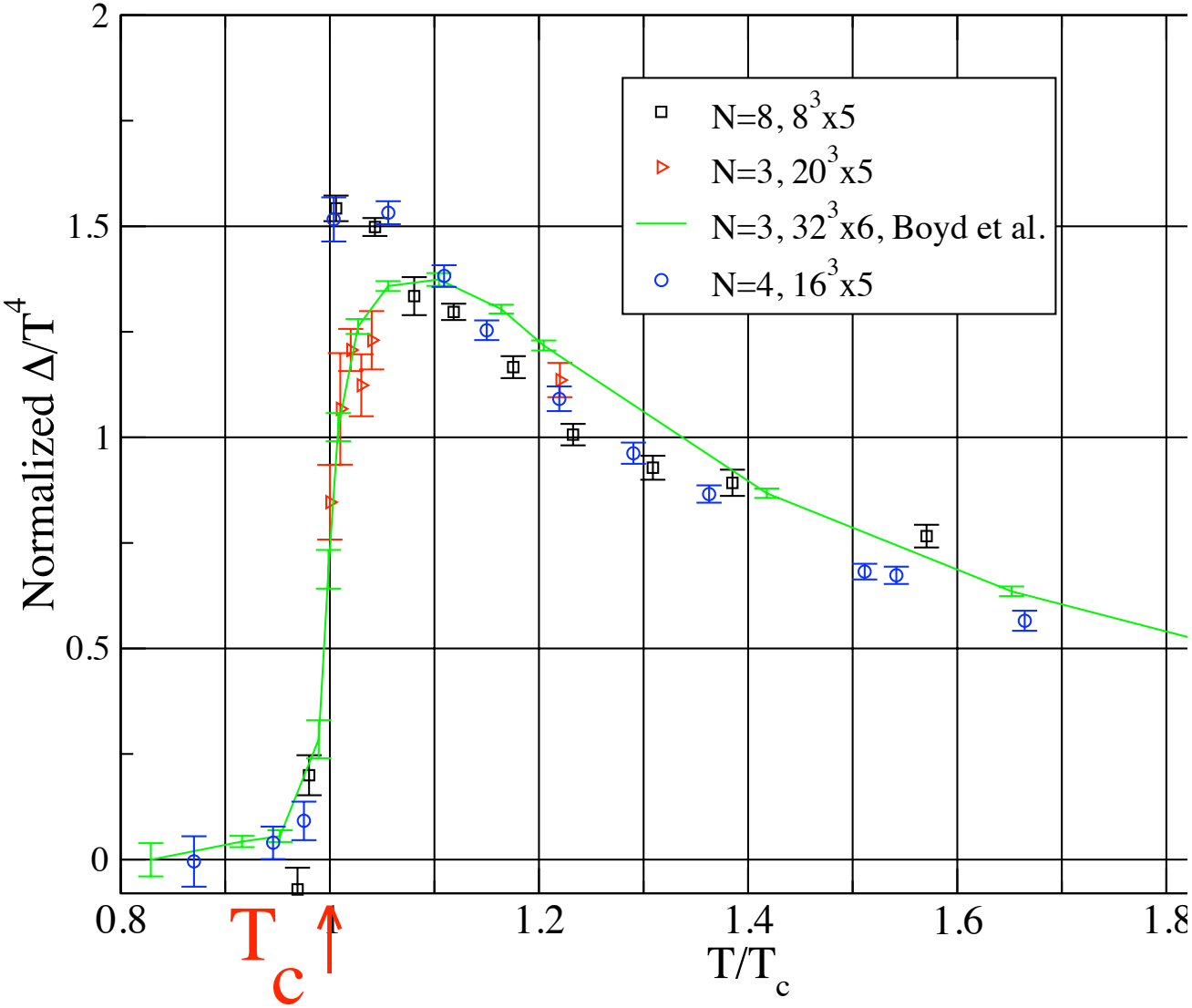
SU(N), no quarks, N= 3, 4, 6, 8, 10, 12.

Deconfining transition first order, latent heat ~ $N^2$.

Hagedorn temperature $T_H \sim 1.116(9)\ T_c$ for $N = \infty$

$$\frac{e-3p}{N^2\ T^4} \sim \text{const.}$$

$$\frac{e-3p}{N^2\ T^4} \uparrow$$




$T_c \uparrow$

$T/T_c \rightarrow$

## 4. Possible phase transitions:

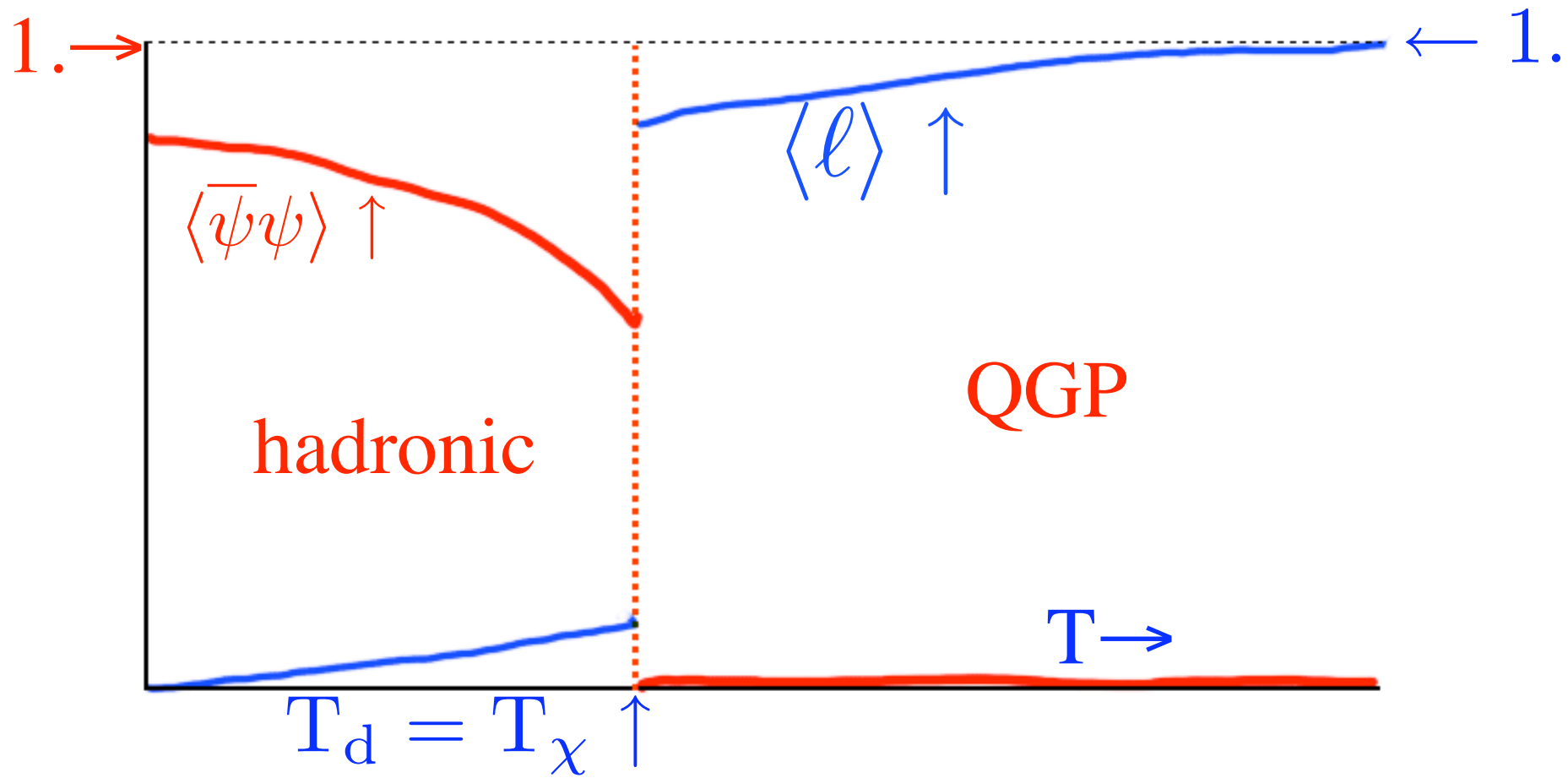
deconfinement versus
chiral symmetry restoration:
*many* possible scenarios!

# One transition

Deconfining and chiral transitions coincide:

go from confined phase with χSB'g to chirally symmetric Quark-Gluon Plasma.

Natural if transition strongly first order. Otherwise, *not* obvious.




Lattice: case in all simulations to date, for quarks in fundamental representation.

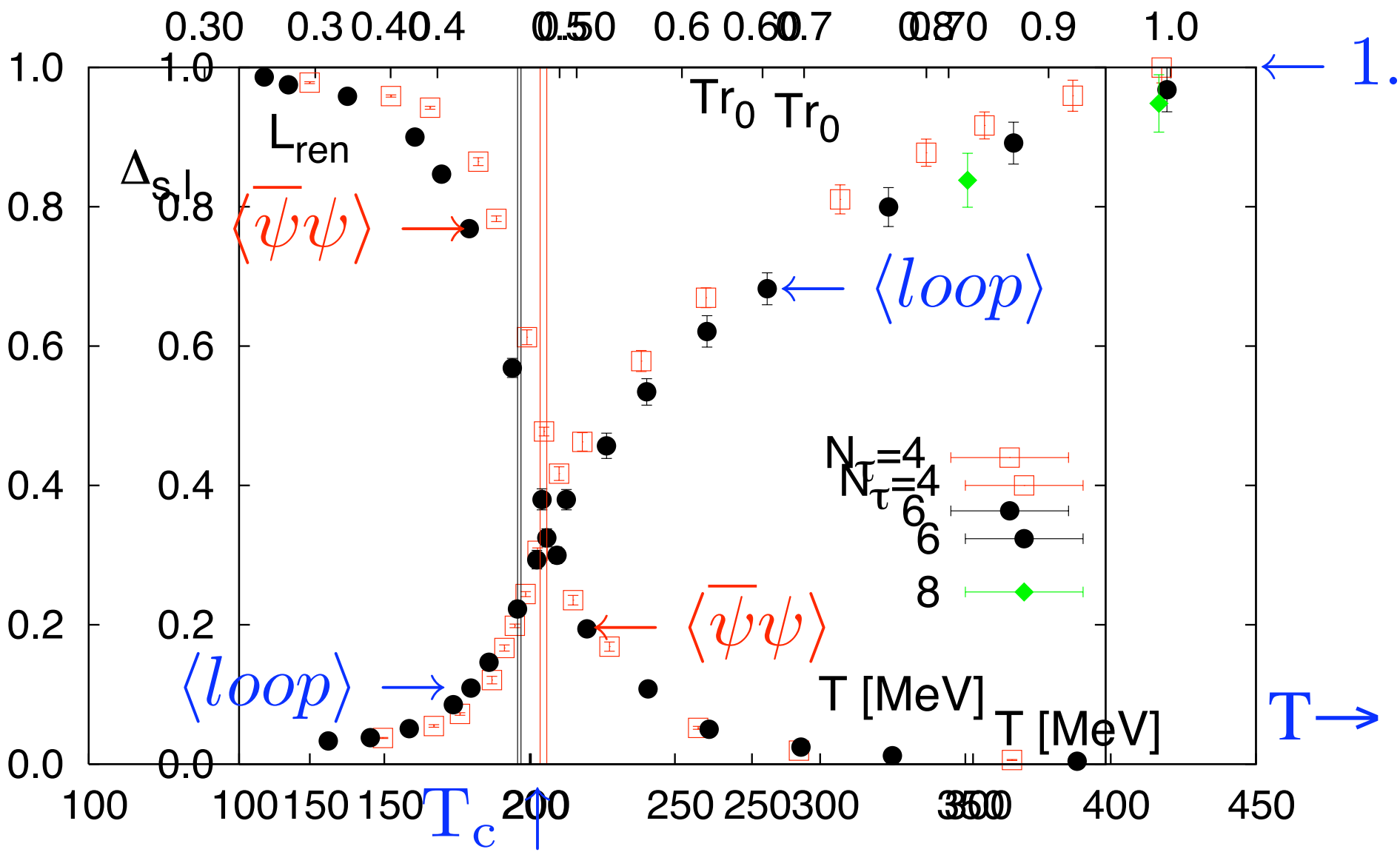
But: perhaps *not* generic?

# Lattice, ~ QCD: one transition

Cheng et al, 0719.0354: chiral order parameter, (renormalized) < *loop* >

No true phase transition, only crossover. *But:*

Within errors, two transitions ~ coincide ($T_c$ from peak in susceptibilities)

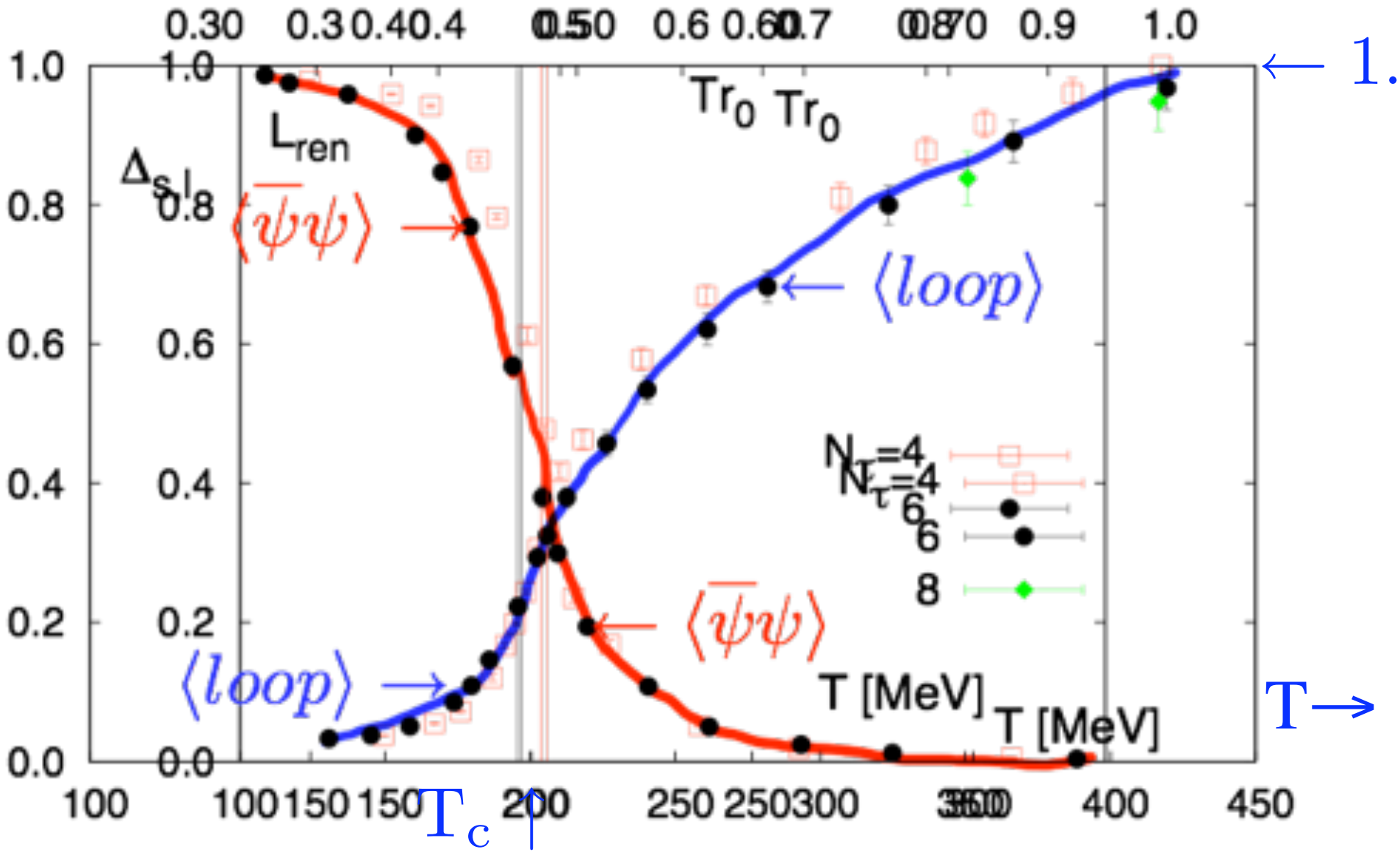


# Lattice, ~ QCD: one transition

Cheng et al, 0719.0354: (renormalized) Polykaov loop, chiral order parameter

No true phase transition, only crossover. *But:*

Within errors, two transitions ~ coincide ($T_c$ from peak in susceptibilities)

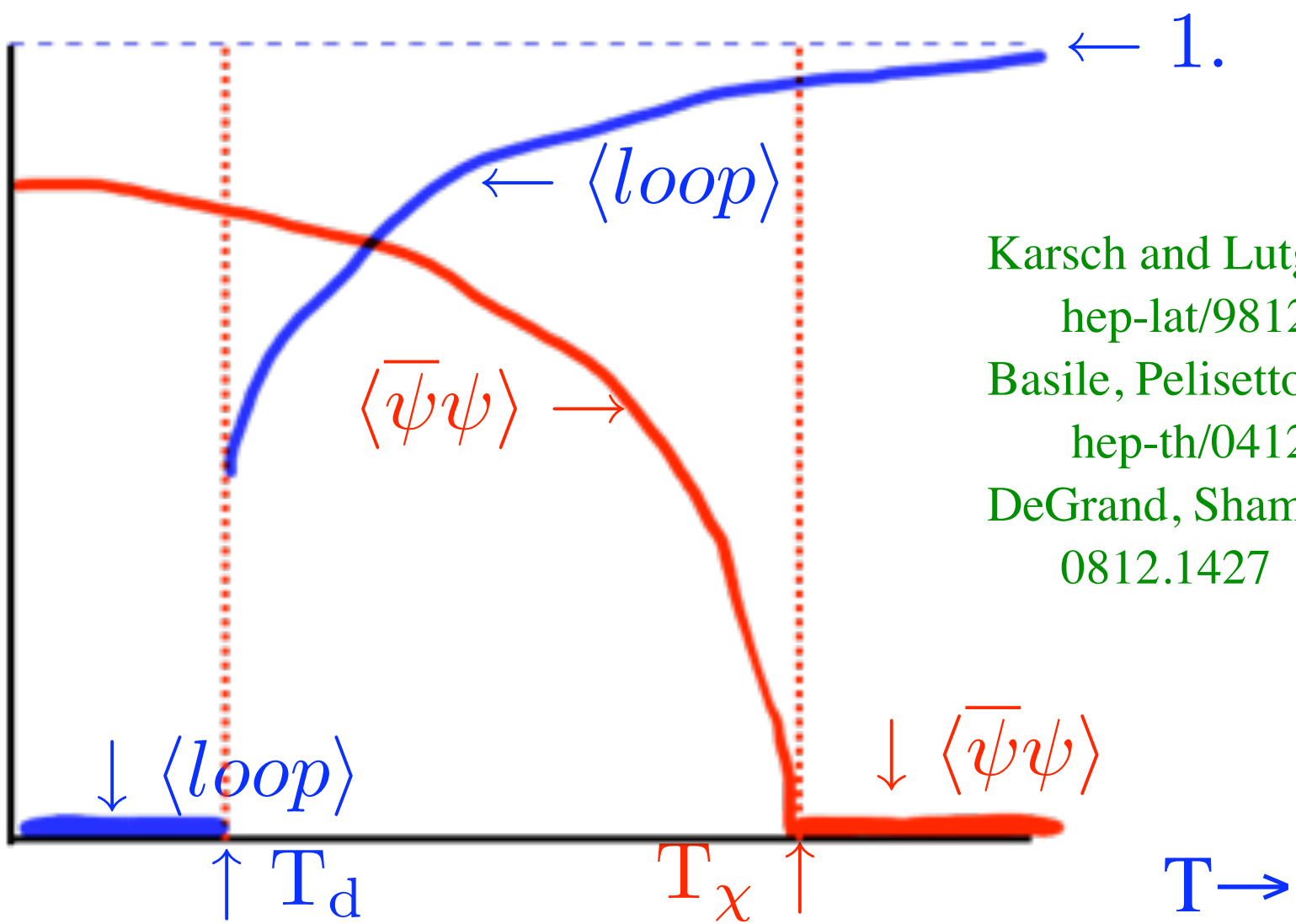


# Adjoint quarks: two transitions, $T_\chi > T_d$

Color adjoint "quarks": $\chi$ sym of 4 Majorana flavors SU(4)->SO(4); O(3) fixed pt

*Exact* Z(3) *and* chiral sym.'s: 1st order trans. at $T_d$, 2nd order chiral at $T_\chi$.

But $T_\chi \sim 8\, T_d$! Due to larger Casimir, and so coupling, of adjoint quarks




Karsch and Lutgemeir, hep-lat/9812023

Basile, Pelisetto, & Vicari, hep-th/0412026

DeGrand, Shamir & Svetitsky, 0812.1427

# Many flavors: one chiral transition?

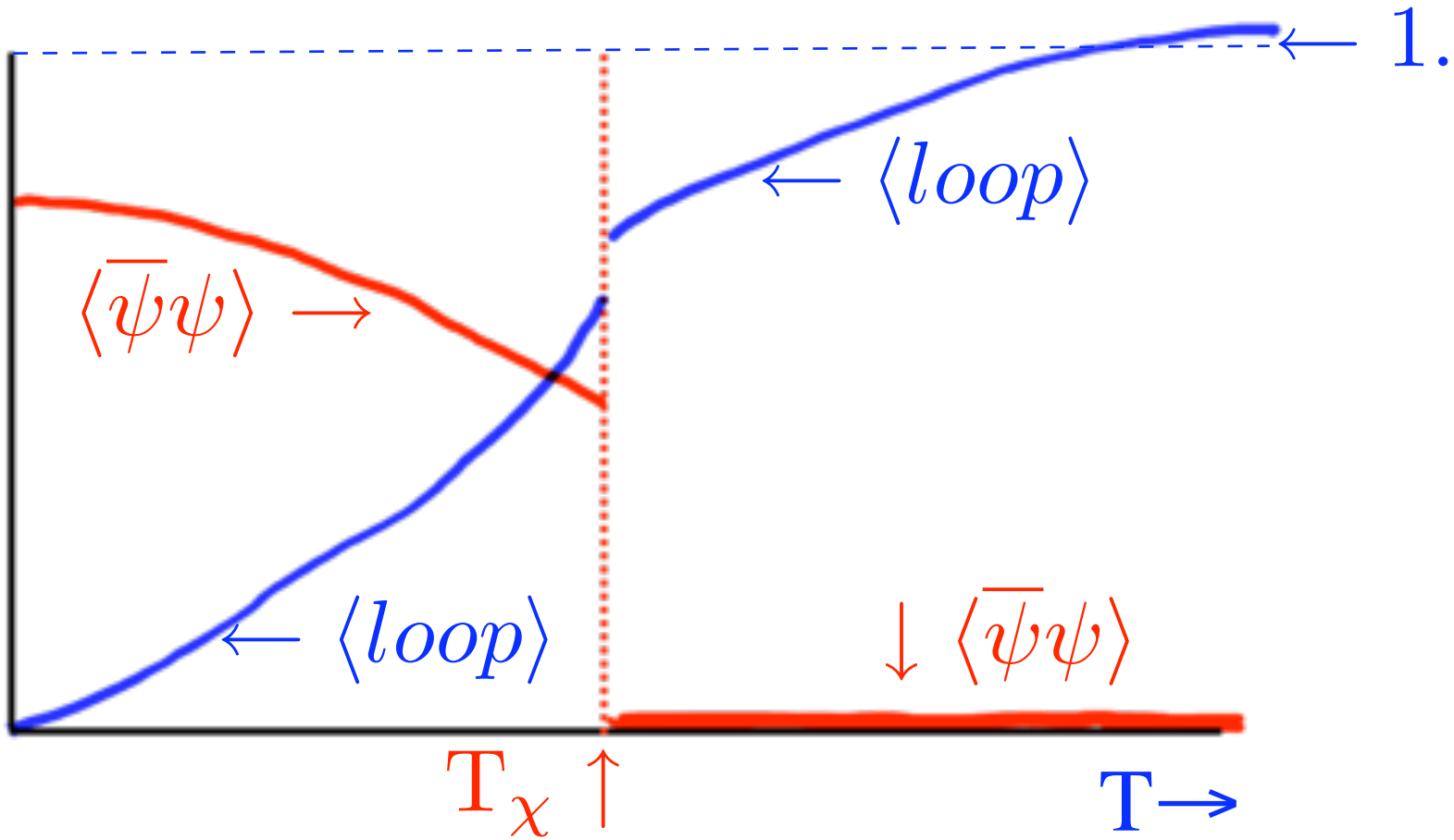
Consider SU(3) color with *many* (light) quarks: *strong* background field for Z(3)

< *loop* > increases from zero, always "large".

At *best* chiral transition, of first order. Drags change in < *loop* > along with it.

*Not* dominated by deconfinement, but by chiral sym.

Appelquist, Fleming, Neil 0901.3766: eight flavors not conformally invariant



# "Quarkyonic" matter: $T_d > T_\chi$

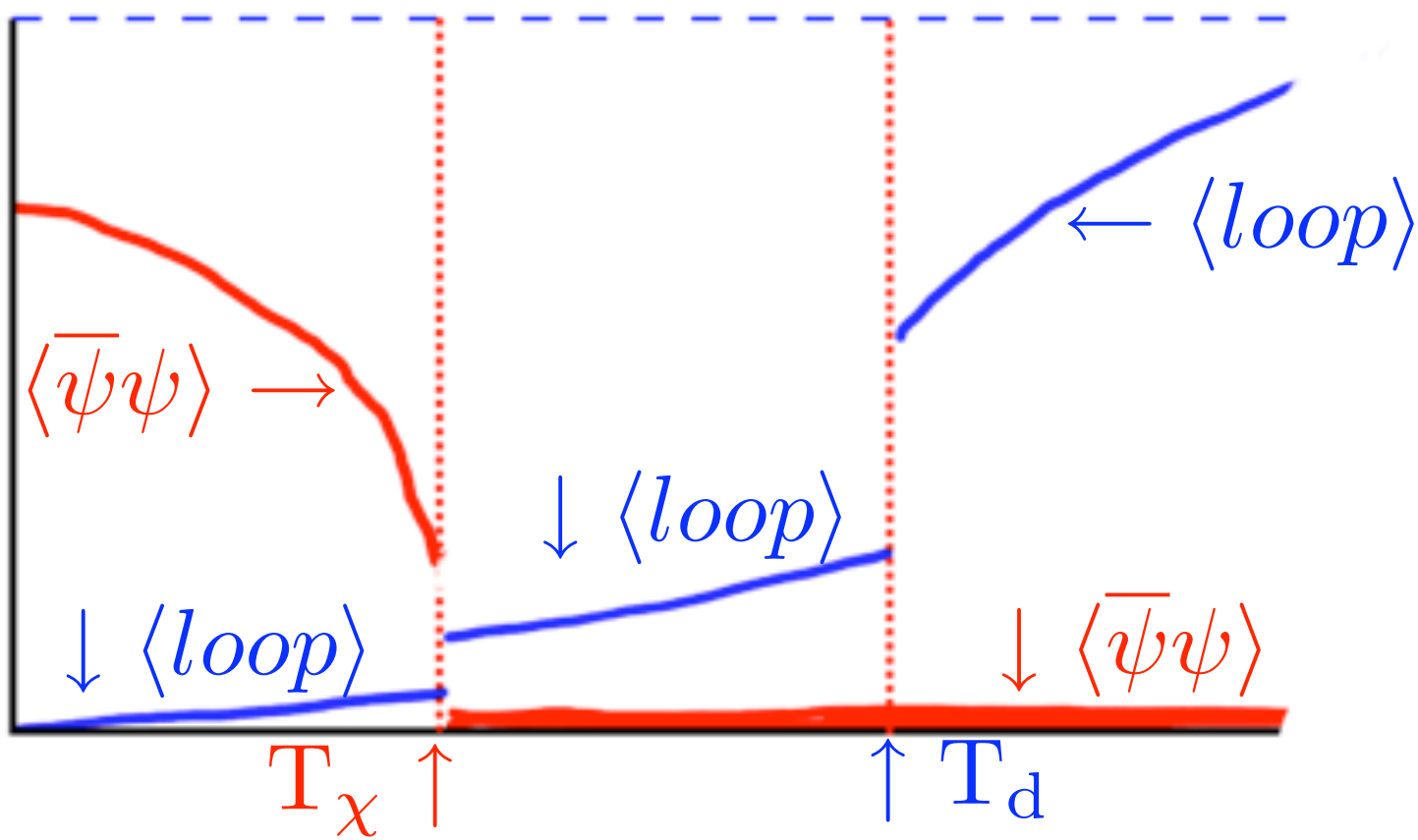
Consider SU(N) color for $N \to \infty$, $N_f$ fixed, at nonzero density.

Possible to have a chirally symmetric, but confined phase: "quarkyonic"

Manifest in Skyrme crystal. Below for 3 colors?

McLerran & RDP, 0706.2191; McLerran, RDP, & Hidaka, 0803.0279

N.B. Anomaly constraints important, RDP, Tytgat, & Trueman hep-ph/9702362



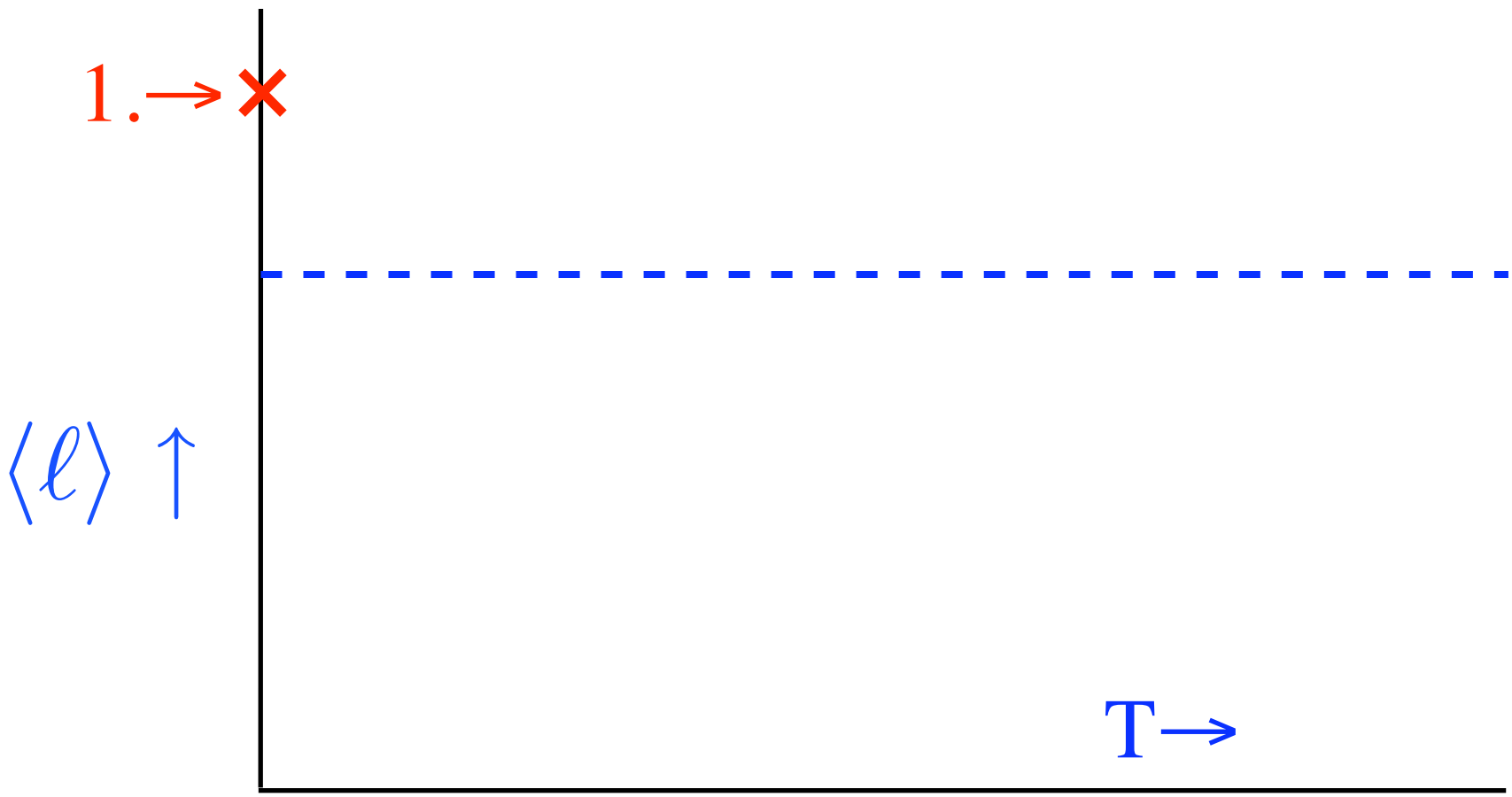
# $\mathcal{N}$ = 4 SU(∞)

AdS/CFT: Can define <loop> = 1 at T = 0 (Polyakov-Maldacena, + scalars)

At T ≠ 0 , <*loop*> = constant (like pressure/$T^4$) : value, vs $g^2$ N?

$\mathcal{N}$ = 4 SU(∞) is *always* deconfined.

Hartnoll & Kumar, hep-th/0610103: also consider <*scalar*>'s

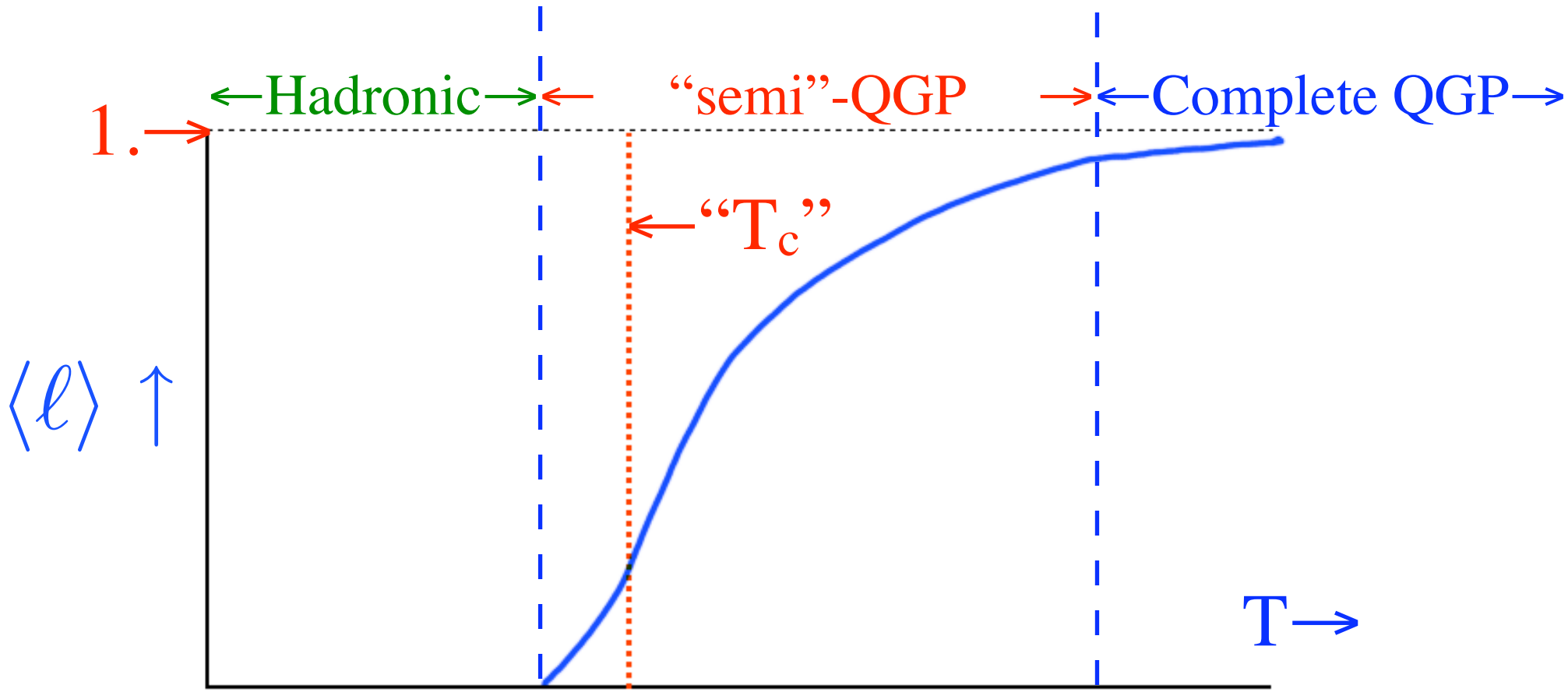


# Punchline: QCD & the "semi"-QGP

*Three* regimes: hadronic, <*loop*> ~ 0.

"Semi"-QGP: <*loop*> ≠ 0, < one. With quarks, both *below* & above $T_c$.
Hadrons *above* $T_c$, quarks & gluons *below* above $T_c$

"Complete" QGP: <*loop*> near one. Usual "perturbative" regime (resummed)



# 5. Is the QCD coupling big at $T_c$?

Perturbative computations at $T \neq 0$

"Helsinki" resummation

# QCD pressure in perturbation theory, T ≠ 0

At high T, computing straightforwardly in perturbation theory in $g^2$:

$$p(T) = p_{ideal}\left(1 + c_2\, g^2 + c_3\, (g^2)^{3/2} + c_4\, (g^2)^2 \log(d/g^2) + \ldots\right)$$

Why (non-analytical) terms, $\sim \sqrt{g^2}$? At high T, QGP = plasma of colored electric charges. Screen static electric fields, through a Debye "mass":

$$m^2_{Debye} = \Pi_{00}(p_0 = \vec{p} = 0) = (N_c + N_f/2)\; g^2\, T^2/3$$

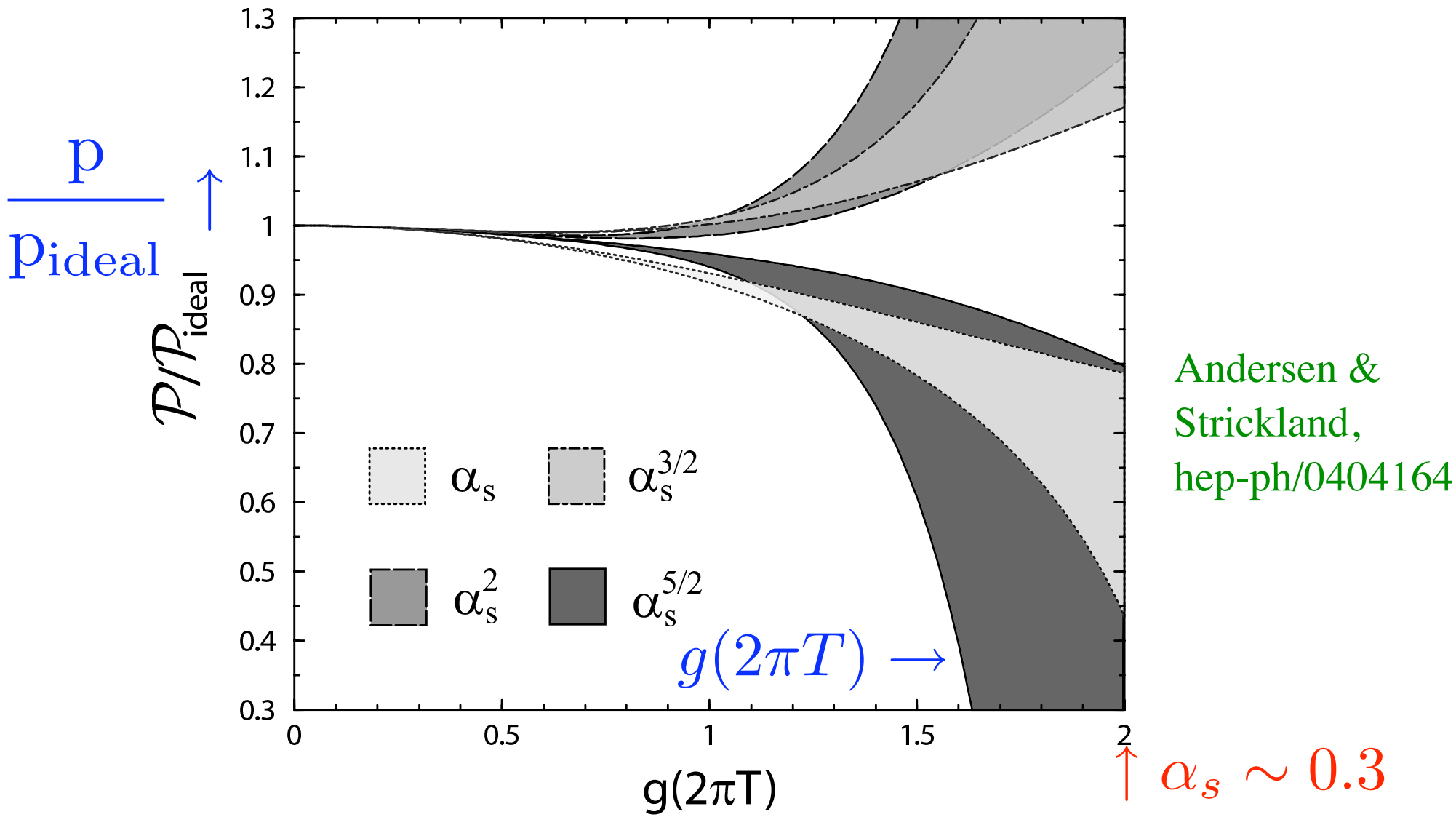
Consider pressure from the Debye modes (static $A_0$):

$$p_{\rm Debye} \sim T \int d^3\vec{p}\, {\rm tr} \log\left(\vec{p}^{\,2} + m_D^2\right) \sim T\, m_D^3 \sim (g^2)^{3/2}\, T^4$$

At $(g^2)^2$, $\log(T/m_D) \sim \log(1/g^2)$. Continue to $\sim (g^2)^3$, when static magnetic modes contribute, $m_{mag} \sim g^2\, T$. Their contribution cannot be determined perturbatively, but can be determined through Lattice simulations.

# Naive perturbation theory fails badly

At T=0, natural expansion parameter is $\alpha_s = g^2/4\pi$. Works well down to $\alpha_s \sim 0.3$, about 1 GeV. Because of the non-analytic terms, fails badly for the pressure even at such (moderate) values:




Andersen & Strickland, hep-ph/0404164

# "Helsinki" program of resummation

Match original theory in 4D, to effective theory in 3D, for r > 1/T

$$\mathcal{L}^{eff} = \frac{1}{2}\,\mathrm{tr}\,G_{ij}^2 \;+\; \mathrm{tr}\,|D_i A_0|^2 + \; m_D^2\,\mathrm{tr}\,A_0^2 \;+\; \kappa\,\mathrm{tr}\,A_0^4$$

$m_{Debye}^2 \sim g^2 T^2$ , $\kappa \sim g^4$, series in $g^2$ .

First step in three: then resum $m_{Debye}$ , then $m_{magnetic}$

Optimal resummation of perturbation theory for small $A_0$

Braaten & Nieto, hep-ph/9501375: Euclidean "energies" $p_0 \sim 2\pi T$.

So natural scale for $\alpha_s = g^2/4\pi$ to "run" is $\alpha_s(2\pi T)$. *Big* factor!

$T_c \sim 175$ MeV: $\alpha_s^{eff}(T_c) \sim$ *non*-perturbative.

$\alpha_s^{eff}(2\pi\, T_c) \sim \alpha_s^{eff}(1.1\text{ GeV}) \sim$ *moderate* coupling.

# Perhaps $\alpha_s$ is *not* so big at $T_c$

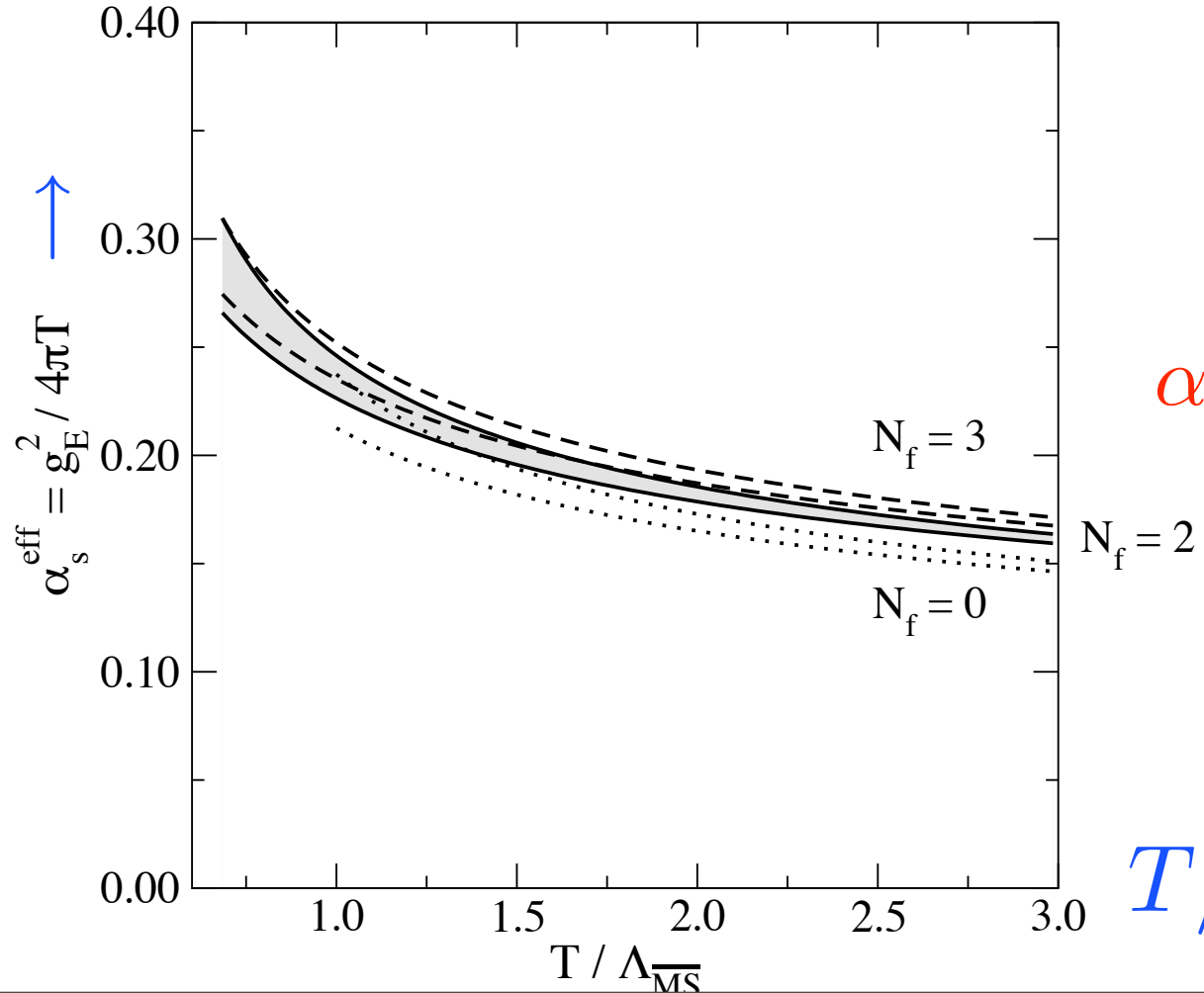
Laine & Schröder, hep-ph/0503061 & 0603048

From two loop calculation, matching original to effective theory:
find factor of 2 π => 7. for pure gauge, 2 π => 9. for $N_f = 3$.

$T_c \sim \Lambda_{MS} \sim 200$ MeV. So $\alpha_s^{eff}(T) \sim \alpha_s^{eff}(2\pi T) \sim 0.3$ at $T_c$: *not* so big

Grey band uncertainty from changing scale by factor 2.

$\alpha_s^{\rm eff}(T) \uparrow$

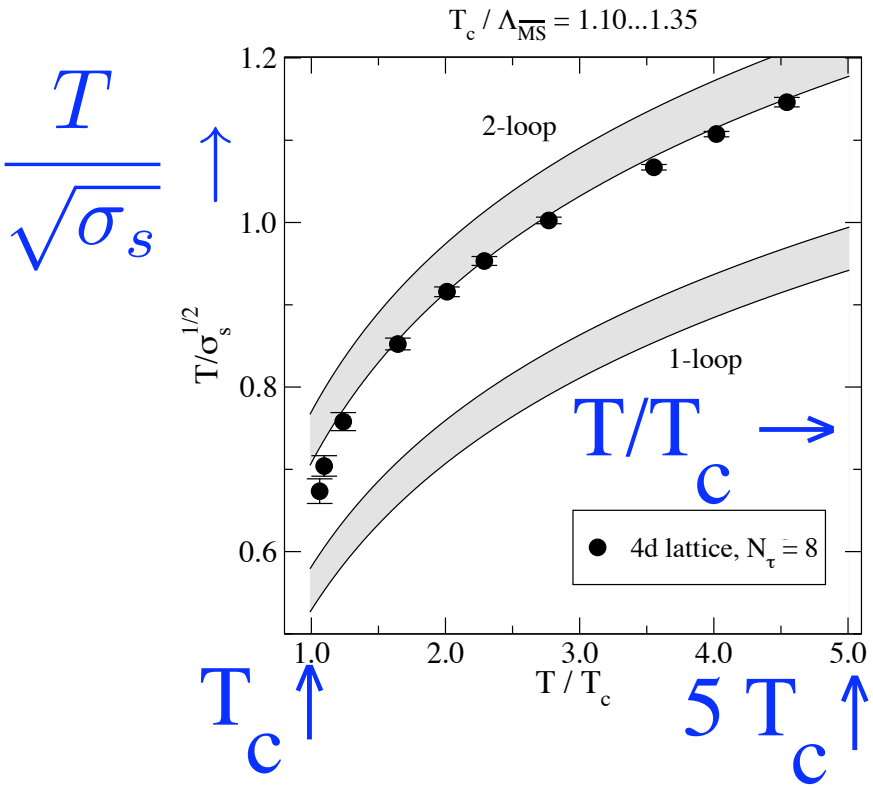



$\alpha_s^{\rm eff}(T_c) \approx 0.3$

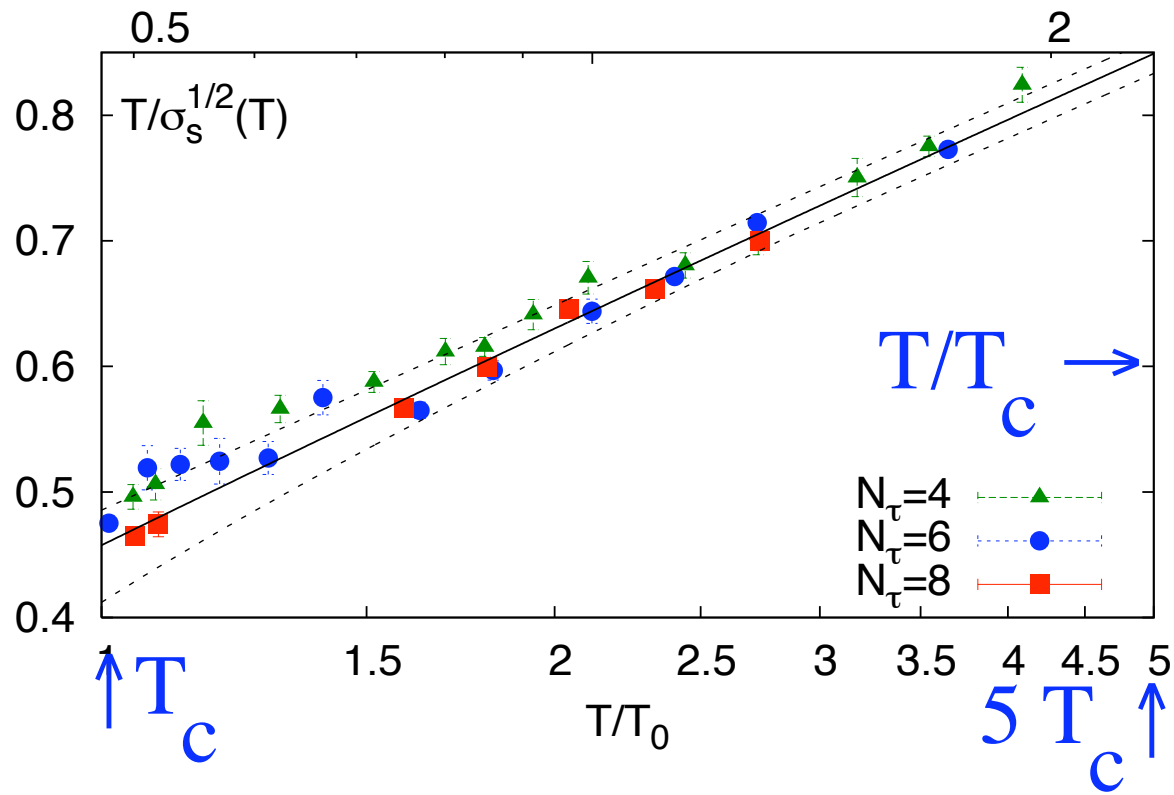
$T/\Lambda_{\overline{\rm MS}} \rightarrow$

# Effective 3D theory works for spatial string tension

Laine & Schröder, hep-ph/0503061




M. Cheng et al, 0806.3264



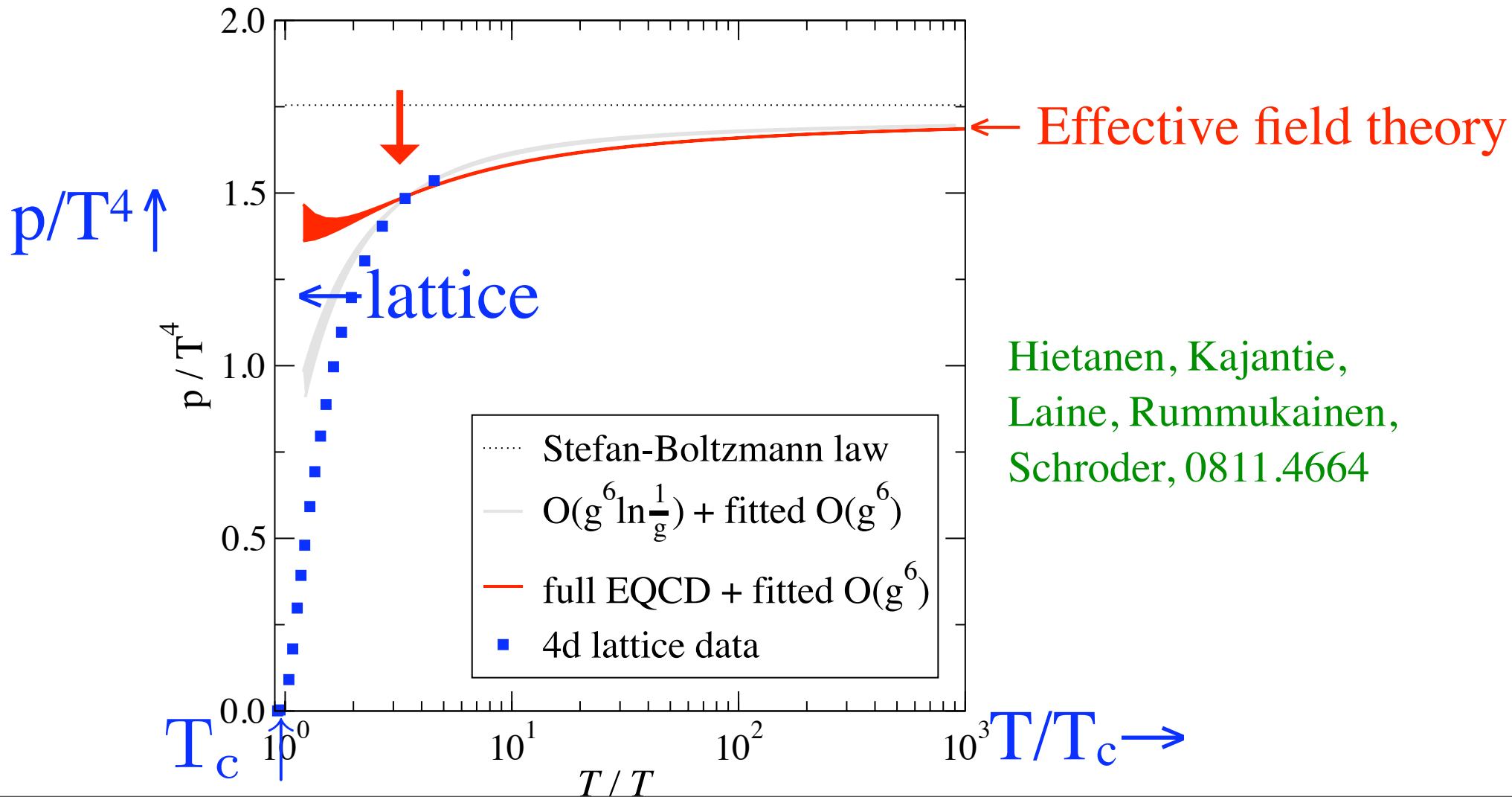

Use $g_{3D}^2(T) = 4\pi\alpha_s(T)\,T$, take $\sigma_{spatial}/g_{3D}^2$ from 3D lattice sim.'s

Compare with direct calc. of $\sigma_{spatial}$ in 4D.

Works down to $T_c$! Need 2-loop calc. Magnetic sector OK to $T_c$

# Resummation fails for pressure below 3 $T_c$

Resummation: includes perturbative computation to *four* loop order, ~ $(g^2)^3$.
Match non-perturbative contribution from static magnetic sector (lattice).
Even so, fails *very* badly below 3 $T_c$:
decrease in pressure near $T_c$ due to *non*-perturbative effects




Hietanen, Kajantie, Laine, Rummukainen, Schroder, 0811.4664

# 6. ‘t Hooft basis of SU(N)

Drawing “birdtrack” diagrams,
or group theory made easy

# Peculiar basis for two colors

Start with the usual Cartan basis.

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = t^{12} \;\; ; \;\; \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = t^{21} \;\; ; \;\; \sigma^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now *add* one *extra* diagonal generator, $-\sigma^3$, in *addition* to $\sigma^3$. Why?
Off-diagonal gen.'s: $t^{12}$ and $t^{21}$ labels the row and column of one nonzero element.
Can one do the same for the diagonal generator(s)? More or less...:

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t^{11} \; ; \; -\sigma^3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t^{22}$$

Usual relations for off-diagonal generators:

$$\mathrm{tr}(t^{12}t^{21}) = 1 \; ; \; \mathrm{tr}(t^{12}t^{11}) = \mathrm{tr}(t^{21}t^{11}) = \ldots = 0$$

But *not* for the diagonal generators:

$$\mathrm{tr}(t^{11})^2 = \mathrm{tr}(t^{22})^2 = \frac{1}{2} \;\; ; \;\; \mathrm{tr}(t^{11}t^{22}) = -\frac{1}{2}$$

Basis is *not* an orthonormal, since we added one *extra* generator.

# " 't Hooft" basis for SU(N)

Off diagonal generators same as Cartan basis. *One* diagonal generator same:

$$t^{NN} = \frac{1}{N}\begin{pmatrix} -\mathbf{1}_{N-1} & 0 \\ 0 & N-1 \end{pmatrix} = \begin{pmatrix} 0_{N-1} & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{N}\mathbf{1}_N$$

*Unlike* Cartan, construct N diagonal generators by permutations of diagonal element above:

$$t^{11} = \frac{1}{N}\begin{pmatrix} N-1 & 0 \\ 0 & -\mathbf{1}_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0_{N-1} \end{pmatrix} - \frac{1}{N}\mathbf{1}_N$$

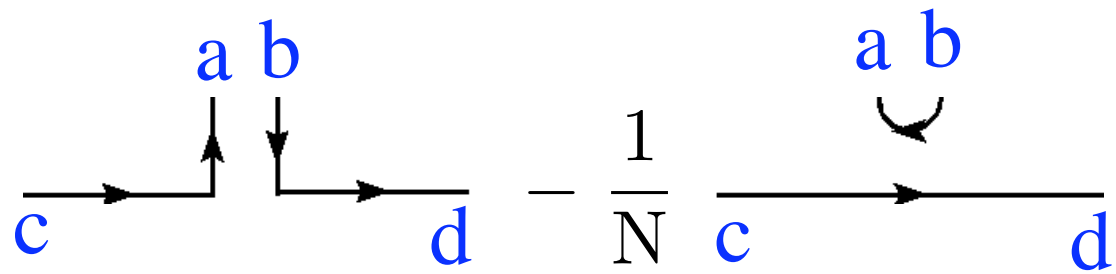
$$t^{22} = \frac{1}{N}\begin{pmatrix} -1 & 0 & 0 \\ 0 & N-1 & 0 \\ 0 & 0 & -\mathbf{1}_{N-2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0_{N-2} \end{pmatrix} - \frac{1}{N}\mathbf{1}_N$$

For N=2, $t^{22} = -\sigma^3$. Now have N diagonal generators, $t^{11}$, $t^{22}$...$t^{NN}$: *one* extra.
Advantage over Cartan: no preferred direction in group space.
Disadvantage: diagonal generators are not orthonormal.
(Off-diagonal generators satisfy usual orthonormality relations.)

# Birdtrack diagrams for the ‘t Hooft basis

Let a,b,c,d... = 1...N represent indices for fundamental representation.
Denote adjoint indices by a pair (ab): $N^2$ of these!
Then the generators can be represented simply by a “birdtrack” diagram:

$$(t^{ab})_{cd} = \delta^a_c\,\delta^b_d - \frac{1}{N}\,\delta^{ab}\,\delta_{cd}$$




First term is the single element of a matrix. The second term is the identity matrix
What is useful is that the generators are obviously *projectors*. Thus the product of two generators is again a projector:

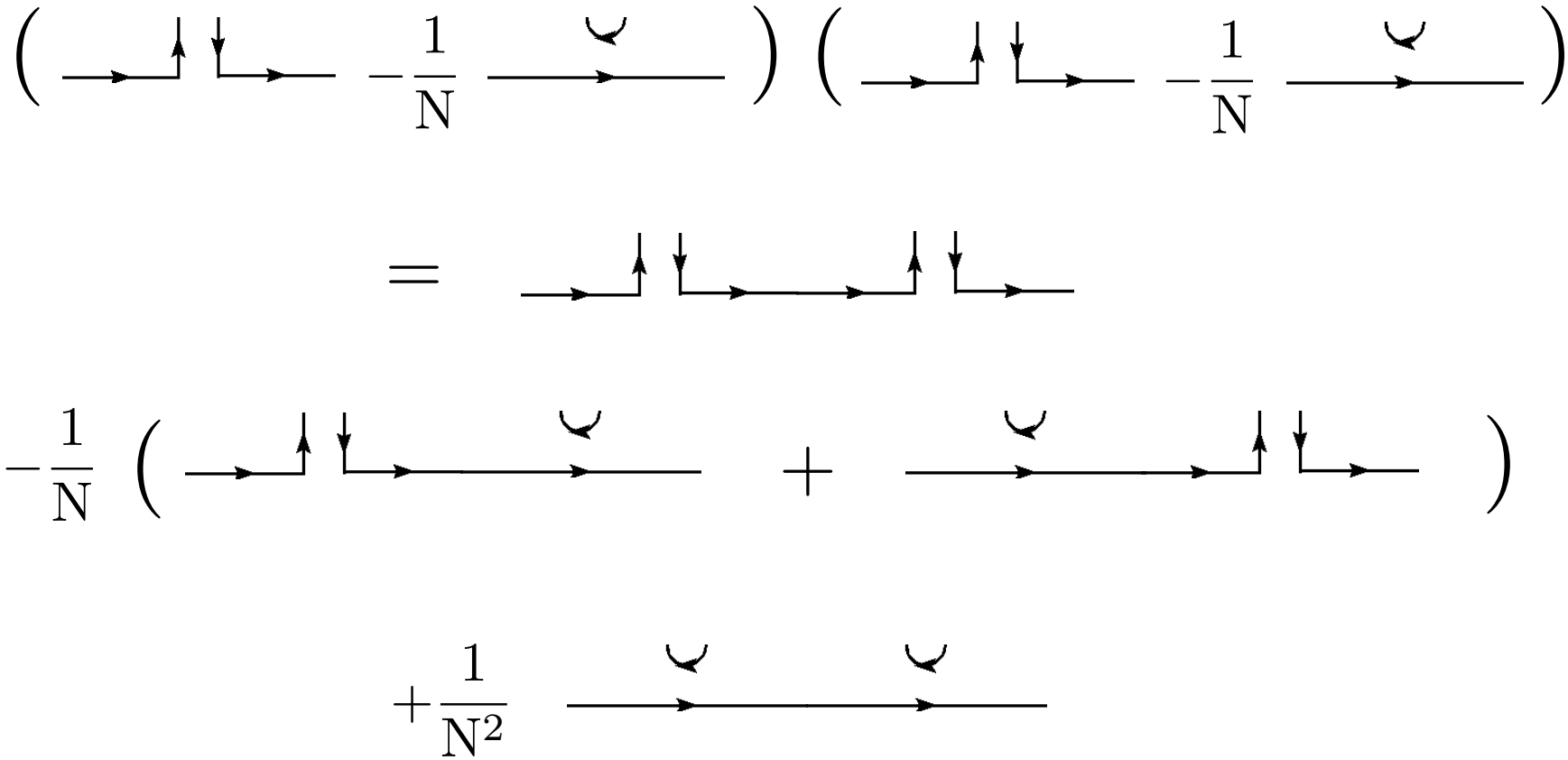
$$\mathrm{tr}(t^{ab}t^{cd}) = (t^{ab})_{dc} \quad : \qquad \mathrm{tr}(t^{ab}\,t^{ba}) = 1\ ,\ a \neq b$$

$$\mathrm{tr}(t^{aa})^2 = 1 - \frac{1}{N} \quad ; \quad \mathrm{tr}(t^{aa}\,t^{bb}) = -\frac{1}{N}\ , a \neq b$$

Raise and lower indices by flipping order; automatic with directed lines.
Normalization of the diagonal generators just reflects a *projector*.

# Group identities from birdtracks

$(t^{ab})_{eg}\,(t^{cd})_{gf} =$

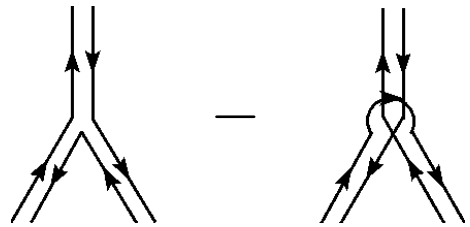



*Just* by drawing arrows, can show the standard relation:

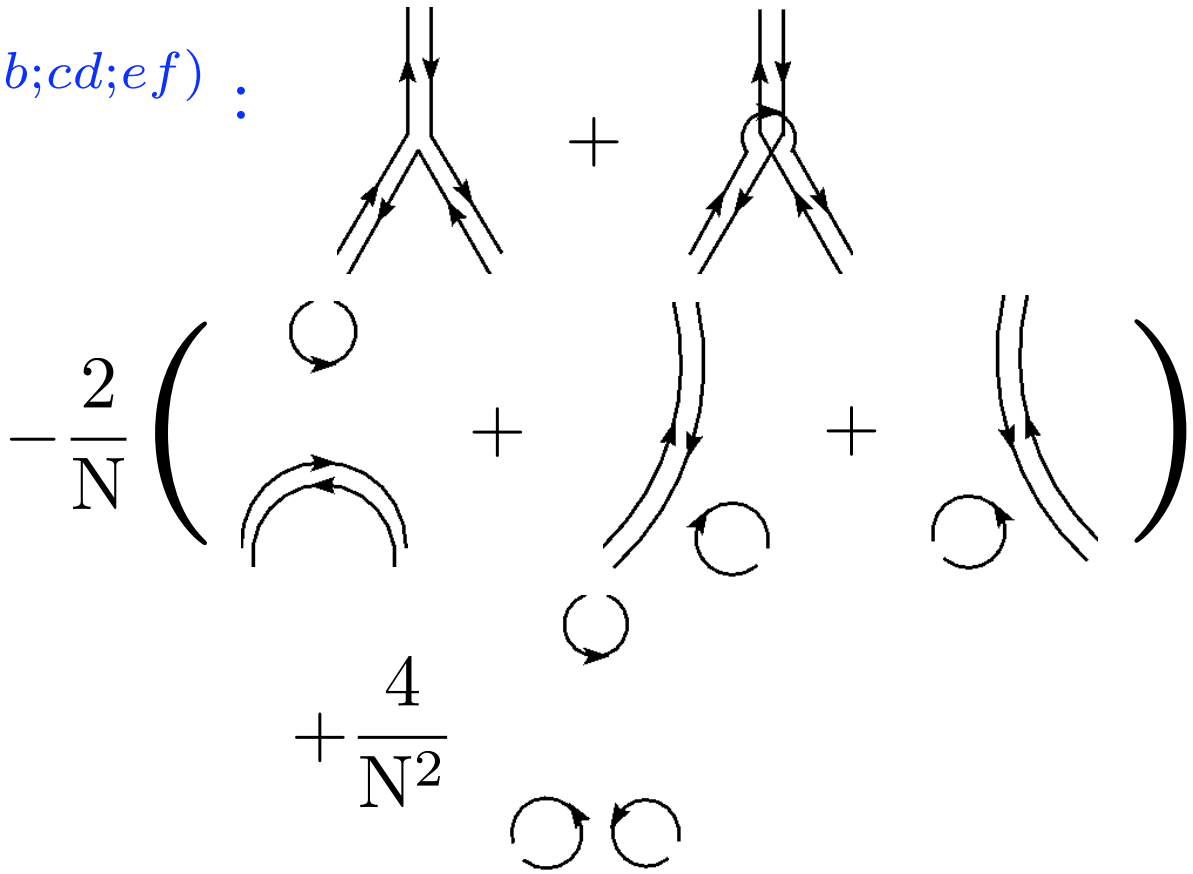
$$\sum_{a,b=1}^{N} (t^{ab}\,t^{ba})_{cd} = \frac{N^2-1}{N}\,\delta_{cd}$$

# Birdtrack structure constants

$f^{(ab;cd;ef)}$ :



$d^{(ab;cd;ef)}$ :



Can derive arbitrary group identities just by doodling.

P. Cvitanovic, http://www.birdtracks.edu

Arbitrary group representations follow similarly, for any group.

N.B.: extracting SU(N) structure constants from U(N).

Useful in a *practical* sense.