QCD phase diagram at $\mu \neq 0$

1. Standard lore:

   One transition, chiral = deconfined, "semicircle"

2. Large number of colors, N_c:

   Two transitions, chiral $\neq$ deconfinement

   "Quarkyonic" matter

   Confined, chirally symmetric baryons: massive, parity doubled.

3. QCD?

   Perhaps: phase intermediate between nuclear matter and "just" quarks

McLerran & RDP, 0706.2191. Hidaka, McLerran, & RDP 0803.0279
Hidaka, Kojo, McLerran, & RDP '09...

The first semicircle

Cabibbo and Parisi '75: Exponential (Hagedorn) spectrum limiting temperature,
or transition to new, "unconfined" phase. One transition.

Punchline today: below for chiral transition, deconfinement splits off at finite μ.

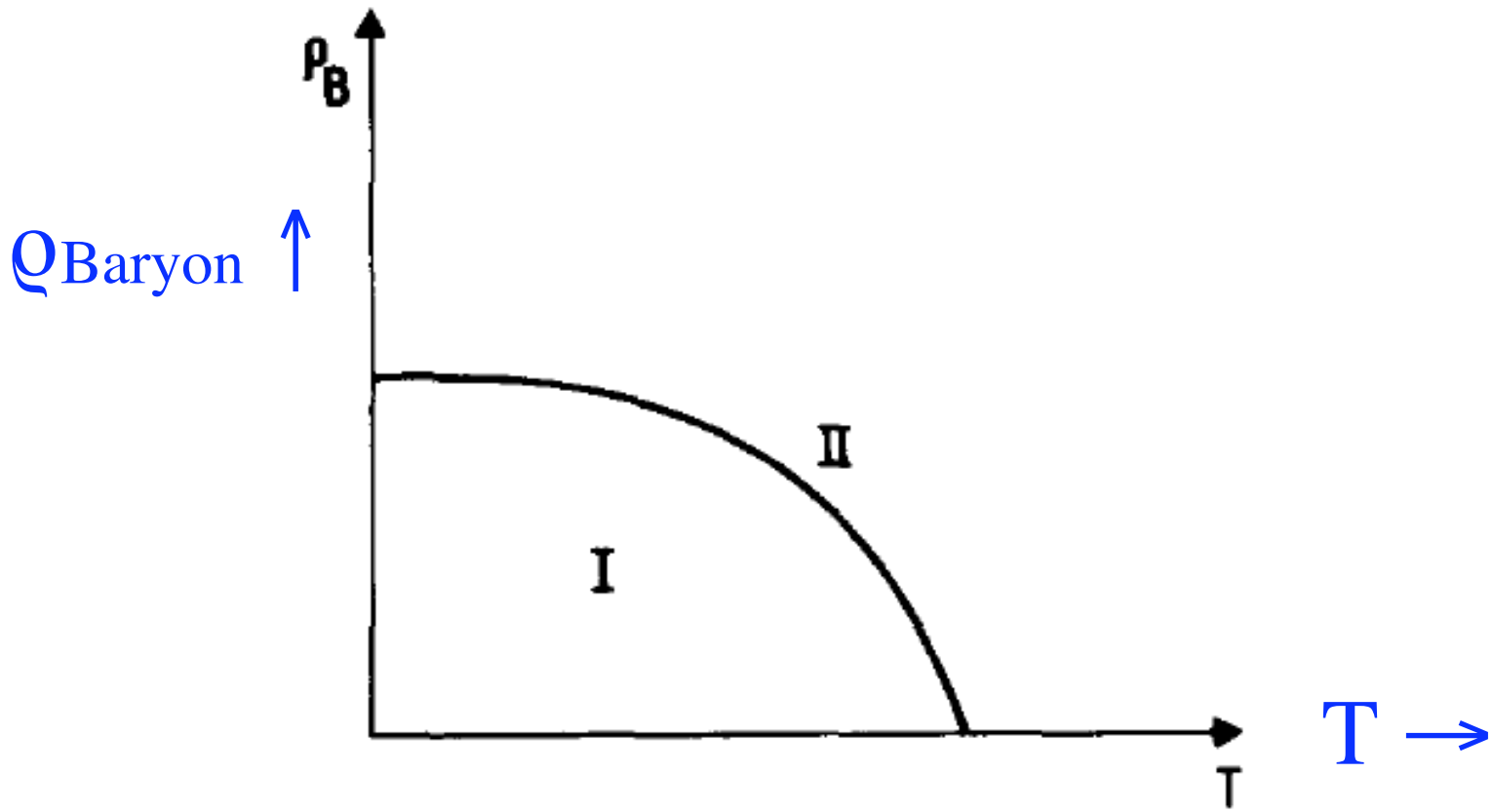


Fig. 1. Schematic phase diagram of hadronic matter. ρ_B is the density of baryonic number. Quarks are confined in phase I and unconfined in phase II.

Two transitions? Casher's argument.

Shuryak '82, RDP '82: Natural to have two transitions:

T_d = deconfinement, T_χ = restoration of chiral symmetry

$T_\chi > T_d$: for $T_\chi > T > T_d$, *deconfined but massive quarks*. Very natural.

Chiral symmetry breaking is due to a (sufficiently) strong coupling.

At T_d , coupling is large enough to deconfine, but not restore chiral sym.

$T_d > T_\chi$: for $T_d > T > T_\chi$, *confined but chirally symmetric quarks (?)*.

Casher '79: confining theories break chiral symmetry.

Meson in rest frame: blue, direction of motion; red: spin.

To be confined, a left moving quark must mix with a right mover.

For massless quarks, chirality = ± helicity: spin = ± momentum

Thus a confined, *massless* quark *must* flip its helicity = ± chirality.

But in QCD, gluon interactions *conserve* chirality. So no confinement!

Chirality is flipped by mass, so ok if chiral symmetry is broken.

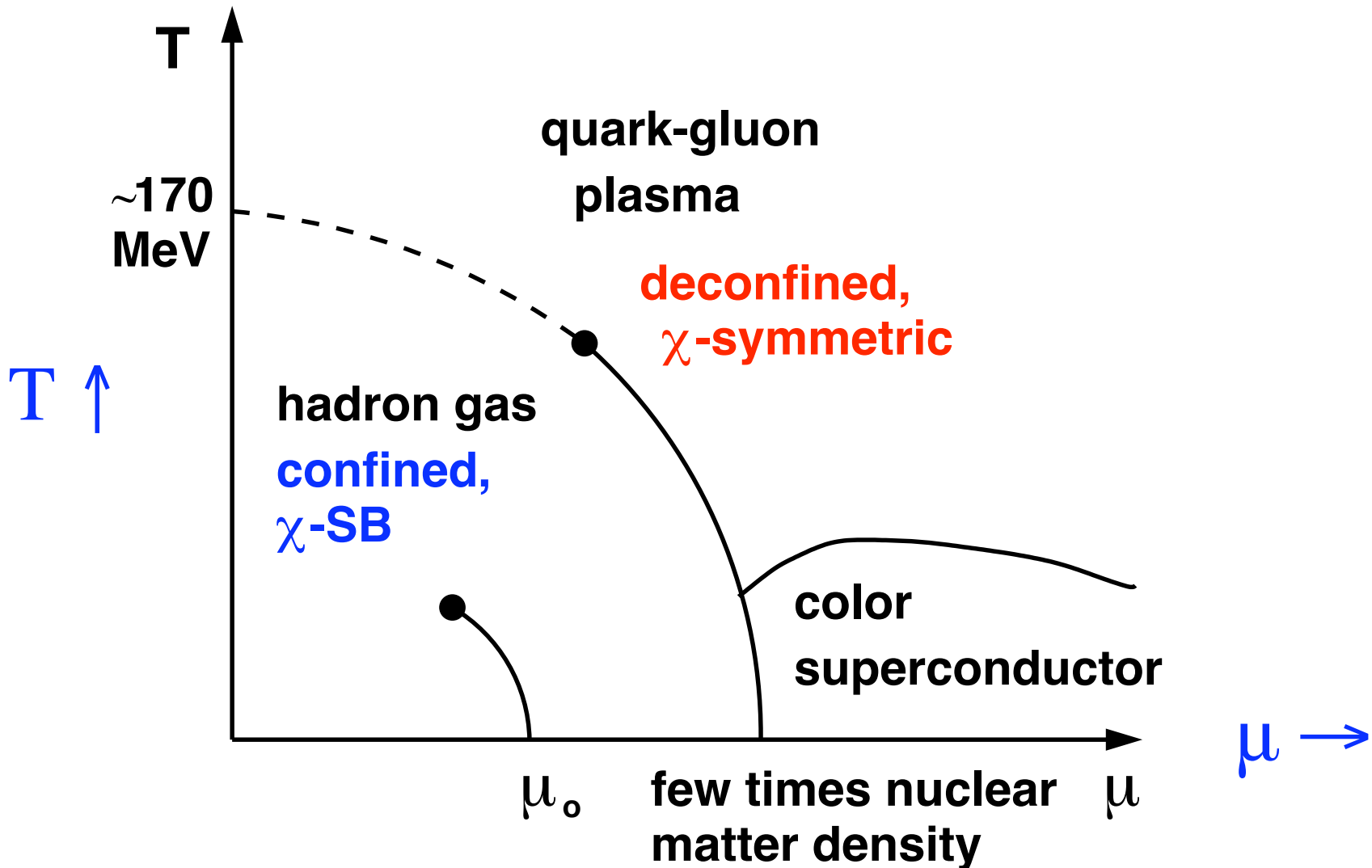
Phase diagram, ~ '06

Lattice, $T \neq 0$, $\mu = 0$: two possible transitions; *one* crossover, *same* T.

Karsch hep-lat/0601013. Remains crossover for $\mu \neq 0$?

Stephanov, Rajagopal, & Shuryak hep-ph/9806219, /9903292, /0010100

Critical end point where crossover turns into first order transition

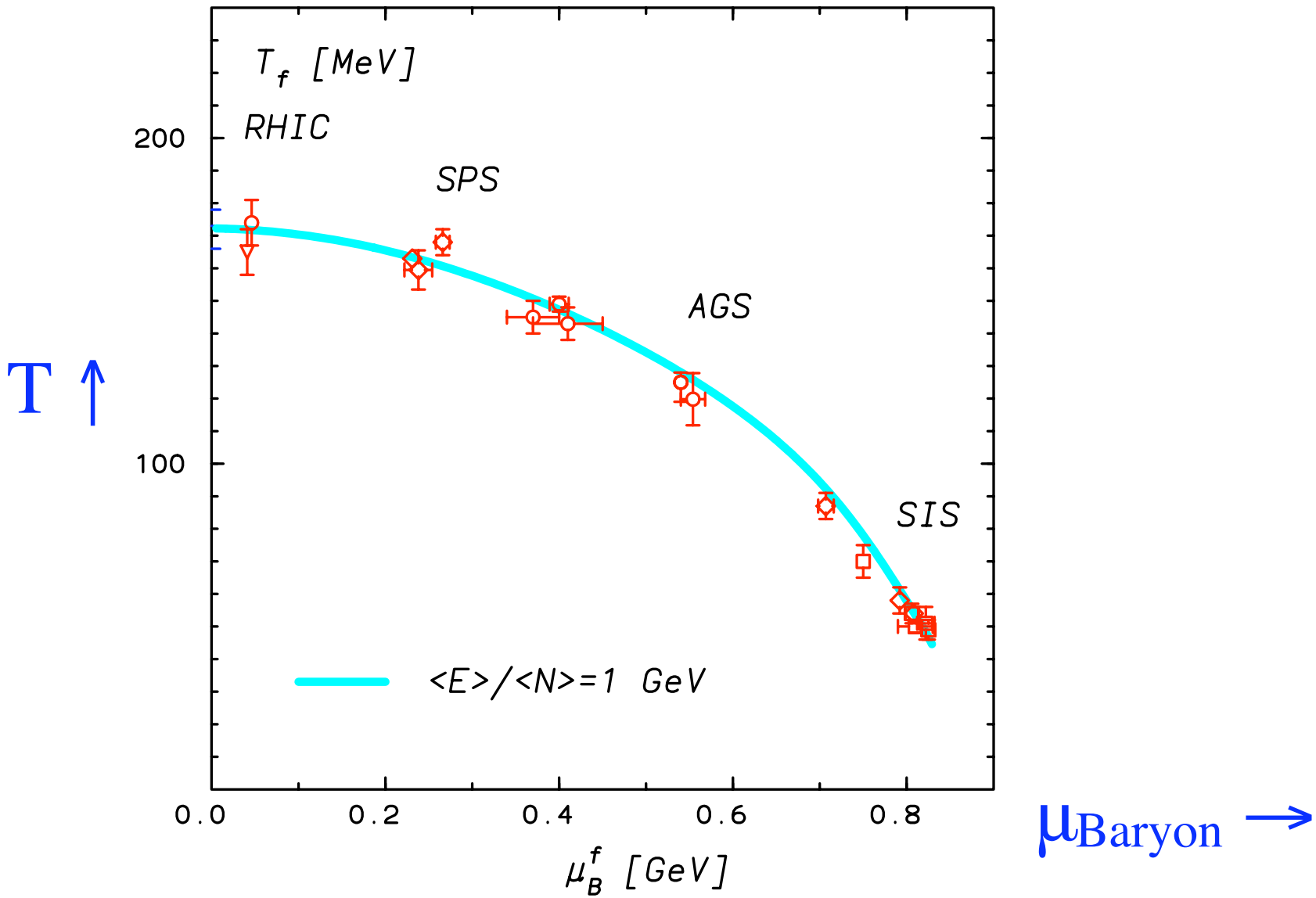


Experiment: freezeout line

Cleymans & Redlich nucl-th/9906065...Kraus, Cleymans, Oeschler, Redlich 0808.0611

Line for chemical equilibriation at freezeout ~ semicircle.

N.B.: for T = 0, goes down to ~ nucleon mass.

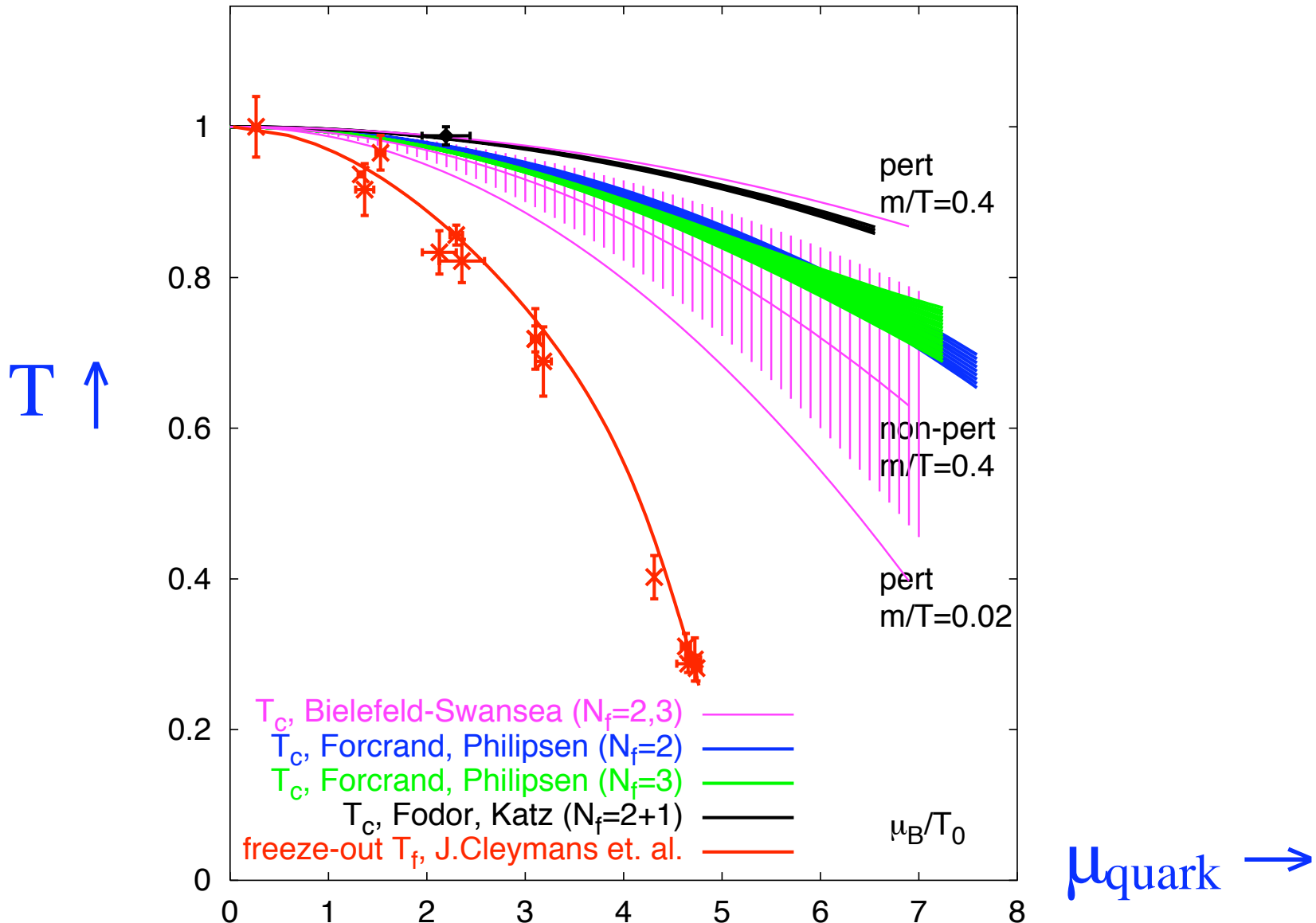


Experiment vs. Lattice

Lattice "transition" appears *above* freezeout line?

Fodor, Katz, & Schmidt hep-lat/0701022; Schmidt '07

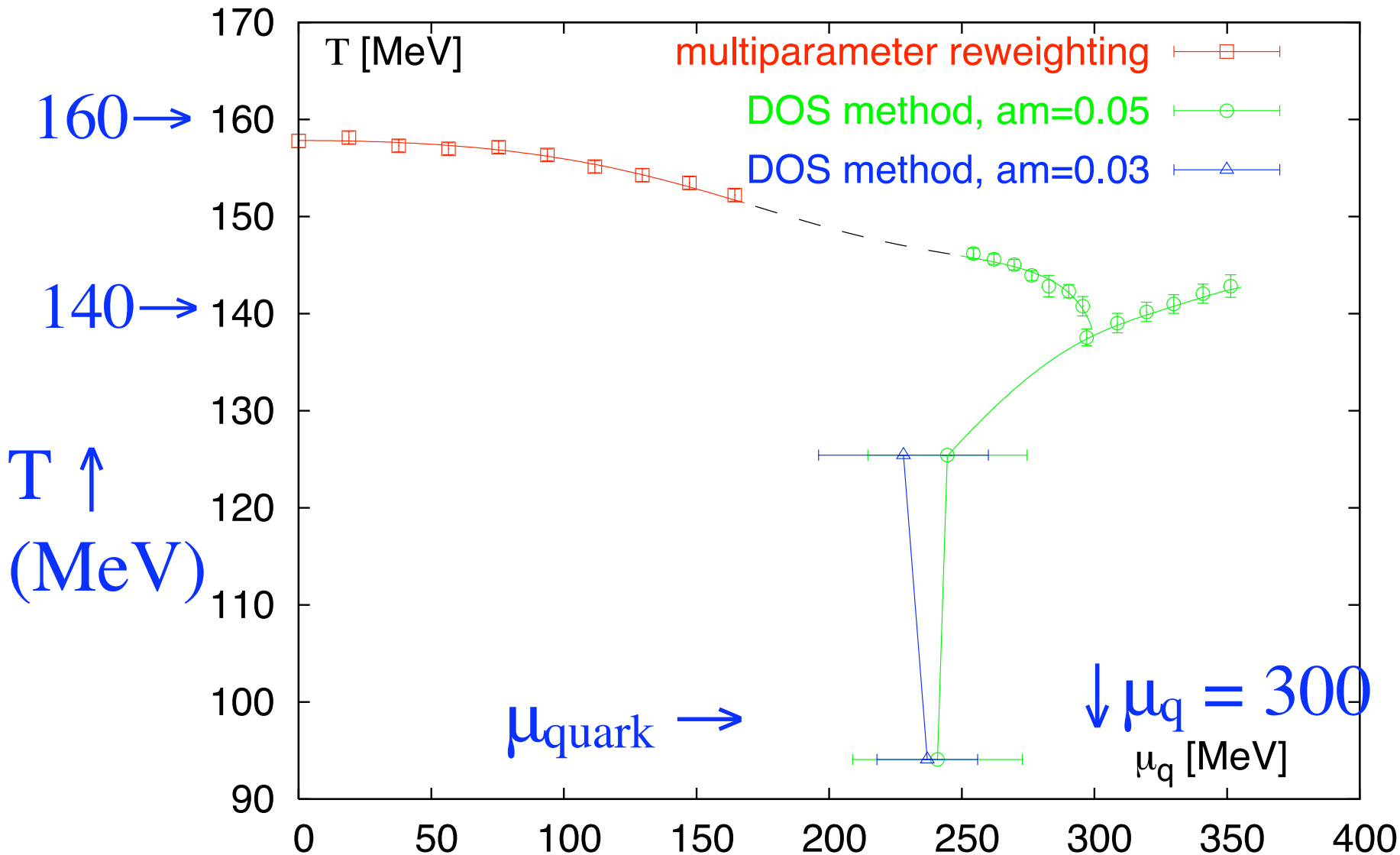
N.B.: *small* change in T_c with μ?



Lattice T_c : *flat* with μ

Rather small change in T_c vs μ? Depends where μ_c is at $T = 0$.

Fodor, Katz, & Schmidt hep-lat/0701022



EoS of nuclear matter

Akmal, Panharipande, & Ravenhall nucl-th/9804027:

Equation of State for nuclear matter, T=0

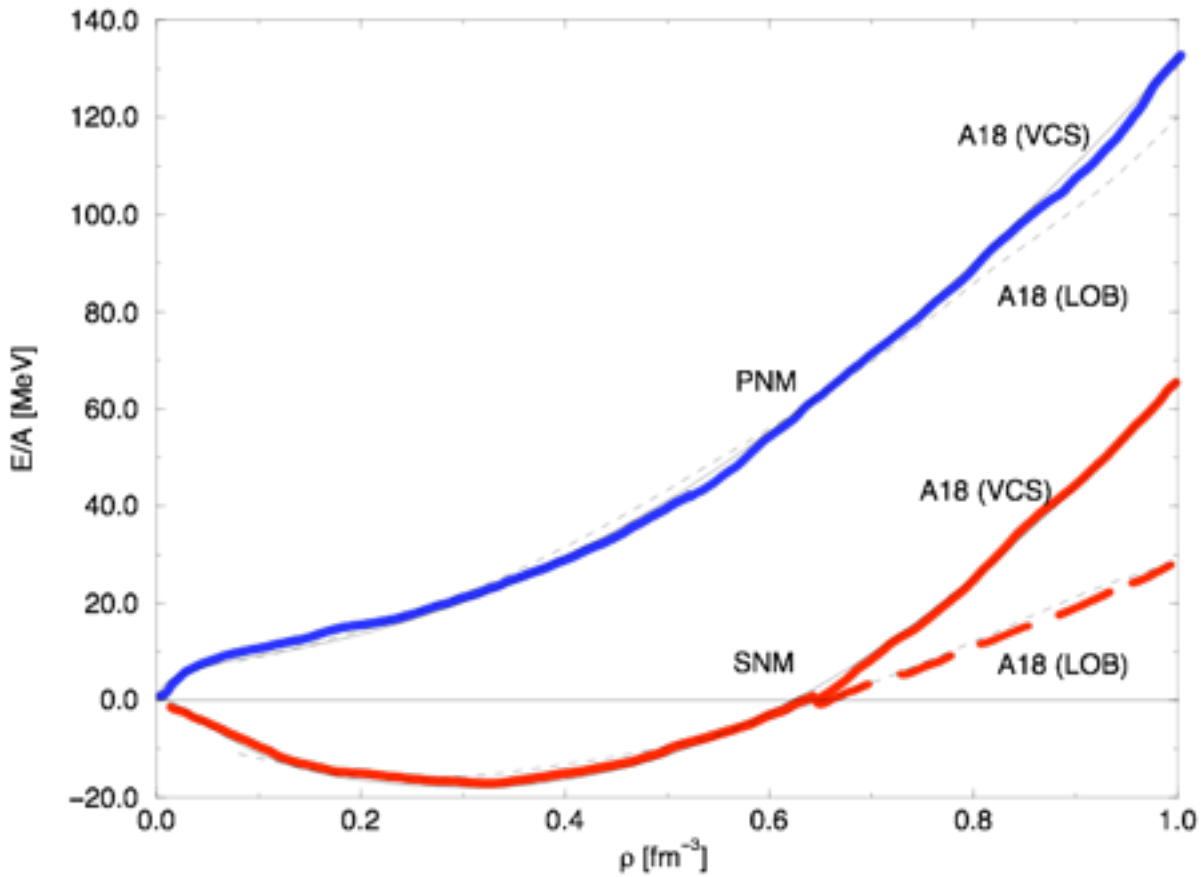
E/A = energy/nucleon. Fits to various nuclear potentials: A18= Argonne 18...

PNM = pure neutron matter. SNM = symmetric nuclear matter (equal #'s n's, p's)

Binding energy of nuclear matter ~ 15 MeV!

Much smaller than any natural hadronic scale: f_π , Λ_{MS}...

E/A ↑



ρ_{Baryon} →

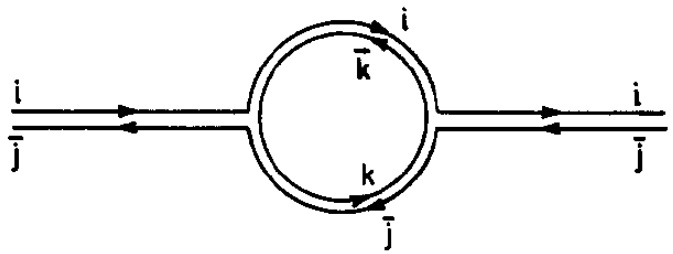
Expansion in large N_c

't Hooft '74: let $N_c \to \infty$, with $\lambda = g^2 N_c$ fixed.

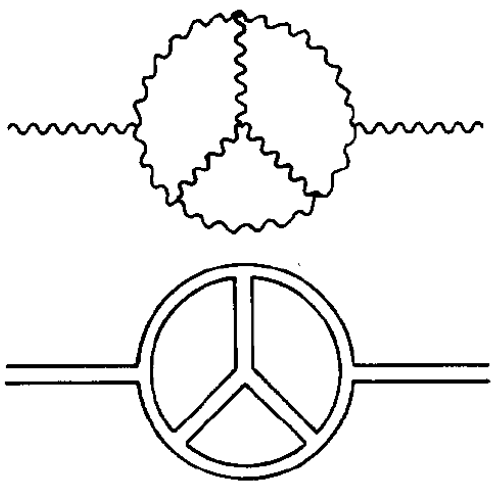
$\sim N_c^2$ gluons in adjoint representation, vs $\sim N_c$ quarks in fundamental rep. $\Rightarrow$

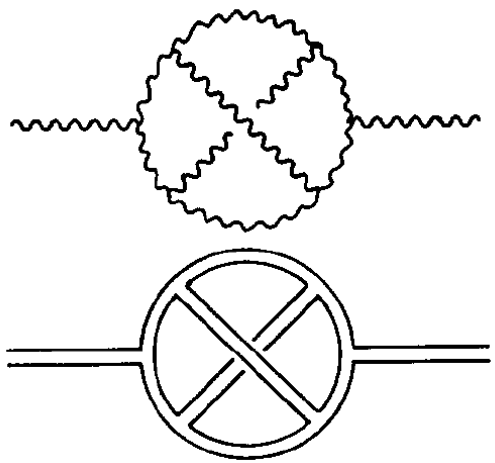
large N_c dominated by *gluons* (iff N_f = # quark flavors *small*)

Double line (birdtrack) notation:



$\sim g^2 N_c = \lambda$



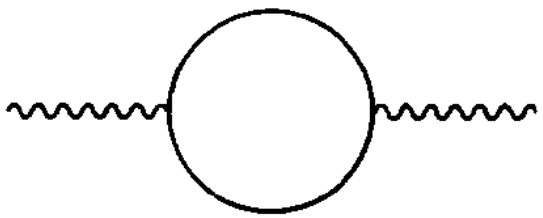


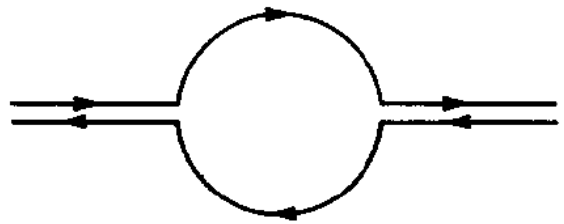
Planar diagram, $\sim \lambda^2$

Non-planar diagram, $\sim \lambda^2 / N_c$

Suppressed by $1/N_c$. Trace terms also $1/N_c$

Quark loops suppressed at large N_c





$$\sim g^2 = \lambda \times \frac{1}{N_c}$$

Quark loops are suppressed at large N_c *if* N_f , # quark flavors, is held fixed

Thus: limit of: large N_c , *small* N_f

Quarks introduced as external sources.

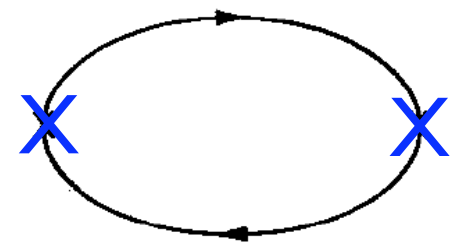
Analogous to "quenched" approximation, expansion about $N_f = 0$.

Veneziano '78: take both N_c and N_f large.
Can use baryon number as order parameter: Hidaka, McLerran, & RDP.

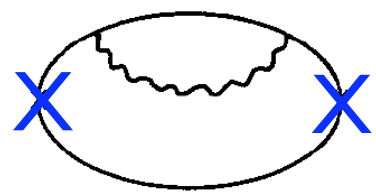
Form factors at large N_c

J ~ (gauge invariant) mesonic current

$$< J(x)J(0) > \sim N_c$$



Infinite # of planar diagrams for $< J\ J >$:



Confinement => sum over mesons, form factors ~ $N_c^{1/2}$

$$< J(x)J(0) > \sim \int d^4p\, e^{ip\cdot x} \sum_n < 0|J|n > \frac{1}{p^2 + m_n^2} < n|J|0 >$$

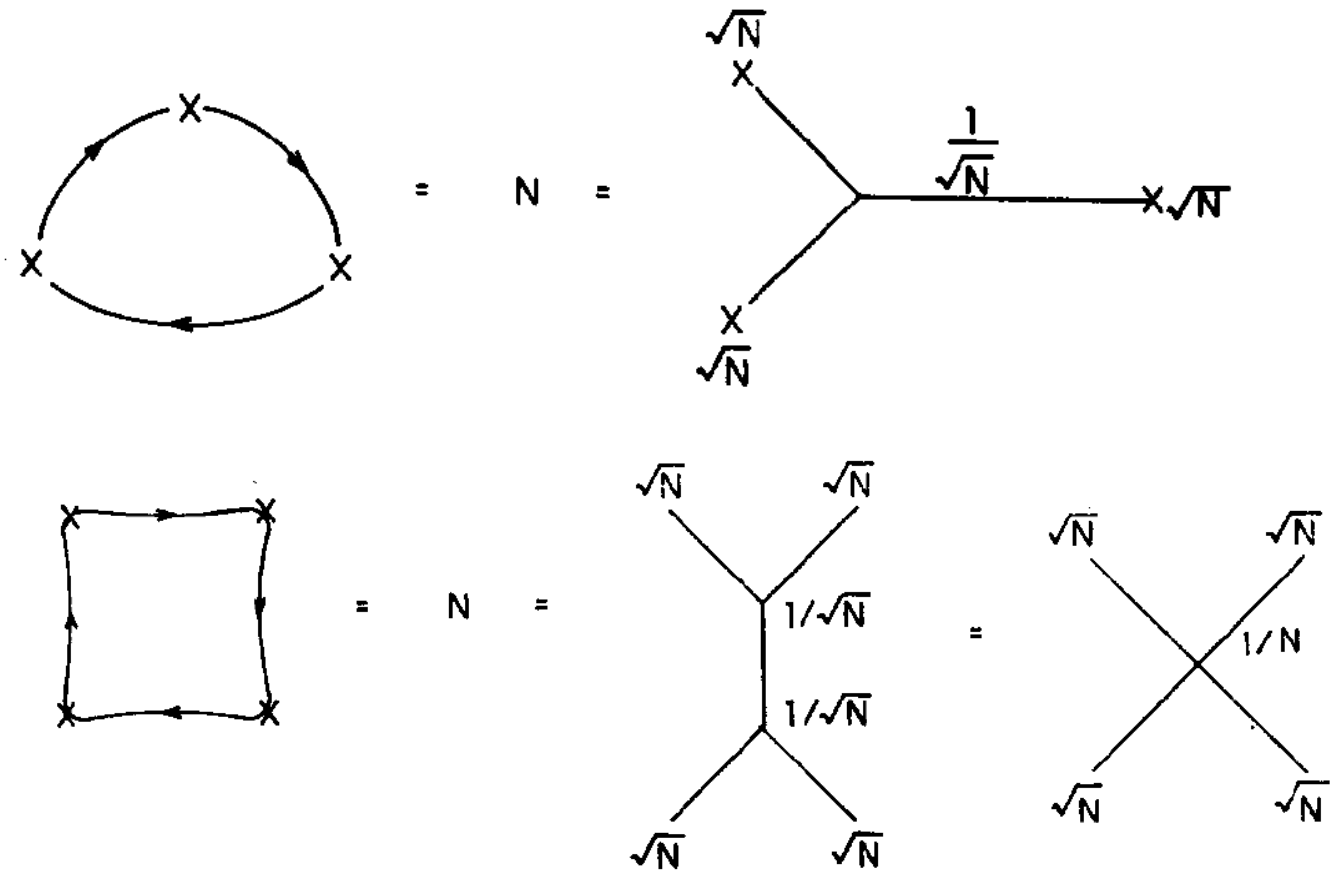
$$< J(x)J(0) > \sim N_c \;\Rightarrow\; < 0|J|n > \sim \sqrt{N_c} \text{ if } m_n \sim 1$$

Mesons & glueballs *free* at $N_c = \infty$

With form factors ~ $N_c^{1/2}$, 3-meson couplings ~ $1/N_c^{1/2}$; 4-meson, ~ $1/N_c$
For glueballs, 3-glueball couplings ~ $1/N_c$, 4-glueball ~ $1/N_c^2$

Mesons and glueballs don't interact at $N_c = \infty$.

Large N limit *always* (some) classical mechanics Yaffe '82

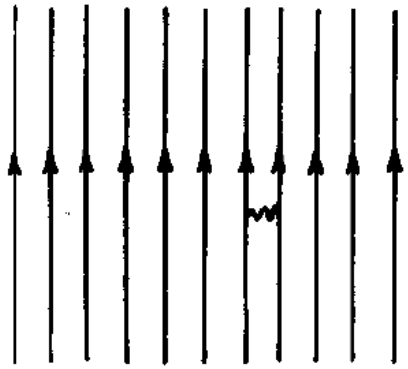


Baryons at large N_c

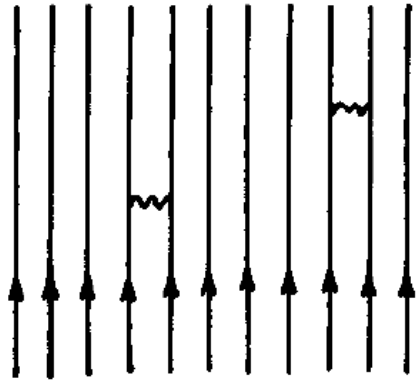
Witten '79: Baryons have N_c quarks, so nucleon mass $M_N \sim N_c \Lambda_{QCD}$.

Baryons like "solitons" of large N_c limit (~ Skyrmion)

Leading correction to baryon mass:



$$g^2 \times N_c \times N_c \sim \lambda N_c$$

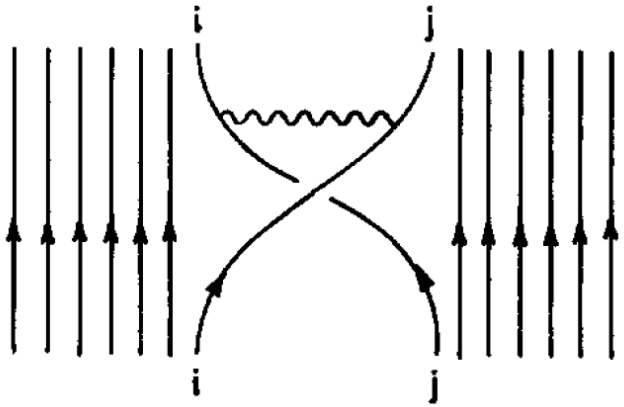


Appears ~ $g^4 N_c^4 \sim \lambda^2 N_c^2$?

No, iteration of average potential, mass still ~ N_c .

Baryons are *not* free at $N_c = \infty$

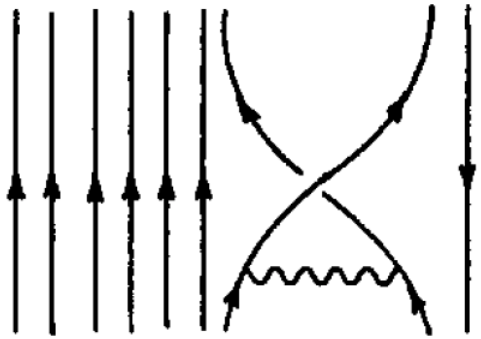
Baryons interact strongly. Two baryon scattering ~ N_c :



$$g^2 \times N_c \times N_c \sim \lambda N_c$$

Scattering of three, four... baryons also ~ N_c

Mesons also interact strongly with baryons, ~ N_c^0 ~ 1



$$g^2 \times N_c \sim \lambda$$

Towards the phase diagram at $N_c = \infty$

As example, consider gluon polarization tensor at zero momentum.
(at leading order, ~ Debye mass^2 , gauge invariant)

$$\Pi^{\mu\mu}(0) = g^2 \left(\left(N_c + \frac{N_f}{2} \right) \frac{T^2}{3} + \frac{N_f \mu^2}{2\pi^2} \right) = \lambda \, \frac{T^2}{3} \, , \; N_c = \infty$$

For $\mu \sim N_c^0 \sim 1$, at $N_c = \infty$ the gluons are *blind* to quarks.

When $\mu \sim 1$, since gluons don't feel quarks, the
deconfining transition temperature is independent of μ! $T_d(\mu) = T_d(0)$

Chemical potential only matters when larger than mass:
$\mu_{Baryon} > M_{Baryon}$. Define $m_{quark} = M_{Baryon}/N_c$; so $\mu > m_{quark}$.

"Box" for $T < T_c$; $\mu < m_{quark}$: confined phase baryon free, since their mass ~ N_c
Thermal excitation ~ $\exp(-m_B/T) \sim \exp(-N_c) = 0$ at large N_c.
So hadronic phase in "box" = mesons & glueballs only, *no* baryons.

Phase diagram at $N_c = \infty$, I

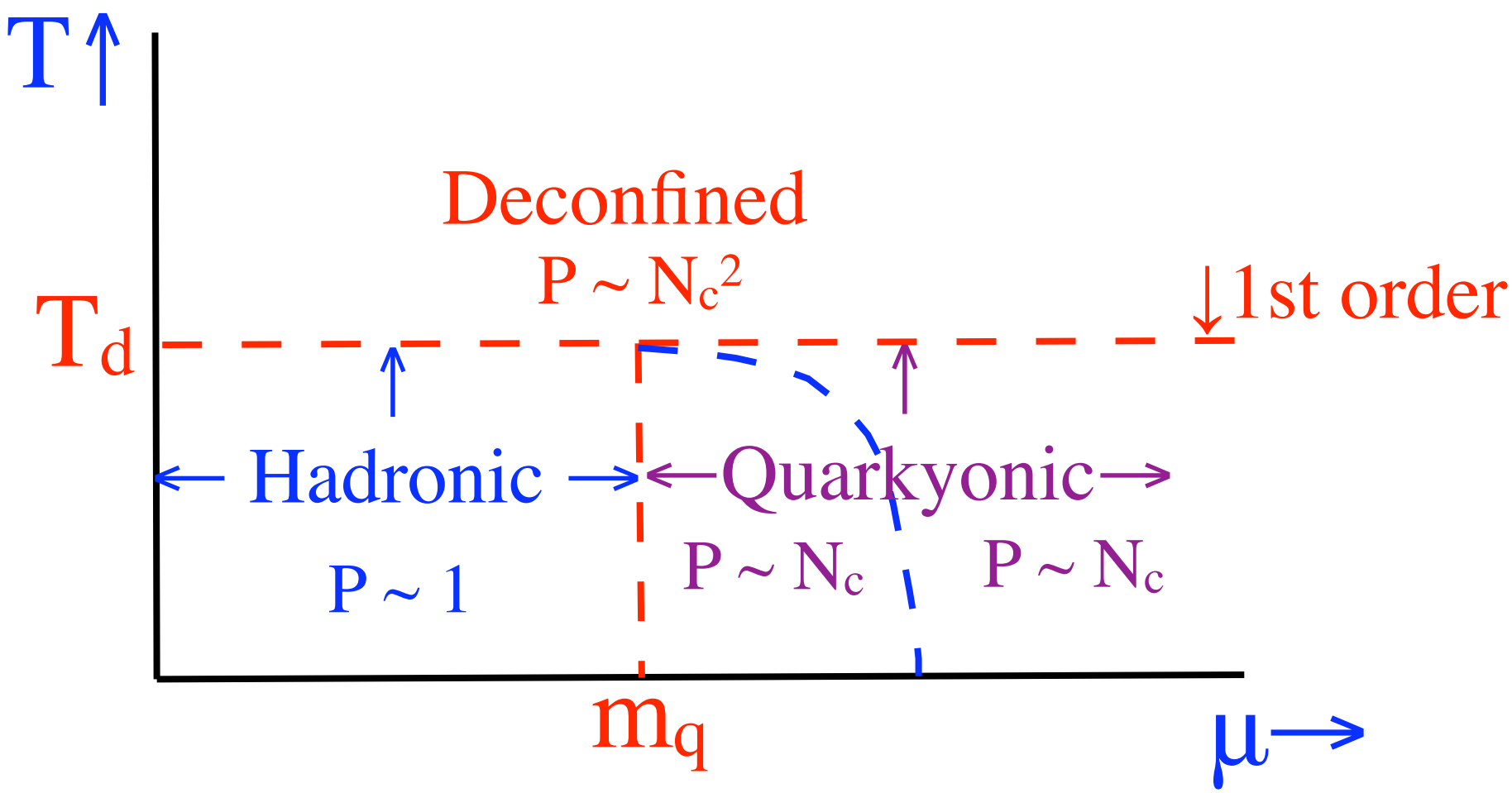
At *least* three phases. At large N_c, can use pressure, P, as order parameter.

Hadronic (confined): $P \sim 1$. Deconfined, $P \sim N_c^2$. Thorn '81; RDP '84...

$P \sim N_c$: quarks or baryonic = "quark-yonic". Chiral symmetry restoration?

L. McLerran & RDP, 0706.2191

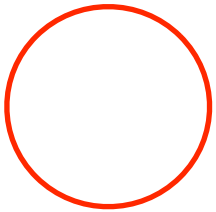
N.B.: mass threshold at m_q neglects (possible) nuclear binding, Son.



Nuclear matter at large N_c

$\mu_{Baryon} = \sqrt{k_F^2 + M^2}$, k_F = Fermi momentum of baryons.
Pressure of ideal baryons density times energy of non-relativistic baryons:

$$P_{\text{ideal baryons}} \sim n(k_F)\,\frac{k_F^2}{M} \sim \frac{1}{N_c}\,\frac{k_F^5}{\Lambda_{QCD}}$$

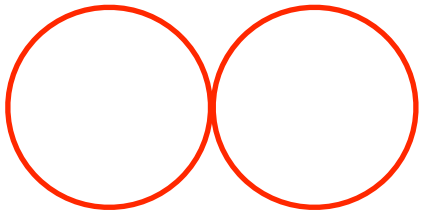


This is small, ~ $1/N_c$. The pressure of the I = J tower of resonances is as small:

$$\delta P_{\text{resonances}} \sim \frac{1}{M}\,\frac{k_F^8}{\Lambda_{QCD}^3} \sim \frac{1}{N_c}\,\frac{k_F^8}{\Lambda_{QCD}^4}$$

Two body interactions are huge, ~ N_c in pressure.

$$\delta P_{\text{two body int.'s}} \sim N_c\,\frac{n(k_F)^2}{\Lambda_{QCD}^2} \sim N_c\,\frac{k_F^6}{\Lambda_{QCD}^2}$$



At large N_c , nuclear matter is dominated by potential, not kinetic terms!
Two body, three body... interactions *all* contribute ~ N_c .

Window of nuclear matter

Balancing $P_{\text{ideal baryons}} \sim P_{\text{two body int.'s}}$, interactions important very quickly,

$$k_F \sim \frac{1}{N_c^2}\,\Lambda_{QCD}$$

For such momenta, only two body interactions contribute.

By the time $k_F \sim 1$, *all* interactions terms contribute $\sim N_c$ to the pressure.

But this is *very* close to the mass threshold,

$$\mu - m_q = \frac{\mu_B - M}{N_c} = \frac{k_F^2}{2MN_c} \sim \frac{1}{N_c^2}\,k_F^2$$

Hence "ordinary" nuclear matter is only in a *very* narrow window.

One quickly goes to a phase with pressure $P \sim N_c$.

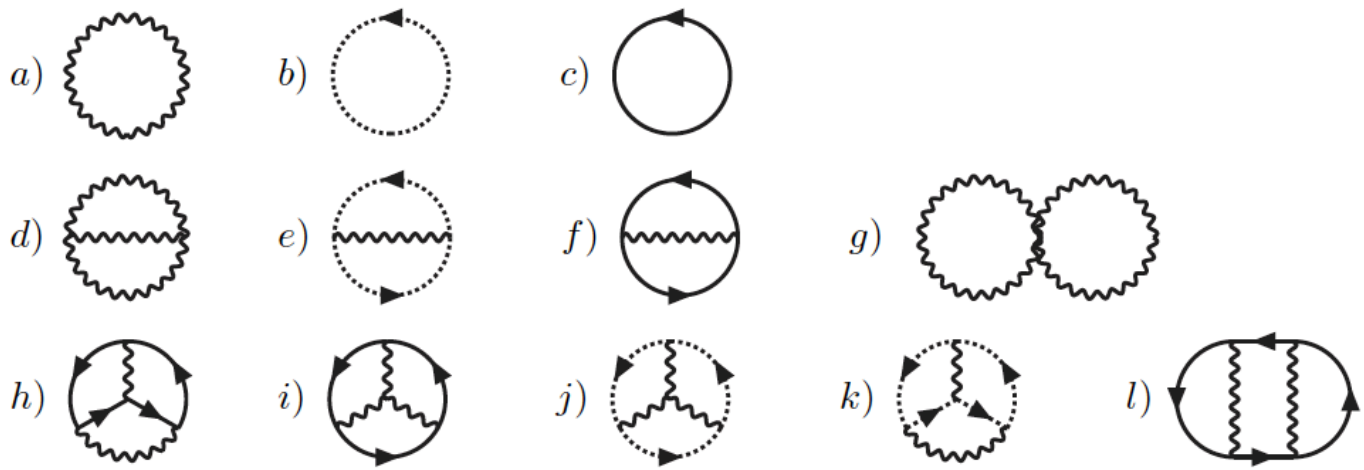
So are they baryons, or quarks?

Perturbative pressure

At high density, $\mu >> \Lambda_{QCD}$, compute $P(\mu)$ in QCD perturbation theory.

To $\sim g^4$, (Freedman & McLerran)[4] '77

Ipp, Kajantie, Rebhan, & Vuorinen, hep-ph/0604060



At $\mu \neq 0$, only diagrams with at least one quark loop contribute. Still...

$$P_{\text{pert.}}(\mu) \sim N_c N_f \; \mu^4 \; F_0(g^2(\mu/\Lambda_{QCD}), N_f)$$

For $\mu >> \Lambda_{QCD}$, but $\mu \sim N_c^0 \sim 1$, calculation reliable.

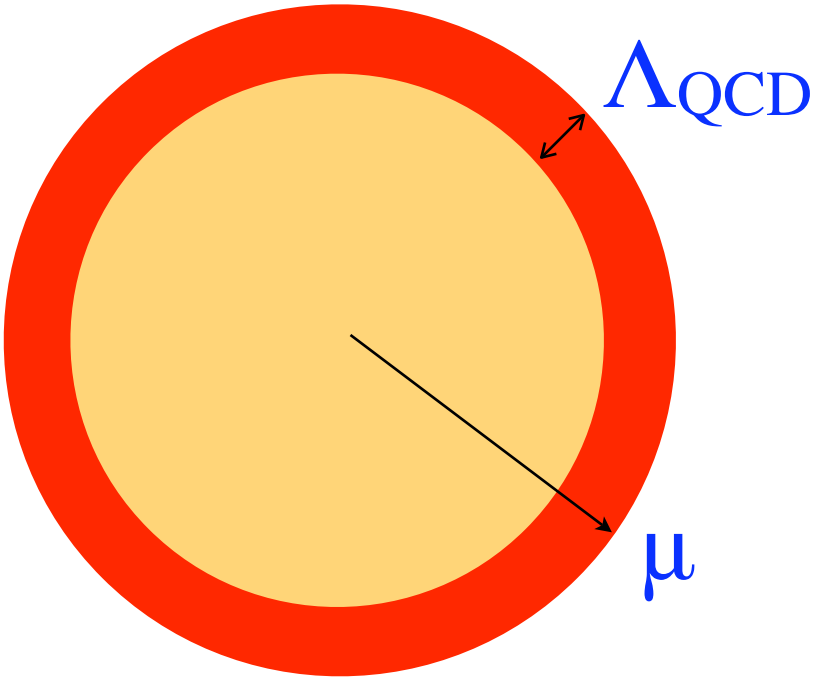
Compute $P(\mu)$ to $\sim g^6$? No "magnetic mass" at $\mu \neq 0$, well defined $\forall$ $(g^2)^n$.

"Quarkyonic" phase at large N_c

As gluons blind to quarks at large N_c, for $\mu \sim N_c^0 \sim 1$, *confined* phase for $T < T_d$

This includes $\mu >> \Lambda_{QCD}$! Central puzzle. We suggest:

To the right: Fermi sea =>



Deep in the Fermi sea, $k << \mu$,
looks like quarks.

But: within $\sim \Lambda_{QCD}$ of the Fermi surface,
confinement => *baryons*

We term combination "*quark-yonic*"

OK for $\mu >> \Lambda_{QCD}$. When $\mu \sim \Lambda_{QCD}$, baryonic "skin" entire Fermi sea.

But what about chiral symmetry breaking?

Skyrmions and $N_c = \infty$ baryons

Witten '83; Adkins, Nappi, Witten '83: Skyrme model for baryons

$$\mathcal{L} = f_\pi^2 \, \mathrm{tr}|V_\mu|^2 + \kappa \, \mathrm{tr}[V_\mu, V_\nu]^2 \ , \ V_\mu = U^\dagger \partial_\mu U \ , \ U = e^{i\pi/f_\pi}$$

Baryon soliton of pion Lagrangian: $f_\pi \sim N_c^{1/2}$, $\kappa \sim N_c$, mass $\sim f_\pi^2 \sim \kappa \sim N_c$.

Above Lagrangian simplest form: presumably (?) *infinite* series in V_μ.

Single baryon: at $r = \infty$, $\pi^a = 0$, $U = 1$. At $r = 0$, $\pi^a = \pi \, r^a/r$.

Baryon number topological: Wess & Zumino '71; Witten '83.

Huge degeneracy of baryons: multiplets of isospin and spin, $I = J$: $1/2 \ldots N_c/2$.

Obvious as collective coordinates of soliton, coupling spin & isospin

Dashen & Manohar '93, Dashen, Jenkins, & Manohar '94:

Baryon-meson coupling $\sim N_c^{1/2}$,

Cancellations from extended $SU(2\,N_f)$ symmetry.

Skyrmion crystals

Kutschera, Pethick & Ravenhall (KPR) '84; Klebanov '85 + ...

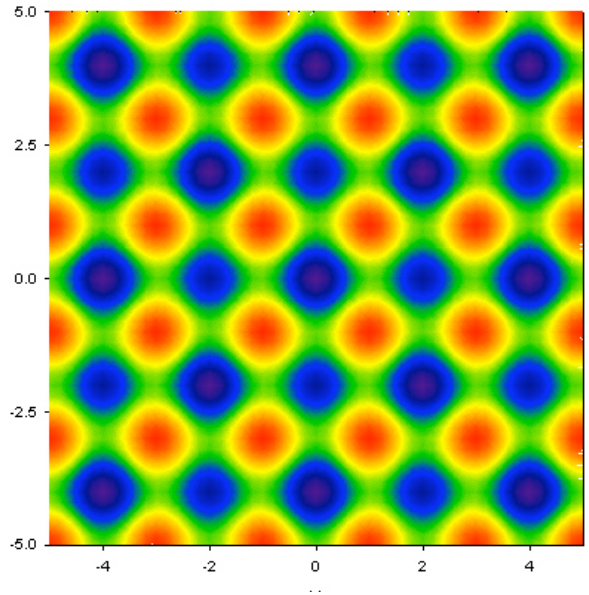
Lee, Park, Min, Rho & Vento, hep-ph/0302019; Park, Lee, & Vento, 0811.3731:

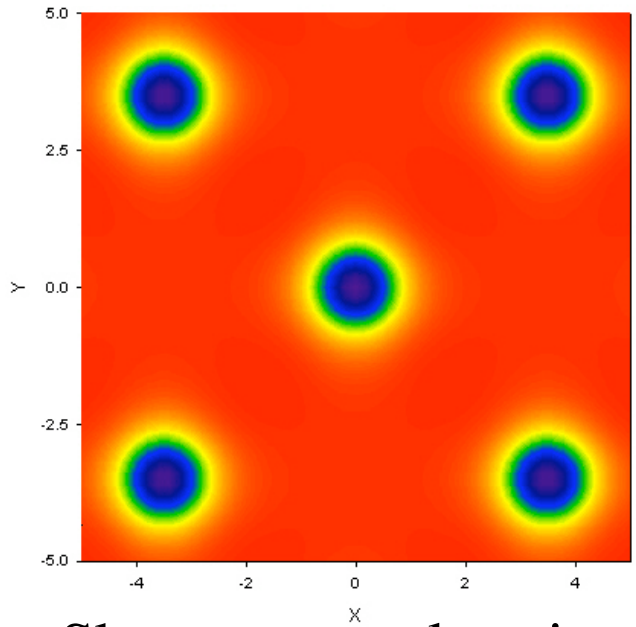
At large N_c, baryons are heavy, so form a crystal.

Form Skyrmion crystal by taking periodic boundary conditions in a box.

Lee+... '03 : box of size L, units of length $1/(\sqrt{\kappa}\, f_\pi)$, plot baryon number density:

L=2.0→





←L=3.5

At low density, chiral symmetry broken by Skyrme crystal, as in vacuum.

But chiral symmetry *restored* at nonzero L (density): $\langle U \rangle = 0$ in *each* cell.

Skyrmion crystals as quarkyonic matter

Why chiral symmetry restoration in a Skyrmion crystal?

Goldhaber & Manton '87: due to "half" Skyrmion symmetry in each cell.

Easiest to understand with "spherical" crystal: sphere instead of cube...

KPR '84, Ruback & Manton '86, Manton '87. Consider the "trivial" map:

$$U(r) = \exp(i\, f(r)\, \hat{r} \cdot \tau)\ ;\ f(r) = \pi \left(1 - \frac{r}{R}\right)$$

Solution has unit baryon number per unit sphere, and so is a crystal.

Solution is minimal when R < $\sqrt{2}$ (* $1/(\sqrt{\kappa}\, f_\pi)$).

Forkel, Jackson, Rho, & Weiss '89 =>

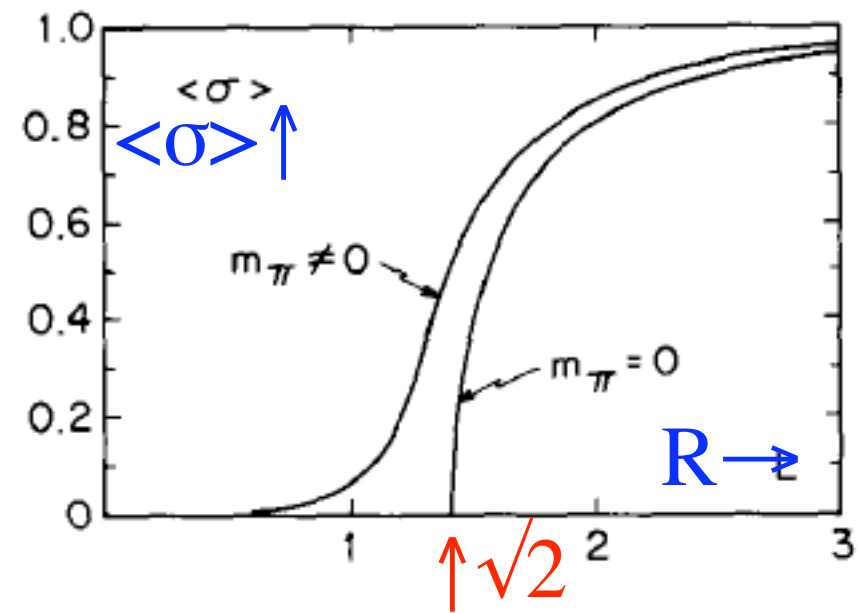
looks like standard chiral transition!

Excitations *are* chirally symmetric.

But Skyrmions are *not* deconfined.

Example of quarkyonic matter,

chirally broken and chirally symmetric.



Casher's argument in a Fermi sea

What about Casher's argument?

In vacuum, can neglect scattering with baryons, as baryon (anti-baryon) pairs are exponentially suppressed, exp(-N).

In a Fermi sea, though, baryon's are there by construction. Thus a massless quark can scatter off of a baryon in the Fermi sea:

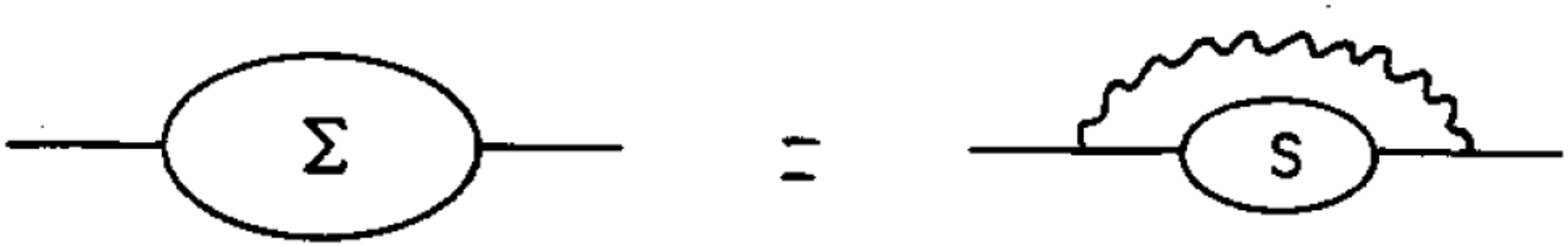
Quark baryon scattering is large. Thus a quark can scatter off of a baryon, in the Fermi sea. This process can flip the chirality, since quark baryon scattering does not conserve chirality or helicity.

Argument special to having a Fermi sea: one needs *some* baryons about.

Standard Casher argument *does* apply at $T \neq 0$, $\mu = 0$, without a Fermi sea.

Schwinger-Dyson equations at large N_c: 1+1 dim.'s

't Hooft '74: as gluons blind to quarks at large Nc, S-D eqs. simple for quark:
Gluon propagator, and gluon quark anti-quark vertex unchanged.
To leading order in $1/N_c$, only quark propagator changes:



't Hooft '74: in 1+1 dimensions, single gluon exchange generates linear potential,

$$g_{2D}^2 \int dk \, \frac{e^{ikr}}{k^2} \sim g_{2D}^2 \, r$$

In vacuum, Regge trajectories of confined mesons. Baryons?

Solution at $\mu \neq 0$? Should be possible, not yet solved.

M. Thies, hep-th/0601049, C. Boehmer, U. Fritsch, S. Kraus, & M. Thies, 0807.2571
Gross-Neveu model has crystalline structure at $\mu \neq 0$

Quarkyonic matter via Schwinger-Dyson

Glozman & Wagenbrunn 0709.3080, 0805.4799; Glozman 0812.1101:

In 3+1 dimensions, confining gluon propagator, $1/(k^2)^2$ as $k^2 \to 0$:

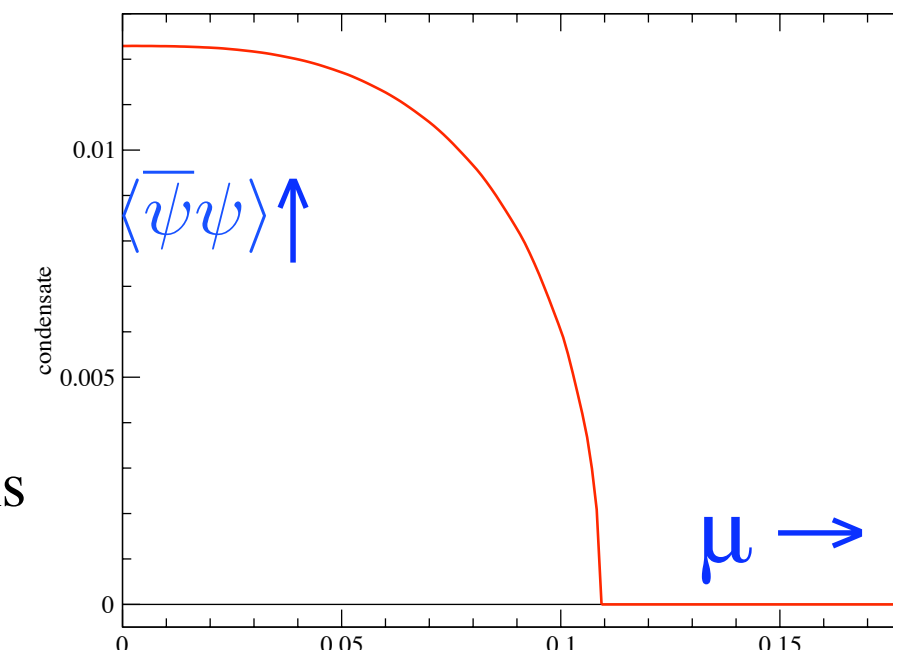
$$g^2 \int d^3k \, \frac{e^{ikr}}{k^2} \left(1 + \frac{\sigma}{k^2}\right) \sim g^2 \, \sigma \, r \, , \; r \to \infty$$

σ = string tension. Very similar to 1+1 dimensions. $\mu = 0$: $\langle \overline{\psi}\psi \rangle = (.23\sqrt{\sigma})^3$

Take Schwinger-Dyon eq. at large N_c: confinement unchanged by $\mu \neq 0$.
Treat μ by usual cutoff in momentum space: for confining system, same as $\mu \neq 0$?

Chiral symmetry restoration: $\mu_\chi = .11\sqrt{\sigma}$
Transition *second* order: not evident.
Also: all infrared divergences *cancel*.

No nuclear matter:
restore chiral symmetry before Fermi sea forms



Asymptotically large μ, grows with N_c

For $\mu \sim (N_c)^p$, $p > 0$, gluons feel the effect of quarks. Perturbatively,

$$P_{\rm pert.}(\mu, T) \sim N_c N_f \; \mu^4 \; F_0 \; , \; N_c N_f \; \mu^2 \; T^2 \; F_1 \; , \; N_c^2 \; T^4 \; F_2 \; .$$

First two terms from quarks & gluons, last only from gluons. Two regimes:

$\mu \sim N_c^{1/4} \Lambda_{QCD}$: $N_c \; \mu^4 \; F_0 \sim N_c^2 \; F_2 \sim N_c^2 >> N_c \; \mu^2 \; F_1 \sim N_c^{3/2}$.

Gluons & quarks contribute equally to pressure; quark cont. T-independent.

$\mu \sim N_c^{1/2} \; \Lambda_{QCD}$: New regime: $m^2_{\rm Debye} \sim g^2 \; \mu^2 \sim 1$, so gluons feel quarks.

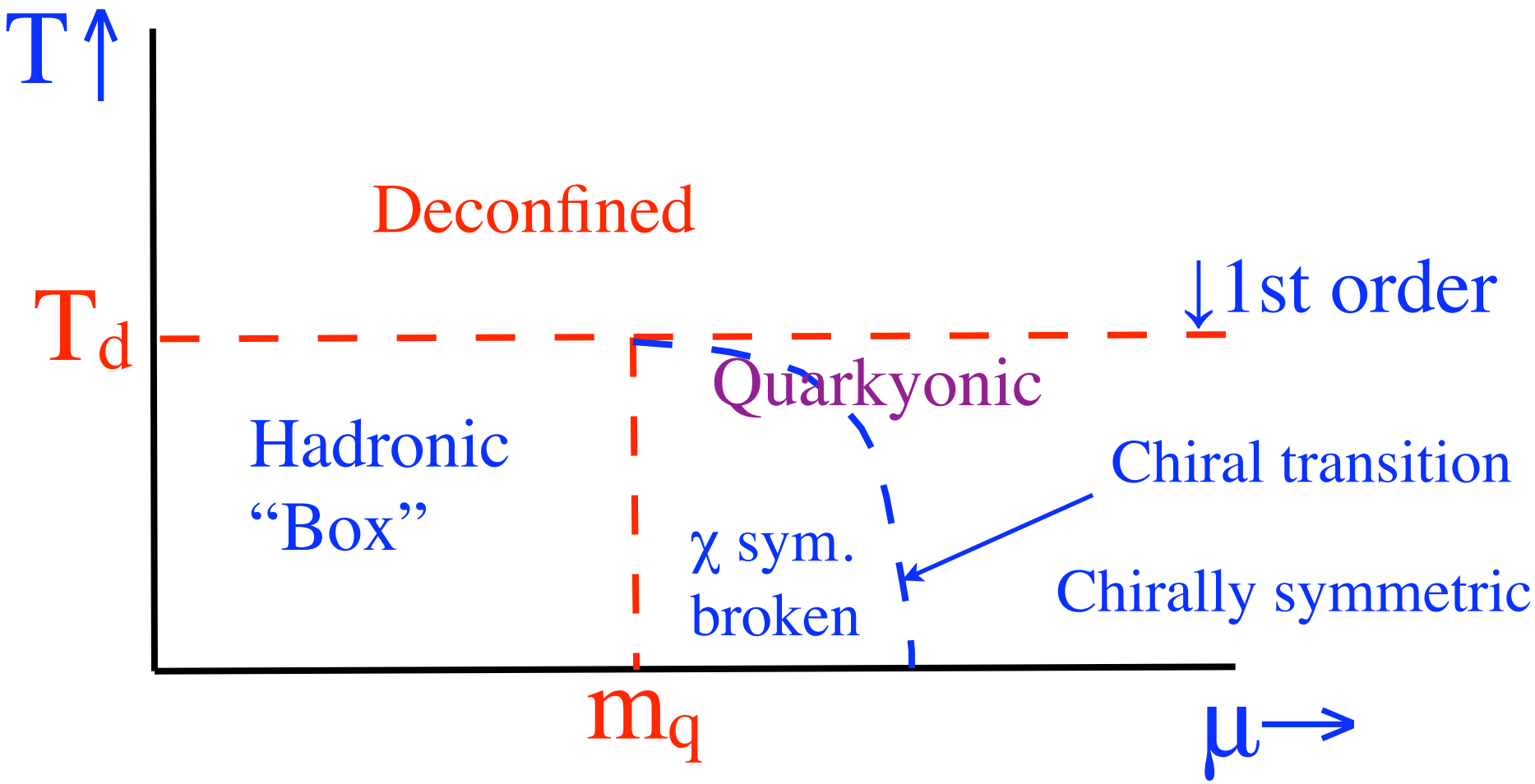
$N_c \; \mu^4 \; F_0 \sim N_c^3 >> N_c \; \mu^2 \; F_1$, $N_c^2 \; F_2 \sim N_c^2$.

Quarks dominate pressure, T-independent.

Eventually, first order deconfining transition can either:

end in a critical point, or bend over to $T = 0$: ?

Phase diagram at $N_c = \infty$, II



We suggest: quarkyonic phase includes chiral trans. Order by usual arguments.

Mocsy, Sannino & Tuominen hep-th/0308135:

splitting of transitions in effective models

But: quarkyonic phase *confined*. Chirally symmetric baryons?

Chirally symmetric baryons: parity doubling

B. Lee, '72; DeTar & Kunihiro '89; Jido, Oka & Hosaka, hep-ph/0110005; Zschiesche et al nucl-th/0608044. Wigner-Weyl mode: *two* baryon multiplets. One usual nucleon, + parity. Other state with - parity. *Perhaps* N*(1535) Transform *opposite* under chiral transformations:

$$\psi_{L,R} \to U_{L,R}\,\psi_{L,R} \;\; ; \;\; \chi_{L,R} \to U_{R,L}\,\chi_{L,R}$$

Chirally symmetric mass

$$m_0\left(\overline{\psi}_L\,\chi_R - \overline{\psi}_R\,\chi_L + \overline{\chi}_R\,\psi_L - \overline{\chi}_L\,\psi_R\right)$$

Also: usual sigma field, $\Phi \to U_L\,\Phi\,U_R^\dagger$,

$$g_1\,\overline{\psi}_L\,\Phi\,\psi_R + g_2\,\overline{\chi}_R\,\Phi\,\chi_L$$

In component form:

$$g_1\,\psi(\sigma + i\gamma_5\overline{\tau}\cdot\overline{\pi})\psi + g_2\,\overline{\chi}(\sigma - i\gamma_5\overline{\tau}\cdot\overline{\pi})\chi$$

N.B.: difference in signs for axial coupling between two fields.

Parity doubled baryons

Letting $< \sigma > = \sigma_0$, diagonalize

$$\begin{pmatrix} g_1\sigma_0 & m_0\gamma_5 \\ -m_0\gamma_5 & g_2\sigma_0 \end{pmatrix} \qquad m_{N^\pm} = \frac{1}{2}\left(\sqrt{(g_1+g_2)^2\sigma_0^2 + 4m_0^2} \mp (g_1-g_2)\sigma_0\right)$$

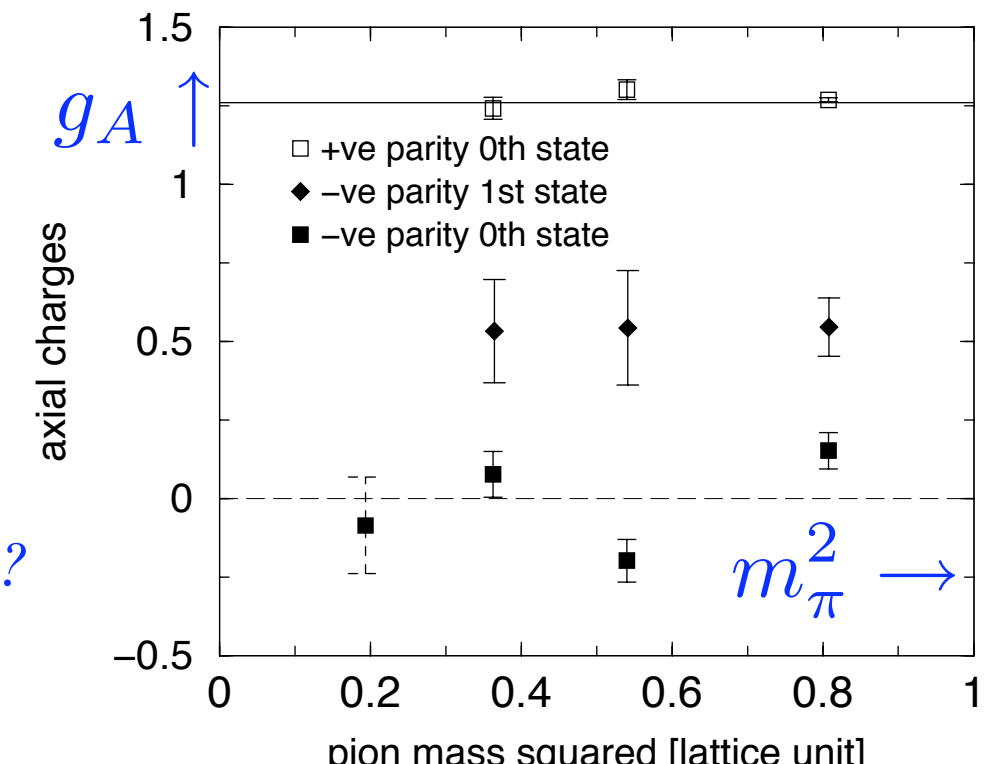
Looks ok: when $\sigma_0 = 0$, states degenerate, with mass m_0.
Can arrange couplings so that $g_1 \sim 13$, $g_2 \sim 3$ are natural; m_0 small, ~ 300 MeV.

But: as parity partner, g_A should be +1 for N, -1 for parity partner.

Takahashi & Kunihiro 0801.4707:
lattice, two flavors dynamical quarks, relatively heavy

$g_A \sim 0$ for parity partner!

Where is the parity partner of the nucleon?



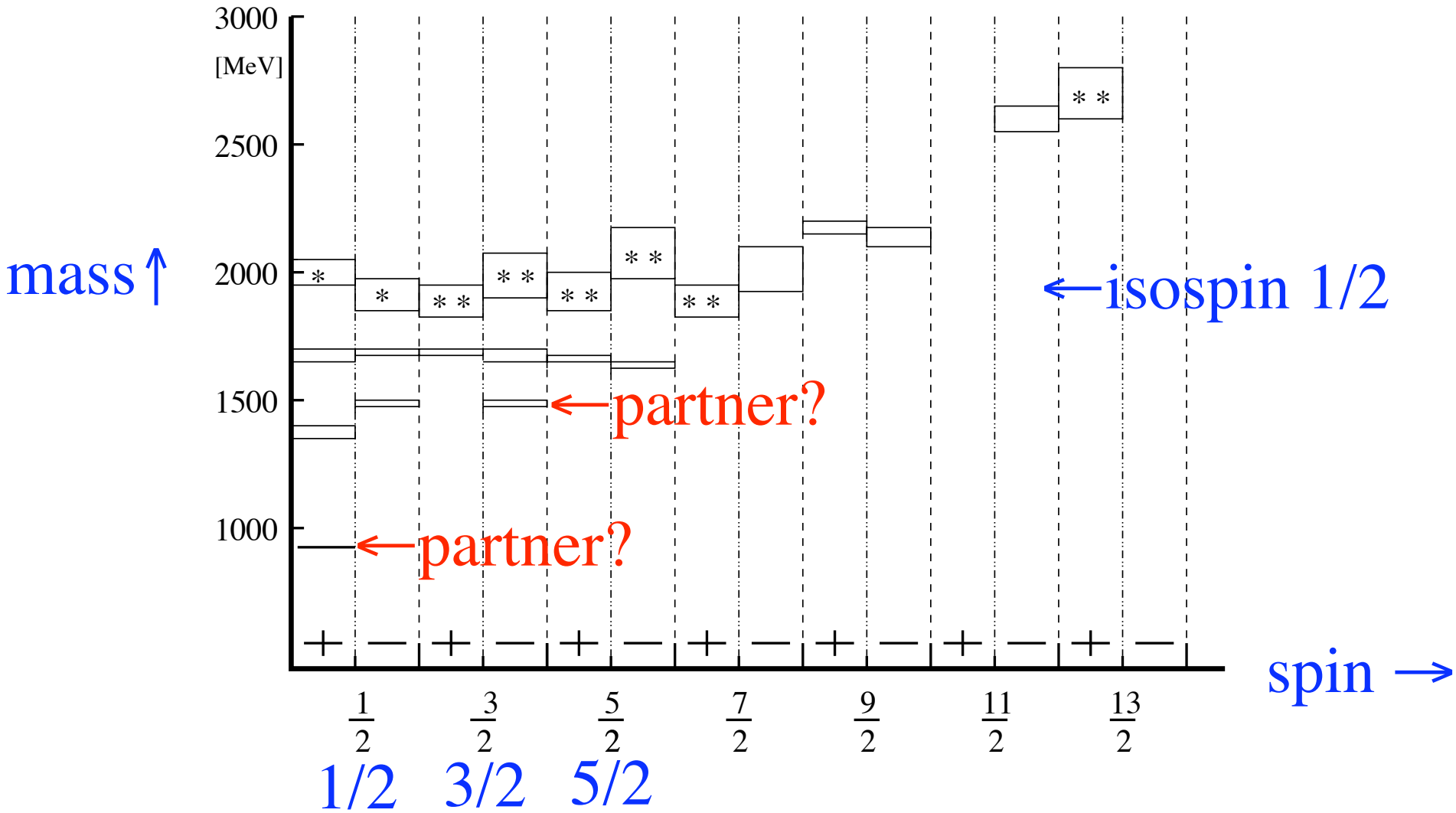
Parity partners in vacuum?

Glozman hep-ph/0701081: Look at baryon mass spectrum in vacuum.

Find *all* states have parity partners *except* the lightest: e.g., nucleon, Δ...

So: either the N & Δ have *no* partner. Or: they are *very* heavy.

Lighest states are clearly special!



‘t Hooft anomaly condition

‘t Hooft ‘79; Coleman & Witten ‘80

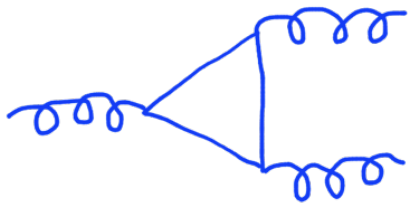
A = flavor matrix, current J:

$$J^\mu_\pm = \overline{\psi}\, A_\pm (1 \pm \gamma_5)\gamma^\mu\, \psi$$

Construct 3-pt fnc, J = $J_+ + J_-$.

$$\Gamma^{\mu\nu\sigma}(P_1, P_2) = \langle J^\mu(P_1) J^\nu(P_2) J^\sigma(Q)\rangle$$

Axial anomaly:



$$Q^\sigma\, \Gamma^{\mu\nu\sigma} = -\frac{g^3 N_c}{\pi^2}\, \mathrm{tr}(A_+^3 - A_-^3)\, \epsilon^{\mu\nu\sigma\kappa}\, P_1^\sigma P_2^\kappa$$

With chiral symmetry breaking, pions work, $\Gamma^{\mu\nu\sigma} \sim Q^\sigma / Q^2$

Massless baryons possible? Define currents as

$$\langle u | J^\mu_\pm | u\rangle = \overline{u}\, \gamma^\mu B_\mu (1 \pm \gamma_5) u$$

Must satisfy same anomaly condition:

$$\mathrm{tr}(B_+^3 - B_-^3) = N_c\, \mathrm{tr}(A_+^3 - A_-^3)$$

Generally, *very* hard to do.

Anomaly is due to ultraviolet effects, and so unaffected by T, μ.

Thus effects of the anomaly are unchanged by a medium? *No*.

Anomalies in vacuum

Usually, relate anomalies to pions. Adler '69: can compute $\pi^0 \to \gamma\gamma$.
Let V = vector current, couples to photons. Conserved.
A = axial vector current, couples to pions. Isospin-3 comp. = axial anomaly:

$$\partial_\mu V^\mu = 0 \quad ; \quad \partial_\mu A^\mu = -\frac{e^2 N_c}{48\pi^2}\, F_{\mu\nu}\widetilde{F}^{\mu\nu}$$

Consider "VVA":

$$\Gamma^{\mu\nu\sigma}(P_1, P_2) = \langle V^\mu(P_1) V^\nu(P_2) A^\sigma(Q)\rangle \; ; \; -Q = P_1 + P_2$$

This satisfies

$$P_1^\mu\, \Gamma^{\mu\nu\sigma} = P_2^\nu\, \Gamma^{\mu\nu\sigma} = 0 \qquad\qquad Q^\sigma\, \Gamma^{\mu\nu\sigma} = -\frac{e^2 N_c}{12\pi^2}\, \epsilon^{\mu\nu\sigma\kappa}\, P_1^\sigma P_2^\kappa$$

Construct

$$\mathcal{T}^{\mu\nu} = e^2 Q^2 \langle V^\mu(P_1) V^\nu(P_2)\pi\rangle \qquad\qquad \epsilon_1^\mu \epsilon_2^\nu\, \mathcal{T}^{\mu\nu} = g_{\pi\gamma\gamma}\, \epsilon^{\mu\nu\sigma\kappa} \epsilon_1^\mu \epsilon_2^\nu P_1^\sigma P_2^\kappa$$

and take out the one pion pole term, using PCAC:

$$\langle 0|A^\sigma|\pi\rangle = iQ^\sigma f_\pi \qquad\qquad \widetilde{\Gamma}^{\mu\nu\sigma} = \Gamma^{\mu\nu\sigma} + f_\pi \frac{Q^\sigma}{Q^2}\mathcal{T}^{\mu\nu}$$

Anomalies in vacuum, cont.'d

Most general amplitude which is Bose symmetric under photon exchange, $P_1\ \mu \leftrightarrow P_2\ \nu$. Also Lorentz invariant.

$$\widetilde{\Gamma}^{\mu\nu\sigma} = T_1 \epsilon^{\mu\nu\sigma\kappa}(P_1 - P_2)^\kappa + T_2(\epsilon^{\mu\sigma\kappa\lambda} P_2^\nu - \epsilon^{\nu\sigma\kappa\lambda} P_1^\mu) P_1^\kappa P_2^\lambda$$
$$+ T_3(\epsilon^{\mu\sigma\kappa\lambda} P_1^\nu - \epsilon^{\nu\sigma\kappa\lambda} P_2^\mu) P_1^\kappa P_2^\lambda$$

Current conservation: $P_1^\mu\ \widetilde{\Gamma}^{\mu\nu\sigma} = P_2^\nu\ \widetilde{\Gamma}^{\mu\nu\sigma} = 0$

$$T_1 + P_1^2\ T_2 + P_1 \cdot P_2\ T_2 = 0$$

Axial anomaly:

$$Q^\sigma\ \widetilde{\Gamma}^{\mu\nu\sigma} = f_\pi \mathcal{T}^{\mu\nu} - \frac{e^2 N_c}{12\pi^2}\ \epsilon^{\mu\nu\sigma\kappa}\ P_1^\sigma P_2^\kappa$$

$$-2T_1 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}$$

Now consider on shell photons and pions: $P_1^2 = P_2^2 = Q^2 = 0$.

Then $T_1 = 0$, which relates pion-photon coupling as:

$$f_\pi\ g_{\pi\gamma\gamma} = \frac{e^2 N_c}{12\pi^2}$$

$T_1 = 0$ is Sutherland-Veltman theorem.

Anomalies in medium

Itoyama & Mueller'83; RDP, Trueman & Tytgat hep-ph/9702362:

Now do the same in a medium. Just have extra vector for rest frame, n^μ

$$\widetilde{\Gamma}^{\mu\nu\sigma} = T_1 \epsilon^{\mu\nu\sigma\kappa}(P_1 - P_2)^\kappa + T_2(\epsilon^{\mu\sigma\kappa\lambda} P_2^\nu - \epsilon^{\nu\sigma\kappa\lambda} P_1^\mu) P_1^\kappa P_2^\lambda$$
$$+ T_3(\epsilon^{\mu\sigma\kappa\lambda} P_1^\nu - \epsilon^{\nu\sigma\kappa\lambda} P_2^\mu) P_1^\kappa P_2^\lambda$$
$$+ T_4\, n\cdot Q\, \epsilon^{\mu\nu\kappa\lambda} P_1^\kappa P_2^\lambda\, n^\sigma + T_5(n\cdot P_2\, \epsilon^{\mu\sigma\lambda\kappa} n^\nu - n\cdot P_1\, \epsilon^{\nu\sigma\lambda\kappa} n^\mu) P_1^\lambda P_2^\kappa$$

Current conservation:

$$T_1 + P_1^2\, T_2 + P_1\cdot P_2\, T_3 + (n\cdot P_1)^2\, T_5 = 0$$

Axial anomaly:

$$-2T_1 + (n\cdot Q)^2 T_4 = f_\pi g_{\pi\gamma\gamma} - \frac{e^2 N_c}{12\pi^2}$$

On the mass shell:

$$f_\pi\; g_{\pi\gamma\gamma} = \frac{e^2 N_c}{12\pi^2} + (n\cdot Q)^2 T_4 + 2(n\cdot P_1)^2 T_5$$

Here f_π and $g_{\pi\gamma\gamma}$ are functions of temperature, density....

The axial anomaly does relate these to amplitudes, but they don't vanish, even on mass shell. Sutherland-Veltman *fails* in a medium.

Verified by explicit calculation to one loop order: at $T \neq 0$, $T_4 = 0$, $T_5 \neq 0$

Banks-Casher and quarkyonic

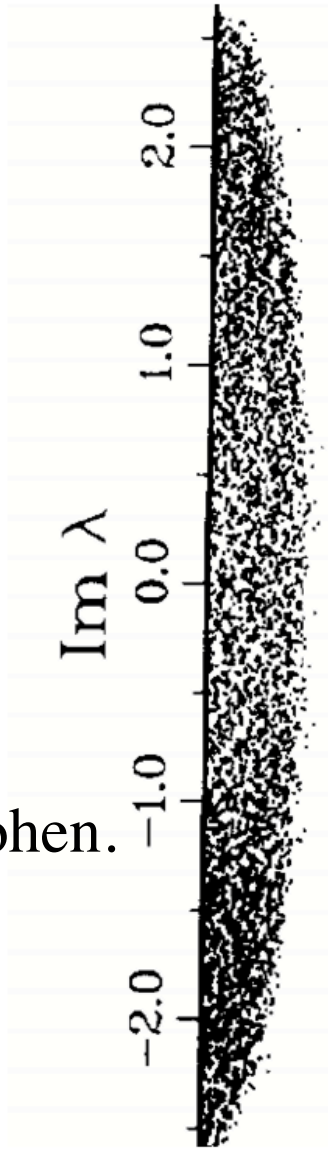
Splittorff & Verbaarschot 0809.5259; Osborn, Splittorff & Verb. 0807.4584...

Use random matrix theory to study the sign problem.
Valid for small box, epsilon regime.
In vacuum, eigenvalues of D purely imaginary:

$$\not{D}\psi = \lambda\psi \;\; ; \;\; \langle\overline{\psi}\psi\rangle = \langle\frac{1}{\not{D}+m}\rangle = \int d\lambda \, \frac{\rho(\lambda)}{\lambda+m}$$

Banks-Casher: $\langle\overline{\psi}\psi\rangle \sim \rho(0)$

At $\mu \neq 0$, eigenvalues of D + i $\mu\,\gamma^0$ complex, spread out into complex plane, as at right.



Epsilon regime shows how to solve "Silver Blaze" problem of Cohen.

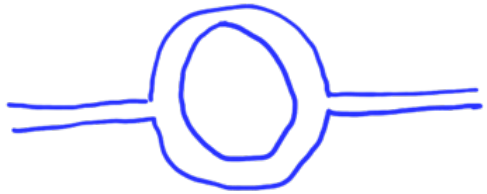
Trivial point: width of distribution determined by μ
Thus for large μ, should obtain chirally symmetric distribution;
i.e., quarkyonic phase

Baryons at Large N_f

Veneziano '78: take *both* N_c and N_f large. Mesons M^{ij} : i,j = 1...N_f .

Thus mesons interact weakly, but there are *many* mesons.

Thus in the hadronic phase, mesons interact *strongly*:



$$\Pi \sim N_f \, g_{3\pi}^2 \sim N_f / N_c$$

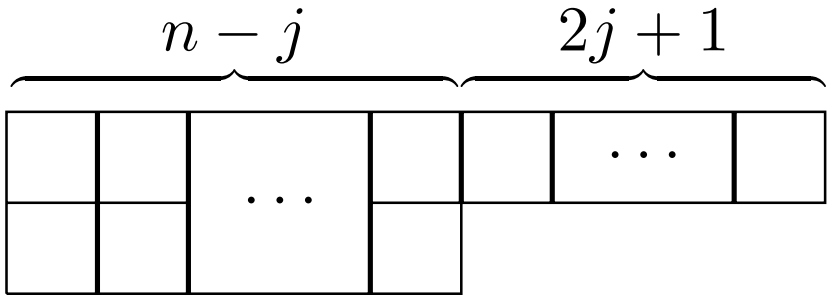
Pressure large in *both* phases:

$\sim N_f^2$ in hadronic phase, $\sim N_c^2$, $N_c N_f$ in "deconfined" phase.

Polyakov loop also nonzero in both phases.

Baryons: lowest state with spin j

has Young tableaux ($N_c = 2n + 1$) =>



$$d_j = \frac{(2j+2)\,(N_f+n+j)!\,(N_f+n-j-2)!}{(N_f-1)!\,(N_f-2)!\,(n+j+2)!\,(n-j)!}$$

Baryons at Large N_f: order parameters

Y. Hidaka, RDP, & L. McLerran, 0803.0279: Use Sterling's formula,

$$d_j \sim \mathrm{e}^{+N_c\, f(N_f/n)}\ ,\ f(x) = (1+x)\log(1+x) - x\log(x)$$

Degeneracy of baryons increases *exponentially*.

Argument is heuristic: baryons are strongly interacting.
Still, difficult to see how interactions can overwhelm exponentially growing spectrum, even for the lowest state.

Use *baryons* as order parameter. At T=0, fluctuations in baryon number,
$\langle B^2\rangle \neq 0$ when $N_c\, f(N_c/n) = m_B/T$, or

$$T_{qk} = f(N_f/n)\,\frac{m_B}{N_c}$$

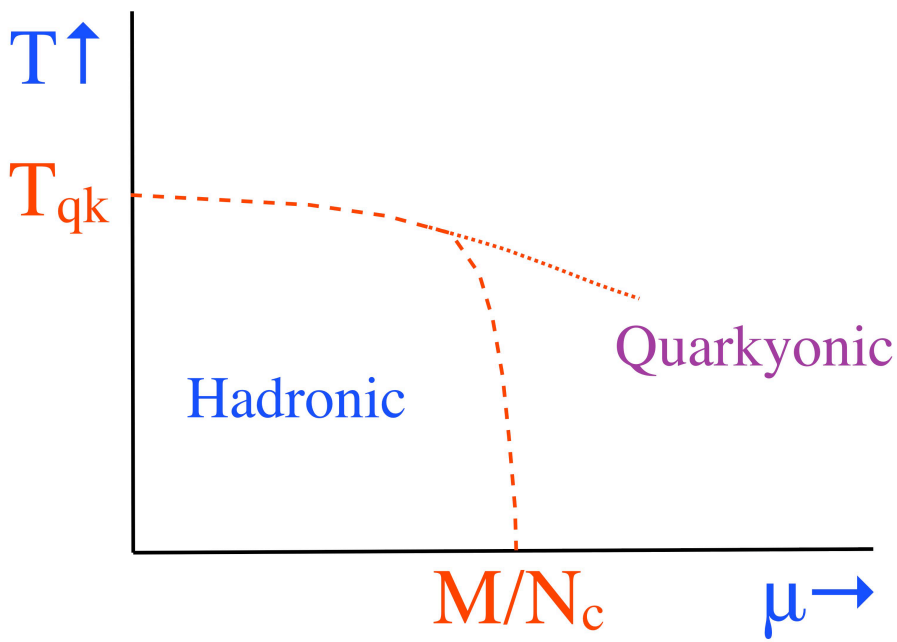
At $\mu \neq 0$, baryon number itself:
$\langle B\rangle \neq 0$ when $N_c\, f(N_c/n) = (m_B - N_c\,\mu)/T$:

$$T_{qk} = f(N_f/n)\left(\frac{m_B}{N_c} - \mu\right)$$

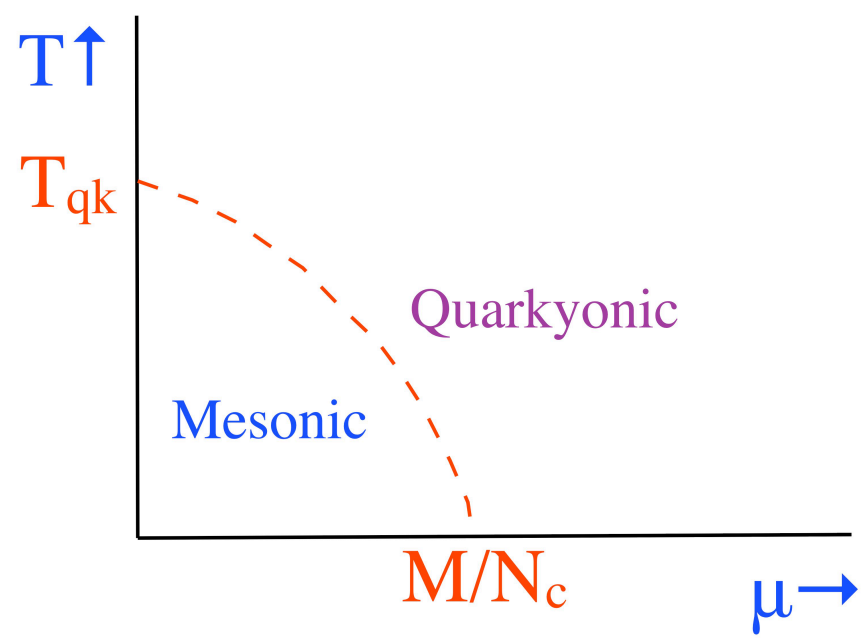
Possible phase diagrams as N_f increases

The "rectangle" for small N_f becomes smoothed.
Eventually, maybe the quarkyonic line merges with that for baryon condensation.
All *VERY* qualitative. Clearly many possible phase diagrams!
With SUSY: condensation of Higgs fields as well.

Small N_f



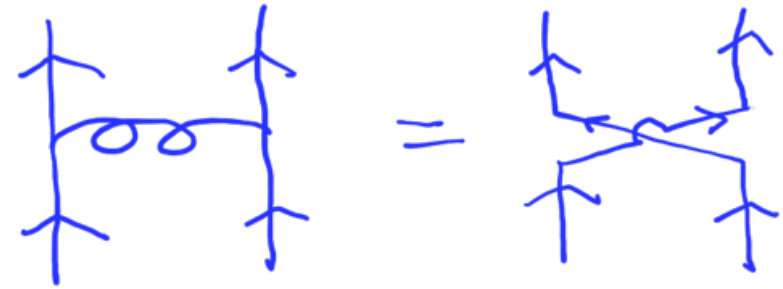
Large N_f



Chiral Density Waves (perturbative)

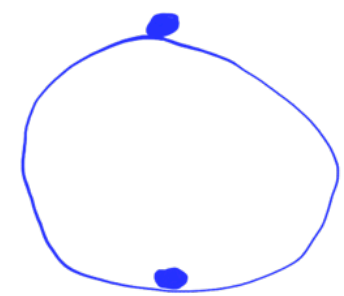
Excitations near the Fermi surface?

At large N_c, color superconductivity suppressed,
$\sim 1/N_c$: pairing into two-index state:



Also possible to have "chiral density waves", pairing of quark and anti-quark:
Deryagin, Grigoriev, & Rubakov '92. Shuster & Son, hep-ph/9905448.
Rapp, Shuryak, and Zahed, hep-ph/0008207.

Order parameter $\langle \overline{\psi}(-\vec{p}_f)\, \psi(+\vec{p}_f) \rangle$
Sum over color, so *not* suppressed by $1/N_c$.



Pair quark at + p_f with anti-quark at - p_f : for a *fixed* direction.
Breaks chiral symmetry, with state varying $\sim \exp(-2\, p_f\, z)$.

Wins over superconductivity in low dimensions. Loses in higher.
Shuster & Son '99: in perturbative regime, CDW only wins for $N_c > 1000\, N_f$

Quarkyonic chiral density waves

Consider meson wave function, with kernel:
Confining potential in 3+1 dimensions like
Coulomb potential in 1+1 dim.s:

$$\int dk_0\, dk_z \int d^2k_\perp \frac{1}{(k_0^2+k_z^2+k_\perp^2)^2} \sim \int dk_0\, dk_z \frac{1}{k_0^2+k_z^2}$$

In 1+1 dim.'s, behavior of massless quarks near Fermi surface maps ~ $\mu = 0$!
Mesons in vacuum naturally map into CDW mesons.
Witten '84: in 1+1 dim.'s, use non-Abelian bosonization for QCD.
a, b= 1...N_c. i,j = 1... N_f.

$$J_+^{ij} = \overline{\psi}^{a,i}\psi^{a,j} \sim g^{-1}\partial_+ g\ ;\ J_+^{ab} = \overline{\psi}^{a,i}\psi^{b,i} \sim h^{-1}\partial_+ h\ .$$

Steinhardt '80. Affleck '86. Frishman & Sonnenschein, hep-th/920717...
Armoni, Frishman, Sonnenschein & Trittman, hep-th/9805155; AFS, hep-th/0011043..
Bringoltz 0901.4035; Galvez, Hietanan, & Narayanan, 0812.3449.

Bosonized quarkyonic matter

After non-Abelian bosonization, action factorizes into sum of g, in SU(N_f), and h, in SU(N_c). Action for g is

$$8\pi\, S_{WZW} = \int d^2z\, \mathrm{tr}\, B_i^2 + 2/3 \int d^3y\, \epsilon^{ijk}\, \mathrm{tr}\, B_i B_j B_k \ , \ B_i = g^{-1}\partial_i g \ .$$

Action for h, is a SU(N_c) gauged WZW model. But: g and h *decouple*!
Spectrum of h complicated, involves massive modes, like usual 't Hooft model.

Spectrum of g is that of usual WZW model, with *massless* modes.

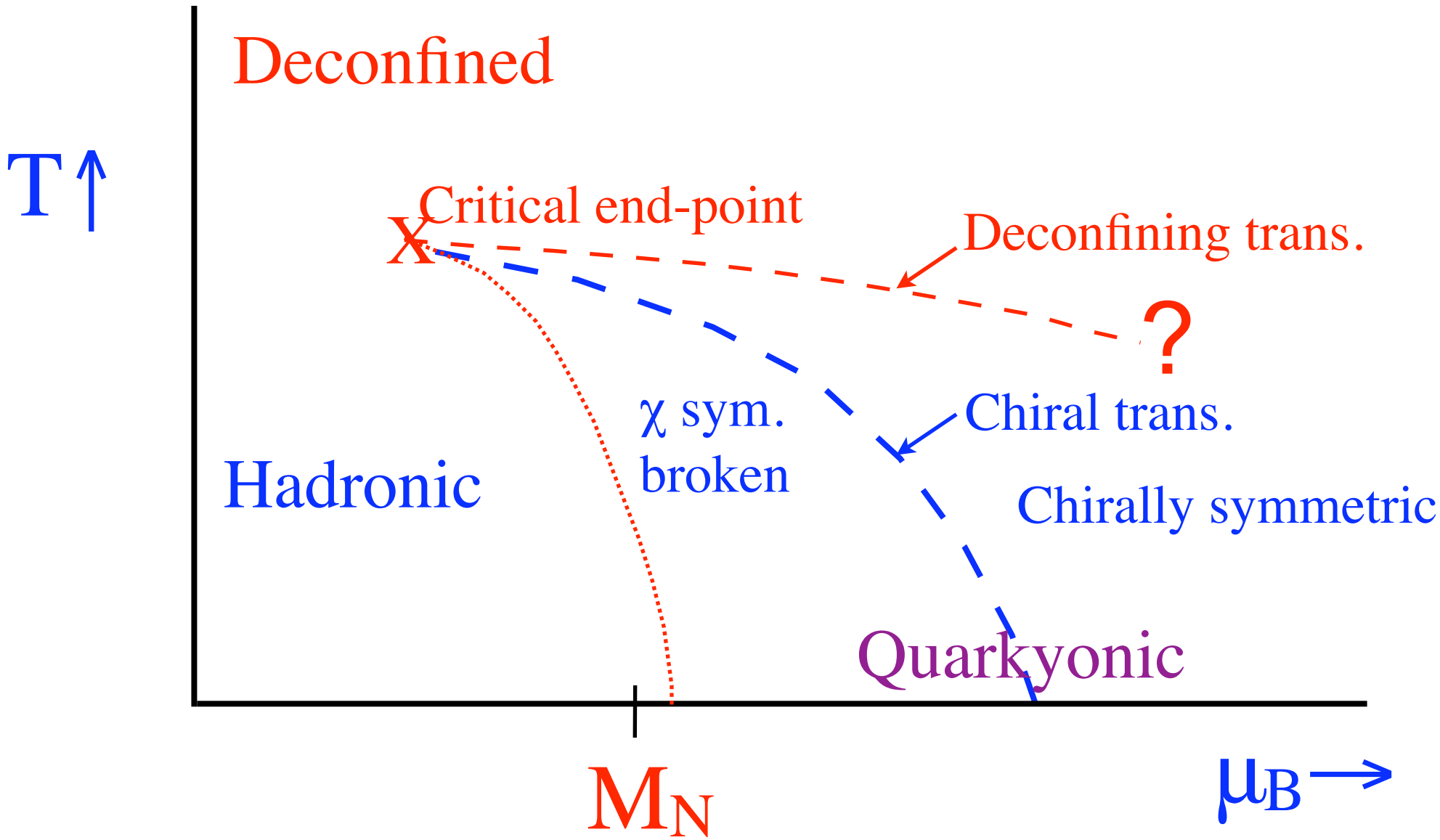
Hence in 1+1 dim.'s, CDW are natural, but with *massless* excitations thereof.

In 3+1 dim.'s: have highly anisotropic state, *somen*-state:
Y. Hidaka, T. Kojo, L. McLerran, & RDP '09...

Chiral condensate ~ Λ_{QCD}^2/μ^2. Length of somen-state large, ~ exp(N_c).
Quantum fluctuations tend to scramble the somen.

Guess for phase diagram in QCD

Pure guesswork: deconfining & chiral transitions split apart at critical end-point?
Line for deconfining transition first order to the right of the critical end-point?
Critical end-point for deconfinement, or continues down to T=0?



6. ‘t Hooft basis of SU(N)

Drawing “birdtrack” diagrams,
or group theory made easy

Peculiar basis for two colors

Start with the usual Cartan basis.

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = t^{12} \;\; ; \;\; \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = t^{21} \;\; ; \;\; \sigma^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now *add* one *extra* diagonal generator, $-\sigma^3$, in *addition* to σ^3. Why?
Off-diagonal gen.'s: t^{12} and t^{21} labels the row and column of one nonzero element.
Can one do the same for the diagonal generator(s)? More or less...:

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t^{11} \; ; \; -\sigma^3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t^{22}$$

Usual relations for off-diagonal generators:

$$\mathrm{tr}(t^{12}t^{21}) = 1 \; ; \; \mathrm{tr}(t^{12}t^{11}) = \mathrm{tr}(t^{21}t^{11}) = \ldots = 0$$

But *not* for the diagonal generators:

$$\mathrm{tr}(t^{11})^2 = \mathrm{tr}(t^{22})^2 = \frac{1}{2} \;\; ; \;\; \mathrm{tr}(t^{11}t^{22}) = -\frac{1}{2}$$

Basis is *not* an orthonormal, since we added one *extra* generator.

" 't Hooft" basis for SU(N)

Off diagonal generators same as Cartan basis. *One* diagonal generator same:

$$t^{NN} = \frac{1}{N}\begin{pmatrix} -\mathbf{1}_{N-1} & 0 \\ 0 & N-1 \end{pmatrix} = \begin{pmatrix} 0_{N-1} & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{N}\mathbf{1}_N$$

Unlike Cartan, construct N diagonal generators by permutations of diagonal element above:

$$t^{11} = \frac{1}{N}\begin{pmatrix} N-1 & 0 \\ 0 & -\mathbf{1}_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0_{N-1} \end{pmatrix} - \frac{1}{N}\mathbf{1}_N$$

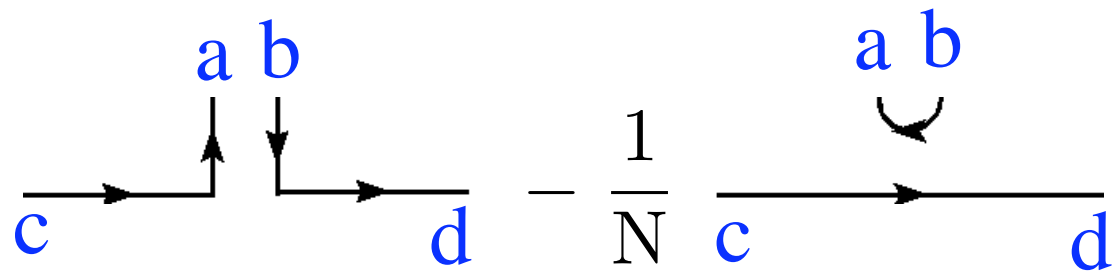
$$t^{22} = \frac{1}{N}\begin{pmatrix} -1 & 0 & 0 \\ 0 & N-1 & 0 \\ 0 & 0 & -\mathbf{1}_{N-2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0_{N-2} \end{pmatrix} - \frac{1}{N}\mathbf{1}_N$$

For N=2, $t^{22} = -\sigma^3$. Now have N diagonal generators, t^{11}, t^{22}...t^{NN}: *one* extra.
Advantage over Cartan: no preferred direction in group space.
Disadvantage: diagonal generators are not orthonormal.
(Off-diagonal generators satisfy usual orthonormality relations.)

Birdtrack diagrams for the ‘t Hooft basis

Let a,b,c,d... = 1...N represent indices for fundamental representation.
Denote adjoint indices by a pair (ab): N^2 of these!
Then the generators can be represented simply by a “birdtrack” diagram:

$$(t^{ab})_{cd} = \delta^a_c\,\delta^b_d - \frac{1}{N}\,\delta^{ab}\,\delta_{cd}$$



First term is the single element of a matrix. The second term is the identity matrix
What is useful is that the generators are obviously *projectors*. Thus the product of two generators is again a projector:

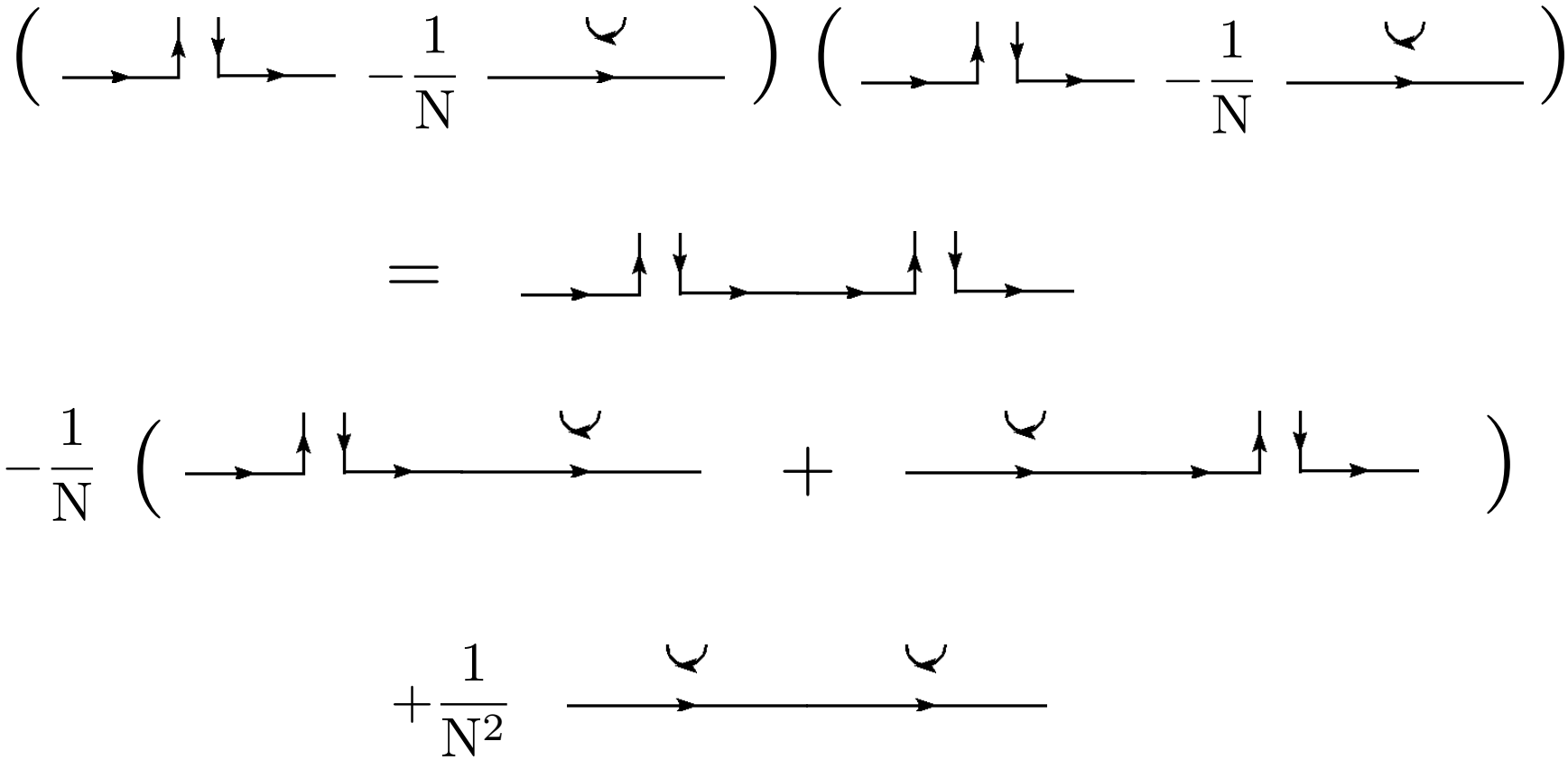
$$\mathrm{tr}(t^{ab}t^{cd}) = (t^{ab})_{dc} \quad : \qquad \mathrm{tr}(t^{ab}\,t^{ba}) = 1\ ,\ a \neq b$$

$$\mathrm{tr}(t^{aa})^2 = 1 - \frac{1}{N} \quad ; \quad \mathrm{tr}(t^{aa}\,t^{bb}) = -\frac{1}{N}\ , a \neq b$$

Raise and lower indices by flipping order; automatic with directed lines.
Normalization of the diagonal generators just reflects a *projector*.

Group identities from birdtracks

$(t^{ab})_{eg}\,(t^{cd})_{gf} =$

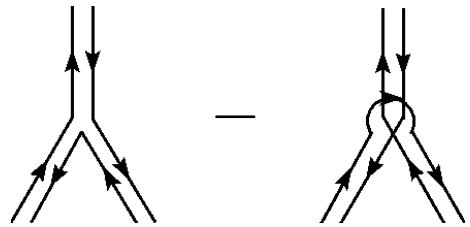


Just by drawing arrows, can show the standard relation:

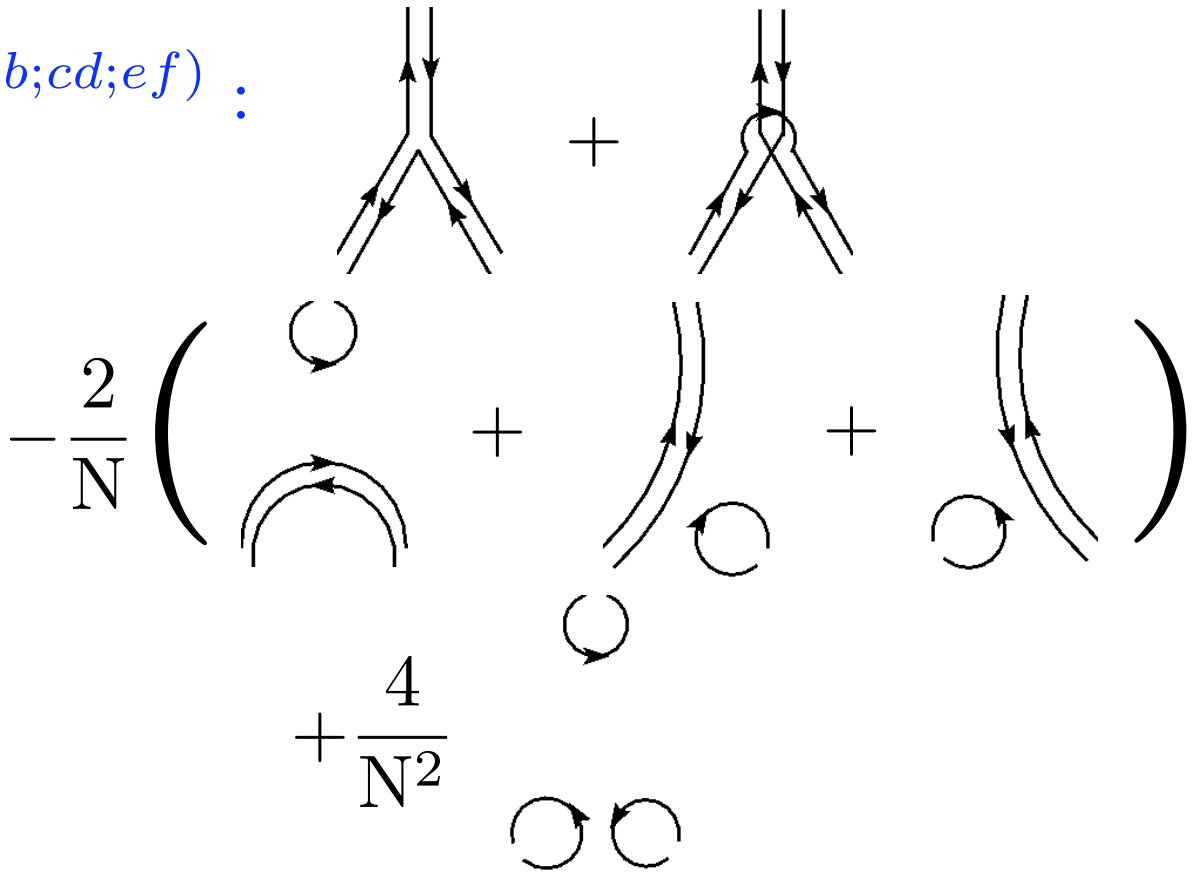
$$\sum_{a,b=1}^{N} (t^{ab}\,t^{ba})_{cd} = \frac{N^2-1}{N}\,\delta_{cd}$$

Birdtrack structure constants

$f^{(ab;cd;ef)}$:



$d^{(ab;cd;ef)}$:



Can derive arbitrary group identities just by doodling.

P. Cvitanovic, http://www.birdtracks.edu

Arbitrary group representations follow similarly, for any group.

N.B.: extracting SU(N) structure constants from U(N).

Useful in a *practical* sense.

6. Effective theory, near T_c, in Euclidean spacetime

Effective theory for large A_0

Lattice shows <*loop*> $\neq 1$ near T_c => large $A_0 \sim T/g$. *Effective theory for large A_0?*

For small A_0, effective theory is just QCD_3 + massive (adjoint) scalar, A_0:

$$\mathcal{L}^{eff} = \frac{1}{2}\,\mathrm{tr}\,G_{ij}^2 \;+\; \mathrm{tr}\,|D_i A_0|^2 + \; m_D^2\,\mathrm{tr}\,A_0^2 \;+\; \kappa\,\mathrm{tr}\,A_0^4$$

Symmetries for large A_0? Certainly, invariance under static gauge transf.'s.

Plus: "large" gauge transformations - spatially constant, time *de*pendent. For SU(N):

$$\Omega(\tau) = \mathrm{e}^{2\pi i \tau T\, t^{NN}} \quad , \quad t^{NN} = \frac{1}{N}\begin{pmatrix} \mathbf{1}_{N-1} & 0 \\ 0 & -(N-1) \end{pmatrix}$$

This $\Omega(\tau)$ is *only* valid c/o quarks: $\Omega(1/T) = e^{2\pi i/N}\,\Omega(0)$: center symmetry

With quarks, consider *strictly* periodic transf: $\Omega_p(\tau) = \Omega^N(\tau)$.

All theories must respect invariance under such *strictly* periodic gauge transf.'s.

For any gauge group, with any matter fields; even for QED.

Effective electric field?

Want 3D effective thy. for large $A_0 \sim T/g$.

Valid for $r > 1/T$, so A_0 varies slowly in space, momenta $p < T$.

Original electric field $E_i^{4D} = D_i A_0 - \partial_0 A_i$. So: $E_i^{3D} = D_i A_0$?

For large gauge transf. $\Omega_p(\tau)$:

$$A_0^{diag} \to A_0^{diag} + \frac{2\pi T}{g} N t^{NN} \ , \ A_i \to \frac{1}{-ig} \Omega_p^\dagger(\tau) A_i \Omega_p(\tau)$$

Constant shift in A_0 , time *de*pendent rotation of A_i .

$D_i A_0 = (\partial_i - i g [A_i ,) A_0$ *not* invariant for A_i^{aN} as $[t^{aN}, t^{NN}] \neq 0$.

Of course, E_i^{4D} *invariant* under $\Omega_p(\tau)$.

$E_i^{3D} = D_i A_0$ at small A_0, but *not* at large A_0!

Periodicity in $q \to q+1$ violated: true only for kinetic, *not* potential terms.

Diakonov & Oswald hep-ph/0303129; hep-ph/0312126; hep-ph/0403108

Megias, Arriola, & Salcedo, hep-ph/0312133

Form E_i^{3D} from Wilson lines?

Electric field of Wilson lines

Wilson line SU(N) matrix, so diagonalize: $\mathbf{L}(x) = \Omega(x)^\dagger e^{i\lambda(x)}\ \Omega(x)$

Static gauge transf.'s: diagonal matrix λ invariant, Ω changes.

Strictly periodic $\Omega_p(\tau)$: $\lambda_a \to \lambda_a + 2\pi \times$ integer: λ_a periodic.

Use just eigenvalues, $E_i^{3D} \sim \partial_i \lambda$? No, $E_i^{3D} \neq D_i A_0$ at small A_0

E_i^{3D} hermitean, so: $$E_i^{3D}(x) = \frac{T}{ig}\mathbf{L}^\dagger(x) D_i \mathbf{L}(x)(1 + c_1|\mathrm{tr}\mathbf{L}|^2 + \ldots)$$

Small A_0 OK, but does *not* fix c_1, c_2...

Large but *abelian* A_0, $A_i = 0$: if $E_i^{3D} = \partial_i A_0$, *must* have $c_1 = c_2 = \ldots = 0$.

Necessary for interfaces to match at *leading* order. Beyond: c_1, c_2 ... $\sim g^2$.

In general, *infinite* number of terms enter.

Calculable perturbatively, match through interfaces, Z(N) or U(1).

$L_{\rm eff}$ of Wilson lines at 0th order

To leading order, $E_i^{3D} = \frac{T}{ig}\, \mathbf{L}^\dagger\, D_i\, \mathbf{L}$

Gauge covariant "average" in time: $\mathbf{L}(\tau) = e^{ig\int_0^\tau A_o(\tau')d\tau'}$; $\mathbf{L} = \mathbf{L}(1/T)$

$$E_i^{3D}/T = \int_0^{1/T} d\tau\, \mathbf{L}(\tau)^\dagger\, \partial_i A_0(\tau)\, \mathbf{L}(\tau) - \mathbf{L}^\dagger [A_i, \mathbf{L}]$$

Lagrangian continuum form of Banks and Ukawa '83, on lattice:

$$\mathcal{L}_{cl}^{eff} = \frac{1}{2}\, \mathrm{tr}\, G_{ij}^2 + \frac{T^2}{g^2}\, \mathrm{tr}\, |\mathbf{L}^\dagger D_i \mathbf{L}|^2$$

To 0th order, Lagrangian for SU(N) principal chiral field.
Non-renormalizable in 3D, but only effective theory for r > 1/T.
Instanton number in 4D = winding number of **L** in 3D

Linear model: *many* more terms. Interfaces match *approximately*, not exactly.

Vuorinen & Yaffe hep-ph/0604100. Kurkela, 0704.1416.
de Forcrand, Kurkela, & Vuorinen, 0801.1566. Korthals-Altes 0810.3325

Confinement & adjoint Higgs phase?

Diagonalize $L = \Omega^\dagger e^{i\lambda} \Omega$

Static gauge transf.'s U: $e^{i\lambda}$ invariant, Ω not: $\Omega \to \Omega \mathcal{U}\ ,\ D_i \to \mathcal{U}^\dagger D_i \mathcal{U}$

Electric field term:

$$\mathrm{tr}\,|\mathbf{L}^\dagger D_i \mathbf{L}|^2 \;=\; \mathrm{tr}\,(\partial_i \lambda)^2 \;+\; \mathrm{tr}\,|[\Omega\, D_i\, \Omega^\dagger, \mathrm{e}^{i\lambda}]|^2$$

1st term same as abelian

2nd term gauge *in*variant coupling of electric & magnetic sectors

$\langle e^{i\lambda} \rangle = 1$: no Higgs phase. True in perturbation theory, order by order in g^2

If $\langle e^{i\lambda} \rangle \neq 1$, Higgs phase,

In weak coupling, diagonal gluons massless, off diagonal massive (a,b = 1...N)

$$m_{ab}^2 = g^2 |\mathrm{e}^{i\lambda_a} - \mathrm{e}^{i\lambda_b}|^2$$

But for 3D theory, gluons couple *strongly*. Effects of Higgs phase?

N.B.: above 't Hooft's abelian projection for Wilson line.

Loop potential, perturbative & not.

U(N): constant **L**, 1 loop order:

$$\mathcal{L}^{eff}_{1\ loop} = -\frac{2\,T^4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^4}\ |\mathrm{tr}\,\mathbf{L}^m|^2\ .$$

Perturbative vacuum $<e^{i\lambda}> = 1$,
stable to leading order, to *any* finite order in g^2 .

Can compute corrections to effective Lagrangian at next to leading order, NLO.
At NNLO, $\sim g^3$, need to resum m_{Debye} . Eventually, $m_{magnetic}$

SU(3) lattice: near T_c , pressure(T) $\sim T^4$ *and* $\sim T^2$.

To represent: add, *by hand:*

$$\mathcal{L}^{eff}_{non-pert.}(\mathbf{L}) = +\,B_f\,T^2\ |\mathrm{tr}\,\mathbf{L}|^2$$

$B_f \sim \#\ T_c^2$ "fuzzy" bag const. Non-pert., infinity of possible terms.

$B_f \neq 0 \Rightarrow <e^{i\lambda}> \neq 1 \Rightarrow$ Higgs phase near T_c

Confinement in $L_{\rm eff}$

SU(N), no quarks: in confined state, all Z(N) charged loops vanish:

$$\langle \mathrm{tr}\, \mathbf{L}^j_{\rm conf} \rangle = 0 \ , \ j = 1 \ldots (N-1)$$

Satisfied by "center symmetric" vacuum:

$$\mathbf{L}_{\rm conf} = \mathrm{diag}(1, z, z^2 \ldots z^{N-1}) \ , \ z = \mathrm{e}^{2\pi i/N}.$$

At finite N, perturbative pressure($\mathbf{L}_{\rm conf}$) *negative*. Not so good.

Large N: pressure($\mathbf{L}_{\rm conf}$) ~ 1, vs. ~ N^2 in deconfined phase.

At N=∞, center sym. state *can* represent confined vacuum.

$\mathbf{L}_{\rm conf}$ familiar from random matrix models:
completely *flat* eigenvalue distribution, from eigenvalue repulsion.

Where does eigenvalue repulsion arise *dynamically*?